Large liquidity expansion of super-hedging costs

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Abstract

We consider a financial market with liquidity cost as in Çetin, Jarrow and Protter [3] where the supply function $\mathbf{S}^\varepsilon(s,\nu)$ depends on a parameter $\varepsilon \geq 0$ with $\mathbf{S}^0(s,\nu) = s$ corresponding to the perfect liquid situation. Using the PDE characterization of Çetin, Soner and Touzi [6] of the super-hedging cost of an option written on such a stock, we provide a Taylor expansion of the super-hedging cost in powers of ε. In particular, we explicitly compute the first term in the expansion for a European Call option and give bounds for the order of the expansion for a European Digital Option.

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1 Introduction

The classical option pricing equation of Black & Scholes is derived under several simplifying assumptions. The "infinite" liquidity of the underlying stock process is one of them. In an attempt to understand the impact of liquidity, Çetin, Jarrow, Protter and collaborators [3, 4, 5] postulated the existence of a supply curve $\mathbf{S}(t,s,\nu)$ which is the price of a share of the stock when one wants to buy ν shares at time t. In the Black & Scholes setting, this price function is taken to be independent of ν corresponding to infinite amount of supply, hence infinite liquidity. In a recent paper, Çetin, Soner and Touzi [6] used this model and studied the liquidity premium in the price of an option written on such a stock with less than infinite liquidity. They characterized the option price by a nonlinear Black & Scholes equation, given in (2.3) below. In this pricing equation the liquidity manifests itself by means of a *liquidity function* ℓ, which is given by

$$\ell(t,s) := \left[4\,\frac{\partial \mathbf{S}}{\partial \nu}(t,s,0)\right]^{-1}, \quad (t,s) \in [0,T] \times \mathbb{R}_+.$$

The liquidity function ℓ measures the level of liquidity of the market. Namely, the larger ℓ is, the more liquid the market is.

The main result of [6] is the characterization of the liquidity premium as the unique viscosity solution of a nonlinear Black-Scholes equation (2.3), which is very similar to the one derived by Barles and Soner [2]. This nonlinear equation can only be solved numerically as no explicit solutions are available. Motivated by this fact, in this paper we obtain rigorous asymptotic expansions for the liquidity premium. For vanilla options with sufficiently regular payoff, this expansion can be calculated explicitly giving further insight into the liquidity effects.

As stated the chief objective of this paper is to analyze the large liquidity effect. Thus, we assume that the supply function depends on a small parameter ϵ

$$\mathbf{S}^\varepsilon(t,s,\nu) := \mathbf{S}(t,s,\varepsilon\nu), \qquad (t,s) \in [0,T] \times \mathbb{R}_+.$$

Then, the corresponding liquidity function is given by

$$\ell^\varepsilon(t,s) := \frac{1}{\varepsilon}\ell(t,s), (t,s) \in [0,T] \times \mathbb{R}_+ .$$

Hence, as ε tends to zero, the market becomes completely liquid. So we expect the price of an option V^ε to converge to the classical Black-Scholes price, v^{BS}, and we are interested in expansions of the form

$$V^\varepsilon = v^{BS} + \varepsilon v^{(1)} + \ldots + \varepsilon^n v^{(n)} + \; + o(\varepsilon^n).$$

Indeed, we prove this type of results and identify the functions $v^{(n)}$ in some cases. In particular, we show that

$$v^{(1)}(t,s) = \int_t^T \; \mathbb{E}_{t,s}\left[\frac{S_u^2\sigma^2(u,S_u)}{4\ell(u,S_u)}\left(v^{BS}_{ss}(u,S_u)\right)^2\right] du. \tag{1.1}$$

This is exactly the liquidity premium of the standard Black-Scholes hedge.

The paper is organized as follows. The problem is introduced in the next section and the approach is formally introduced in Section 3. Under a strong smoothness assumption, full expansion is obtained in Section 4. A quick convergence result is proved in Section 5. The Call option is studied in Section 6 and the Digital option in the final section.

2 The general setting

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a Brownian motion W with completed canonical filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0,T]\}$, where $T > 0$ is fixed maturity. The marginal price process S_t is defined by the stochastic differential equation

$$\frac{dS_t}{S_t} = \sigma(t, S_t) dW_t,$$

where σ is assumed to be bounded, Lipschitz-continuous and uniformly elliptic.
Given a continuous portfolio strategy Y with finite quadratic variation process $\langle Y \rangle$, the *small time liquidation value of the portfolio* is given by

$$dZ_t^{\varepsilon,Y} = Y_t dS_t - \left[4\ell^\varepsilon(t,S_t)\right]^{-1} d\langle Y\rangle_t = Y_t dS_t - \varepsilon \left[4\ell(t,S_t)\right]^{-1} d\langle Y\rangle_t.$$

The dependence of the process Z on its initial condition is suppressed for simplicity.
Given a function $g : \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfying

$$g \quad \text{is bounded from below and} \quad \sup_{s>0} \frac{g(s)}{1+s} \ < \ \infty, \tag{2.1}$$

the super-hedging cost is defined by

$$V^\varepsilon(t,s) \quad := \quad \inf \left\{ z \ : \ Z_t^{\varepsilon,Y} = z \text{ and } Z_T^{\varepsilon,Y} \geq g\left(S_T\right) \ \mathbb{P}\text{-}as \text{ for some } Y \in \mathcal{A}_{t,s} \right\}, \tag{2.2}$$

where the time origin is removed to t and the initial condition for the price process is $S_t = s$. We refer to [6] for the precise definition of the set of admissible strategies $\mathcal{A}_{t,s}$.
This problem is similar to the super-replication problem studied extensively in [7, 8, 9, 17, 18, 19, 20]. In the above setting, it is shown in Çetin, Soner and Touzi [6] that the value function of the super-hedging problem is the unique viscosity solution of the following nonlinear equation,

$$-V_t^\varepsilon + \hat{H}^\varepsilon\left(t, s, V_{ss}^\varepsilon\right) \ = \ 0, \quad \text{on} \quad [0,T) \times (0,\infty), \tag{2.3}$$

satisfying the terminal condition $V^\varepsilon(T,.) = g$ and the growth condition

$$-C \ \leq \ V^\varepsilon(t,s) \ \leq \ C(1+s), \ (t,s) \in [0,T] \times \mathbb{R}_+, \text{ for some constant } C > 0\,. \tag{2.4}$$

Here, $\hat{H}^\varepsilon$ denotes the elliptic majorant of the first guess operator H^ε:

$$\hat{H}^\varepsilon(t,s,\gamma) := \sup_{\beta \geq 0} \ H^\varepsilon(t,s,\gamma+\beta),$$
$$H^\varepsilon(t,s,\gamma) := -\frac{1}{2} s^2 \sigma^2(t,s)\gamma - \varepsilon[4\ell(t,s)]^{-1} s^2 \sigma^2(t,s)\gamma^2.$$

By direct calculation, it follows that

$$\hat{H}^{\varepsilon}(t,s,\gamma) = -\frac{1}{2}s^2\sigma^2(t,s)\left[\gamma + \left(\gamma + \frac{\ell(t,s)}{\epsilon}\right)^- + \frac{\epsilon}{2\ell(t,s)}\left(\gamma + \left(\gamma + \frac{\ell(t,s)}{\epsilon}\right)^-\right)^2\right].$$

For $\varepsilon = 0$, both $\hat{H}^{\varepsilon}$, H^{ε} coincides with the following standard elliptic operator,

$$\hat{H}^0(t,s,\gamma) = H^0(t,s,\gamma) = -\frac{1}{2}s^2\sigma^2(t,s)\gamma, \qquad (t,s,\gamma) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}.$$

Hence, the equation (2.3) reduces to the linear Black-Scholes equation

$$-\frac{\partial v^{BS}}{\partial t} - \frac{1}{2}s^2\sigma^2(t,s)v^{BS}_{ss} = 0. \tag{2.5}$$

We recall the well-known fact that its unique solution, v^{BS}, is the Black-Scholes price,

$$v^{BS}(t,s) = \mathbb{E}_{t,s}\left[g(S_T)\right], \qquad (t,s) \in [0,T] \times \mathbb{R}_+,$$

where we used the notation $\mathbb{E}_{t,s} = \mathbb{E}[\,\cdot \mid S_t = s]$.

3 Formal calculations and Assumptions

It is formally clear that as the market becomes more liquid, V^{ε} should converge to the Black-Scholes price v^{BS}. Indeed, this is proved in Section 5. We are also interested in a Taylor expansion of V^{ε} in the parameter ε, i.e.,

$$V^{\varepsilon}(t,s) = v^{BS}(t,s) + \varepsilon v^{(1)}(t,s) + \varepsilon^{(2)}v^2(t,s) + \ldots + \varepsilon^n v^{(n)}(t,s) + o(\varepsilon^n), \tag{3.1}$$

where $o(\varepsilon^n)$ is the standard notation, indicating that $o(\varepsilon^n)/\varepsilon^n$ converges to zero as ε tends to zero.
Indeed, under sufficient regularity

$$v^{(n)}(t,s) = \frac{1}{n!}\left.\frac{\partial^n V^{\varepsilon}(t,s)}{\partial \varepsilon^n}\right|_{\varepsilon=0}.$$

Thus, formally differentiate the equation (2.3) n-times with respect to ε and then set ε to zero. Using the above formal definition of $v^{(n)}$, we arrive at,

$$0 = -v^{(n)}_t - \frac{1}{2}s^2\sigma^2(t,s)v^{(n)}_{ss} - F_n(t,s), \tag{3.2}$$

$$F_n(t,s) = \frac{s^2\sigma^2(t,s)}{4\ell(t,s)}\sum_{k=0}^{n-1}\left[v^{(k)}_{ss}(t,s)\; v^{(n-1-k)}_{ss}(t,s)\right], \tag{3.3}$$

where we set $v^{(0)} := v^{BS}$. For all $n \geq 1$, the terminal data is $v^{(n)}(T,\cdot) \equiv 0$, so that the Feymann-Kac formula yields

$$v^{(n)}(t,s) = \sum_{k=0}^{n-1} \mathbb{E}_{t,s}\left[\int_t^T \left(\frac{S_u^2\sigma^2}{4\ell}v^{(k)}_{ss}v^{(n-1-k)}_{ss}\right)(u,S_u)du\right]. \tag{3.4}$$

In particular, $v^{(1)}$ is given as in (1.1).

The above calculations yield a rigorous proof when the pay-off is sufficiently regular. We will prove this in Section 4. On the other hand, for some discontinuous pay-offs the above functions may not be finite. For instance, for a digital option, $v^{(1)} \equiv \infty$. Indeed, if we take

$$g(s) := \mathbb{1}_{s \geq K}, \qquad \sigma(t,s) \equiv \sigma \qquad \text{and} \qquad \ell(t,s) \equiv \ell,$$

we compute that

$$\begin{aligned} v^{(1)}(t,s) &= \frac{1}{8\pi\ell\sigma^2} \int_t^T \frac{(u-t) e^{-\left(\frac{1}{\sigma\sqrt{T+u-2t}} \ln\left(\frac{s}{K}\right) + \frac{\sigma}{2} \frac{T-2u+t}{\sqrt{T+u-2t}}\right)^2}}{(T-u)^{\frac{3}{2}} (T+u-2t)^{\frac{3}{2}}} \\ &+ \frac{1}{8\pi\ell\sigma^2} \int_t^T \frac{e^{-\left(\frac{1}{\sigma\sqrt{T+u-2t}} \ln\left(\frac{s}{K}\right) + \frac{\sigma}{2} \frac{T-2u+t}{\sqrt{T+u-2t}}\right)^2}}{\sqrt{T-u}(T+u-2t)^{\frac{3}{2}}} \left(\frac{\ln\left(\frac{s}{K}\right)}{\sigma\sqrt{T+u-2t}} + \frac{\sigma}{2} \frac{T-2u+t}{\sqrt{T+u-2t}} \right)^2 . \end{aligned}$$

The first term above is actually $+\infty$ because of the non-integrability of $(T-u)^{-3/2}$ near T.

In such cases, the expansion is not valid and a careful study of the behavior of V^ε near the terminal data is needed. This will be done in Section 7. However, we first prove the full expansion in the "smooth" case. Then, in Section 6, we consider the Call option proving the expansion up to $n = 2$. Clearly, this later result extends to all Put options. Also, remarks on other payoffs and higher expansions are given in Remarks 6.2 and 6.1.

4 Expansion for smooth pay-offs

In this section, we prove the expansion under the assumption that there is a constant $\hat{C}$ so that

$$\begin{aligned} -\hat{C} \leq v^{(n)}(t,s) \leq \hat{C}(1+s), &\qquad \left|(s^2+1) v^{(n)}_{ss}(t,s)\right| \leq \hat{C}, \\ |F_n(t,s)| \leq \hat{C}, &\qquad \forall\ (t,s) \in [0,T] \times \mathbb{R}_+,\ n = 1, 2, \dots . \end{aligned} \tag{4.1}$$

Clearly, this is an implicit assumption on the pay-off g. Essentially, it holds for all smooth pay-offs growing at most linearly. In particular, (4.1) holds if $\sigma(t,s) \equiv \sigma$, $\ell(t,s) \equiv \ell$ and if there exists a constant C so that

$$-C \leq g(s) \leq C(1+s), \quad \left|(s^2+1) \frac{\partial^n}{\partial s^n} g(s)\right| \leq C, \qquad \forall\ s \in \mathbb{R}_+,\ n = 2, 3, \dots .$$

This is proved by using the homogenity of the Black-Scholes equation and differentiating it repeatedly.

Following the techniques developed in the papers [13, 11, 14, 15, 16], for an integer $n \geq 0$ we define,

$$V^{\varepsilon,n}(t,s) := \frac{V^\varepsilon(t,s) - \sum_{k=0}^{n-1} \varepsilon^k v^{(k)}(t,s)}{\varepsilon^n}\ , \tag{4.2}$$

where as before we set $v^{(0)} = v^{BS}$.

Theorem 4.1 *Assume* (4.1). *Then, for every* $n = 1, 2, \ldots$, *there are constants* C_n *and* $\varepsilon_0 > 0$ *so that for every* $\varepsilon \in (0, \varepsilon_0]$, *and* $n = 1, 2, \ldots$,

$$v^{BS}(t,s) \le V^\varepsilon(t,s) \le v^{\varepsilon,n}(t,s) := \sum_{k=0}^{n-1} [\varepsilon^k v^{(k)}(t,s)] + \varepsilon^n C_n (T-t). \tag{4.3}$$

In particular, as $\varepsilon \downarrow 0$, V^ε *converges to the Black-Scholes price* v^{BS} *uniformly on compact sets. Moreover, for every* $n \ge 1$, $V^{\varepsilon,n}$ *converges to* $v^{(n)}$, *again uniformly on compact sets.*

Proof. Clearly, $v^{BS} \le V^\varepsilon$. We continue by proving the upper bound. Let $v^{\varepsilon,n}$ be as in (4.3) with a constant C_n to be determined below. Using (3.2), we calculate that

$$\begin{aligned} -v_t^{\varepsilon,n}(t,s) \quad &+ \quad \hat{H}^\varepsilon(t,s,v_{ss}^{\varepsilon,n}(t,s)) \ge -v_t^{\varepsilon,n}(t,s) + H^\varepsilon(t,s,v_{ss}^{\varepsilon,n}(t,s)) \\ &= \quad -v_t^{\varepsilon,n} - \frac{1}{2}s^2\sigma^2 v_{ss}^{\varepsilon,n} - \frac{\varepsilon s^2\sigma^2}{4\ell(t,s)}\,(v_{ss}^{\varepsilon,n})^2 \\ &= \quad \varepsilon^n C_n + \sum_{k=1}^{n-1} [\varepsilon^k \; F_k(t,s)] - \frac{\varepsilon s^2\sigma^2}{4\ell(t,s)}\,(v_{ss}^{\varepsilon,n})^2 . \end{aligned}$$

In view of (3.3),

$$\frac{\varepsilon s^2\sigma^2}{4\ell(t,s)}\,(v_{ss}^{\varepsilon,n})^2 - \sum_{k=1}^{n-1} [\varepsilon^k \; F_k(t,s)] = \varepsilon^n F_n(t,s) + \varepsilon^{n+1}\frac{s^2\sigma^2}{4\ell(t,s)}\; g^\varepsilon(t,s),$$

where $g^\varepsilon(t,s)$ is a quadratic function $v_{ss}^{(k)}(t,s)$ for $k \le n$ and possibly powers of ε. Hence by (4.1), there is a constant C_n,

$$\left| \sum_{k=1}^{n-1} [\varepsilon^k \; F_k(t,s)] - \frac{\varepsilon s^2\sigma^2}{4\ell(t,s)}\,(v_{ss}^{\varepsilon,n})^2 \right| \le \varepsilon^n C_n.$$

Hence, we conclude that $v^{\varepsilon,n}$ is a supersolution of (2.3). Moreover, by (4.1), $-C \le v^{\varepsilon,n}(t,s) \le C(1+s)$. Then, by the comparison theorem for (2.3) (Theorem 6.1 of [6]), we conclude that $V^\varepsilon(t,s) \le v^{\varepsilon,n}(t,s)$.
In particular, this estimate implies the convergence of V^ε to v^{BS}. To prove the convergence of $V^{\varepsilon,n}$, we first observe that

$$V^\varepsilon = \sum_{k=0}^{n} [\varepsilon^k v^{(n)}(t,s)] + \varepsilon^n V^{\varepsilon,n}.$$

Using the equations (2.3) and (3.2), we conclude that $V^{\varepsilon,n}$ is a viscosity solution of

$$-V_t^{\varepsilon,n} - \frac{1}{2}s^2\sigma^2(t,s)V_{ss}^{\varepsilon,n} + F^{\varepsilon,n}\,(t,s,V_{ss}^{\varepsilon,n}) = 0, \qquad (t,s) \in [0,T) \times \mathbb{R}_+,$$

where

$$F^{\varepsilon,n}(t,s,\gamma) := \frac{1}{\varepsilon^n}\;\left[\hat{H}^\varepsilon(t,s,v_{ss}^{\varepsilon,n}(t,s) + \varepsilon^n\gamma) + \frac{1}{2}s^2\sigma^2 v_{ss}^{\varepsilon,n} + \sum_{k=1}^{n-1}\;\varepsilon^k F^k(t,s)\right].$$

Tedious but a straightforward calculation shows that

$$\lim_{(t',s',\gamma',\varepsilon)\to(t,s,\gamma,0)} F^{\varepsilon,n}(t',s',\gamma') \quad = \quad F^n(t,s),$$

where F_n is as in (3.3). Then, by the classical stability results of viscosity solutions [1, 10, 12], the Barles-Perthame semi-relaxed limits

$$\underline{v}^{(n)}(t,s) := \liminf_{(t',s',\varepsilon)\to(t,s,0)} V^{\varepsilon,n}(t',s') \text{ and } \overline{v}^{(n)}(t,s) := \limsup_{(t',s',\varepsilon)\to(t,s,0)} V^{\varepsilon,n}(t',s'),$$

are, respectively, a viscosity supersolution and a subsolution of the equation (3.2) satisfied by $v^{(n)}$. Moreover it follows from (4.3) that

$$\underline{v}^{(n)}(T,\cdot) = \overline{v}^{(n)}(T,\cdot) = 0 = v^{(n)}(T,\cdot).$$

We now use the comparison result for the linear partial differential equation (3.2), and conclude that $\underline{v}^{(n)} \geq \overline{v}^{(n)}$. Since

$$\underline{v}^{(n)}(t,s) \leq \liminf_{\varepsilon\to 0} V^{\varepsilon,n}(t,s) \leq \limsup_{\varepsilon\to 0} V^{\varepsilon,n}(t,s) \leq \overline{v}^{(n)}(t,s)$$

on $[0,T]\times\mathbb{R}_+$, this proves that $\underline{v}^{(n)} = \overline{v}^{(n)} = v^{(n)}$. Hence, $V^{\varepsilon,n}$ converges to the unique solution $v^{(n)}$, uniformly on compact sets.

□

5 A general convergence result

In this section, we prove an easy convergence result under the following general assumption. We assume that

$$cs^2 \leq \ell(t,s), \tag{5.1}$$

for some constant and

Assumption 5.1 *There is a decreasing sequence of smooth approximation $g_m \geq g$ of the pay-off g satisfying* (4.1) *with $n = 1, 2$. Let $v_m^{(n)}$, F_m^n be the previously defined functions with pay-off g_m. Then, $F_m^1(t,s) \leq c_m$ for some constant c_m.*

This assumption is satisfied by all Lipschitz or for all bounded pay-offs.

Theorem 5.1 *Assume* (2.1), (5.1) *and that Assumption 5.1 holds true. Then, as the liquidity parameter goes to infinity, or equivalently as $\varepsilon \downarrow 0$, V^ε converges to the Black-Scholes price v^{BS}.*

Proof. Let c_m be as above and set

$$u^\varepsilon(t,s) := v_m^{BS}(t,s) + \varepsilon c_m(T-t).$$

As in the proof of Theorem 4.1, we can show that u^ε is a super-solution of (2.3). Hence, $V^\varepsilon \le u^\varepsilon$. Therefore,

$$\limsup_{\varepsilon \downarrow 0} \ V^\varepsilon(t,s) \le v_m^{BS}(t,s).$$

By (2.1), $v_m^{BS}(t,s)$ converges to $v^{BS}(t,s)$. Since $V^\varepsilon \ge v^{BS}$, this proves the convergence of V^ε to v^{BS}.

□

6 First order expansion for convex payoffs

One major limitation of our previous result is that the Call pay-off does not satisfy the Assumption (4.1). Therefore, in this section, we prove the first term in the Taylor expansion (3.1), i.e.,

$$V^\varepsilon(t,s) = v^{BS}(t,s) + \varepsilon v^{(1)}(t,s) + o(\varepsilon), \tag{6.1}$$

for convex payoffs satisfying weaker assumptions than (4.1). In particular, we will show that call options verify those assumptions.

6.1 The general result

In order to capitalize on the results we have already obtained for smooth payoffs, we will also consider a regularized version of our problem

$$\begin{aligned} &-V_t^{\epsilon,\alpha} + \widehat{H}^\varepsilon(t,s,V_{ss}^{\epsilon,\alpha}) = 0, \ \text{for } (t,s) \in [0,T) \times \mathbb{R}_+, \\ &V^{\varepsilon,\alpha}(T,s) = \widehat{g}_\alpha(s), \end{aligned} \tag{6.2}$$

where $\widehat{g}_\alpha(s) = \phi_\alpha * g(s)$ with $\phi_\alpha(\cdot) := \frac{1}{\alpha}\phi(\frac{\cdot}{\alpha})$ and ϕ is a positive, symmetric bump function on $\mathbb{R}$, compactly supported in $[-1,1]$ and satisfying

$$\int_{-1}^{1} \phi(u) du = 1.$$

By convexity of g, for all $\alpha > 0$ we have $\widehat{g}_\alpha \ge g$, so that by monotony of our problem

$$V^\varepsilon \le V^{\varepsilon,\alpha}.$$

Thus, since the main idea of our proof is to find a super-solution of (2.3), we see that it is enough to find a super-solution of (6.2). Let $v^{BS,\alpha}$ and $v^{(1),\alpha}$, respectively, be the Black-Scholes price and the first-order expansion term for the regularized option. We now state our assumptions

Assumption 6.1 (i) $v^{BS} + v^{BS,\alpha} + v^{(1)} + v^{(1),\alpha} < +\infty$.

(ii) *As α tends to 0 we have*

$$v^{BS,\alpha}(t,s) = v^{BS}(t,s) + O(\alpha^2),$$
$$v^{(1),\alpha}(t,s) = v^{(1)}(t,s) + o(1).$$

(iii) *There exists a constant c_* independent of s, $T-t$ and α and $(\nu,\beta) \in [0,1] \times [1/2,1]$ such that $1 < 2\beta + \nu < 2$ and*

$$\frac{s^2\sigma^2}{4\ell}(v^{(1),\alpha}_{ss}(t,s))^2 \leq \frac{c_*}{(T-t)^{1-\nu}\alpha^{2+2\nu}}, \qquad s\left|v^{BS,\alpha}_{ss}(t,s)\right| \leq \frac{c_*}{(T-t)^{1-\beta}\alpha^{2\beta-1}}.$$

This assumption will be proved to be verified by Call options payoffs in subsection 6.2.

Let $V^{\epsilon,1}$ be as (4.2), i.e.

$$V^{\epsilon,1}(t,s) := \frac{V^\epsilon(t,s) - v^{BS}(t,s)}{\epsilon}.$$

Theorem 6.1 *Let Assumption 6.1 hold true and let $a \in (\frac{1}{2}, \frac{1}{2\beta+\nu})$. Then for every $(t,s) \in [0,T] \times \mathbb{R}_+$ we have,*

$$v^{BS} \leq V^\varepsilon \leq v^{BS,\epsilon^a} + \epsilon v^{(1),\epsilon^a} + c_*(T-t)^{\beta+\frac{\nu-1}{2}}\epsilon^{2-a(\nu+2\beta)} + c_*(T-t)^\nu \epsilon^{3-2a(1+\nu)}.$$

Moreover, $V^\varepsilon \to v^{BS}$, $V^{\varepsilon,1} \to v^{(1)}$ uniformly on compact sets, and (6.1) holds true.

Proof. It is clear that $V^\varepsilon \geq v^{BS}$. To prove the reverse inequality, we start by following a technique similar to the one used in the proof of Theorem 4.1. Set

$$v^{\varepsilon,2} := v^{BS,\epsilon^a} + \epsilon v^{(1),\epsilon^a} + c_*(T-t)^{\beta+\frac{\nu-1}{2}}\epsilon^{2-a(\nu+2\beta)} + c_*(T-t)^\nu \epsilon^{3-2a(1+\nu)}.$$

We calculate that for $(t,s) \in [0,T) \times \mathbb{R}_+$

$$\begin{aligned}
&-v^{\varepsilon,2}_t + \hat{H}^\varepsilon(t,s,v^{\varepsilon,2}_{ss}) \geq -v^{\varepsilon,2}_t + H^\varepsilon(t,s,v^{\varepsilon,2}_{ss}) \\
&= \frac{c_*\epsilon^{2-a(\nu+2\beta)}}{(T-t)^{1-\beta-\frac{\nu-1}{2}}} + \frac{c_*\epsilon^{3-2a(1+\nu)}}{(T-t)^{1-\nu}} - v^{BS,\epsilon^a}_t - \varepsilon v^{(1),\epsilon^a}_t - \frac{1}{2}s^2\sigma^2 v^{\varepsilon,2}_{ss} - \frac{\varepsilon s^2\sigma^2}{4\ell}\left(v^{\varepsilon,2}_{ss}\right)^2 \\
&= \frac{c_*\epsilon^{2-a(\nu+2\beta)}}{(T-t)^{1-\beta-\frac{\nu-1}{2}}} + \frac{c_*\epsilon^{3-2a(1+\nu)}}{(T-t)^{1-\nu}} - \frac{s^2\sigma^2}{4\ell}(v^{(1),\epsilon^a}_{ss})^2\varepsilon^3 - \frac{s^2\sigma^2}{2\ell}v^{BS,\epsilon^a}_{ss}v^{(1),\epsilon^a}_{ss}\varepsilon^2.
\end{aligned}$$

In view of Assumption 6.1(iii), this quantity is always positive. We now analyze the terminal condition. In view of the conditions imposed on a, β and ν

$$v^{\varepsilon,2}(T,s) = v^{BS,\epsilon^a}(T,s) = \widehat{g}_{\varepsilon^a}(s).$$

Hence, $v^{\varepsilon,2}$ is a super-solution of (6.2) and therefore of (2.3). Then, by the comparison theorem for (2.3) (proved in [6]), we conclude that $V^\varepsilon(t,s) \leq v^{\varepsilon,2}(t,s)$.

We now let ϵ go to 0 in the above inequalities. This proves that V^ε converges to v^{BS} uniformly on compact sets.

Finally, by Assumption 6.1(ii)

$$0 \leq V^{\varepsilon,1}(t,s) \leq v^{(1)}(t,s) + o\left(\varepsilon^{\min\{1-a(2\beta+\nu),2-2a(1+\nu)\}}\right) + O\left(\varepsilon^{2a-1}\right),$$

where it is clear with our conditions on a, β and ν that the $o(\cdot)$ and $O(\cdot)$ above go to 0 as ε tends to 0.

Using this estimate, we then prove the convergence of $V^{\varepsilon,1}$ exactly as in Theorem 4.1. $\square$

Remark 6.1 Higher expansions can be proved similarly, provided that we extend Assumption 6.1 for $n \geq 2$.

6.2 Expansion for the Call option

In this section, we take

$$g(s) = (s-K)^+, \qquad \sigma(t,s) \equiv \sigma, \qquad \ell(t,s) \equiv \ell,$$

and we verify that Assumptions 6.1(ii) and 6.1(iii) are satisfied, since Assumption 6.1(i) is trivial.

Straightforward but tedious calculations using the Feynman-Kac formula yield

$$\begin{aligned}
v_{ss}^{BS,\alpha}(t,s) &= \frac{1}{\sigma s\sqrt{2\pi\tau}} \int_{-1}^{1} \phi(u) \exp\left(-\frac{1}{2} d_1(s, K+\alpha u, \tau)^2\right) du, \\
v^{(1),\alpha}(t,s) &= \frac{1}{8\ell\pi} \int_0^{\tau} \int_{-1}^{1} \int_{-1}^{1} \frac{\phi(x)\phi(y) h_\alpha(\tau, v, s, K, x, y)}{\sqrt{v(2\tau - v)}} dx dy dv,
\end{aligned}$$

where

$$\begin{aligned}
\tau &= T - t, \\
d_1(s,k,t) &= \frac{1}{\sigma\sqrt{t}} \ln(s/k) + \frac{1}{2}\sigma\sqrt{t}, \\
\delta(\tau, v, s, k) &= \frac{1}{\sigma\sqrt{2\tau - v}} \ln(s/k) - \frac{\sigma}{2} \frac{\tau - 2v}{\sqrt{2\tau - v}}, \\
h_\alpha(\tau, v, s, k, x, y) &= \exp\left(-\delta(\tau, v, s, k)^2 + \frac{\delta(\tau, v, s, k)}{\sigma\sqrt{2\tau - v}} \left(\log\left(1 + \frac{\alpha x}{k}\right) + \log\left(1 + \frac{\alpha y}{k}\right)\right)\right) \\
&\quad \times \exp\left(-\frac{\tau}{2\sigma^2 v(2\tau - v)} \left(\log\left(1 + \frac{\alpha x}{k}\right) - \log\left(1 + \frac{\alpha y}{k}\right)\right)^2\right) \\
&\quad \times \exp\left(-\frac{1}{\sigma^2(2\tau - v)} \log\left(1 + \frac{\alpha x}{k}\right) \log\left(1 + \frac{\alpha y}{k}\right)\right).
\end{aligned}$$

The following two propositions, whose proof is relagated to the appendix, ensure that Assumptions 6.1(ii) and 6.1(iii) are satisfied

Proposition 6.1 *There exists a constant c_*, independent of s, τ and α so that for all $(\nu, \beta) \in [0,1] \times [1/2, 1]$:*

$$s\left|v_{ss}^{BS,\alpha}(t,s)\right| \leq \frac{c_*}{\tau^{1-\beta}\alpha^{2\beta-1}}, \qquad \frac{s^2\sigma^2}{4\ell}(v_{ss}^{(1),\alpha}(t,s))^2 \leq \frac{c_*}{\tau^{1-\nu}\alpha^{2+2\nu}}.$$

Proposition 6.2 *As α tends to 0 we have the following expansions*

$$v^{BS,\alpha}(t,s) = v^{BS}(t,s) + \alpha^2 \frac{e^{-\frac{1}{2}d_0(s,K,\tau)^2}}{2K\sigma\sqrt{2\pi\tau}} \int_{-1}^{1} \phi(v)v^2 dv + O(\alpha^4),$$

$$v^{(1),\alpha}(t,s) = v^{(1)}(t,s) - \alpha \frac{e^{-\frac{1}{2}d_0(s,K,\tau)^2}}{8K\sigma\ell\sqrt{2\pi\tau}} \int_{-1}^{1}\int_{-1}^{1} \phi(x)\phi(y)|x-y|dxdy + o(\alpha),$$

where $d_0(s,k,\tau) = \frac{1}{\sigma\sqrt{\tau}}\ln(s/k) - \frac{1}{2}\sigma\sqrt{\tau}$.

Remark 6.2 It is not hard to show that the results of Propositions 6.1 and 6.2 hold for all convex linear combination of call or put options. However, we cannot use the above proof for, say, a call spread option whose payoff is neither convex nor concave.

6.3 Numerical Experiments

In order to have a better grasp of the liquidity effects, we also solved numerically (with simple finite difference methods) the PDE (2.3). We represent below the behaviour of the liquidity premium (that is to say $V^\varepsilon - v^{BS}$) when the time to maturity t and the spot price vary

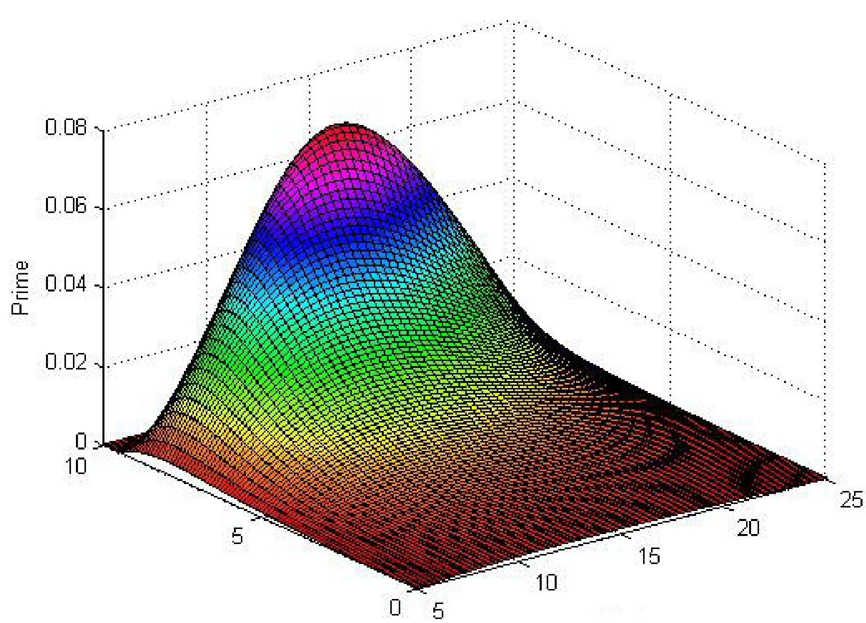

Figure 1: Call liquidity premium - $T = 10$, $K = 15$, $\sigma = 0.5$, $\epsilon = 0.1$, $\ell = 1$

In the above figure, the liquidity effect is strongly marked for ATM options and disapears quickly for ITM and OTM options. This was to be expected. Indeed, our calculations showed that the liquidity effect is, for the first order, driven by the Γ of the call option (see (A.1)), which explodes for ATM options near maturity. Moreover, with our set of parameters, the first order correction is at most 0.06 for a BS price of 8.56, which means that the hedge against liquidity risk is not that expensive when the illiquidity is not too strong.

We now compare the real liquidity premium with its first-order expansion term.

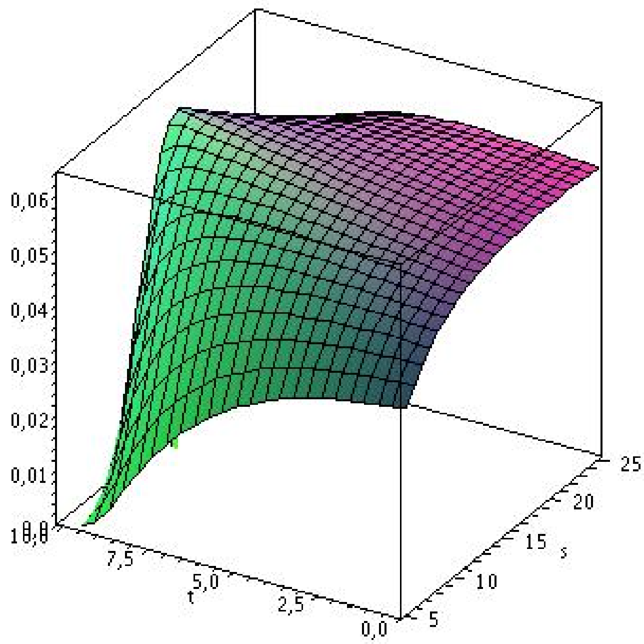

Figure 2: Call first order liquidity premium - $T = 10$, $K = 15$, $\sigma = 0.5$, $\epsilon = 0.1$, $\ell = 1$

A rapid examination of the above figure shows that the first order approximation remains excellent as long as we do not go too far from the maturity time T and we stay close to the money $s = K$. Otherwise, the first order overvalues the liquidity premium.

7 Digital Option

In this section, we analyze the specific example of a Digital option in the context of Black-Scholes model with constant liquidity parameter

$$g(s) := \mathbb{1}_{s \geq K}, \qquad and \qquad \sigma(t,s) \equiv \sigma, \qquad \ell(t,s) \equiv \ell.$$

7.1 Theoretical bounds

As pointed out earlier, for the Digital option, the first-order term that we obtained formally is equal to $+\infty$. Thus, the expansion (3.1) is no longer valid and our aim in this section is to find bounds for the first-order of the expansion. We start by approximating the option by a sequence of regularized call spreads. Then the original problem (2.3) is replaced by

$$\begin{aligned} -V_t^{\epsilon,\alpha} + \widehat{H}^{\varepsilon}(t,s,V_{ss}^{\epsilon,\alpha}) &= 0, \text{ for } (t,s) \in [0,T) \times \mathbb{R}_+, \\ V^{\varepsilon,\alpha}(T,s) &= \widehat{g}_\alpha(s), \end{aligned} \tag{7.1}$$

where $\widehat{g}_\alpha(s) = \phi_\alpha * g_\alpha(s)$ with $g_\alpha(s) = \frac{(s-K+2\alpha)^+ - (s-K+\alpha)^+}{\alpha}$.

Since ϕ_α has compact support in $[-\alpha,\alpha]$, notice that $\widehat{g}_\alpha \geq g$. Then, since the terminal condition is smooth, it follows from the comparison principle that

$$V^{\varepsilon}(t,s) \leq V^{\epsilon,\alpha}(t,s), \text{ for } (t,s,\alpha) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+^*. \tag{7.2}$$

With the same notations as in the previous section, we directly calculate using again the Feynman-Kac formula that

$$\begin{aligned} v_{ss}^{BS,\alpha}(t,s) &= \frac{1}{\sigma s\alpha\sqrt{2\pi\tau}} \int_{-1}^{1} \phi(u) \left(e^{-\frac{1}{2}d_1(s,K+\alpha u-2\alpha,\tau)^2} - e^{-\frac{1}{2}d_1(s,K+\alpha u-\alpha,\tau)^2} \right) du, \\ v^{(1),\alpha}(t,s) &= \frac{1}{8\ell\pi\alpha^2} \int_0^{\tau} \int_{-1}^{1} \int_{-1}^{1} \frac{\phi(x)\phi(y)\widehat{h}_\alpha(\tau,v,s,K,x,y)}{\sqrt{v(2\tau-v)}} dxdydv, \end{aligned}$$

where

$$\widehat{h}_\alpha(\tau,v,s,K,x,y) = \sum_{1\leq i,j\leq 2} h_\alpha(\tau,v,s,K,x-i,y-j).$$

Then, we have the two following propositions which are proved exactly as in the call option case (since the functions involved here are essentially the same)

Proposition 7.1 *There exists a constant c_*, independent of s, τ and α so that for all $(\nu,\beta) \in [0,1] \times [1/2,1]$*

$$s\left|v_{ss}^{BS,\alpha}(t,s)\right| \leq \frac{c_*}{\tau^{1-\beta}\alpha^{2\beta}}, \qquad \frac{s^2\sigma^2}{4\ell}(v_{ss}^{(1),\alpha}(t,s))^2 \leq \frac{c_*}{\tau^{1-\nu}\alpha^{6+2\nu}}.$$

Proposition 7.2 *As α tends to 0 we have the following expansions:*

$$v^{BS,\alpha}(t,s) = v^{BS}(t,s) + \frac{3}{2}\alpha\frac{e^{-\frac{1}{2}d_0(s,K,\tau)^2}}{K\sigma\sqrt{2\pi\tau}} + O(\alpha^2),$$

$$v^{(1),\alpha}(t,s) = \alpha^{-1}\frac{e^{-\frac{1}{2}d_0(s,K,\tau)^2}}{8K\sigma\ell\sqrt{2\pi\tau}} \int_{-1}^{1}\int_{-1}^{1} \phi(x)\phi(y)(|x-y-1|+|x-y+1|-2|x-y|)dxdy + o(\alpha^{-1}).$$

Define $V^{\epsilon,1,c}$ by

$$V^{\epsilon,1,c}(t,s) := \frac{V^{\epsilon}(t,s) - v^{BS}(t,s)}{\epsilon^c}.$$

Theorem 7.1 *Let $(\beta,\nu) \in [1/2,1] \times [0,1]$ be such that $\gamma := \frac{2\beta+\nu-1}{2\beta+\nu+4} \in (0,1)$ and set $a := \frac{2}{5}(1-\gamma)$. Then for all $(t,s) \in [0,T] \times \mathbb{R}_+$,*

$$v^{BS} \leq V^{\varepsilon} \leq v^{BS,\epsilon^a} + \epsilon v^{(1),\epsilon^a} + c_*(T-t)^{\beta+\frac{\nu-1}{2}}\epsilon^{2-3a-a(\nu+2\beta)} + c_*(T-t)^{\nu}\epsilon^{3-2a(3+\nu)}.$$

In particular, V^{ε} converges to v^{BS}, uniformly on compact sets and

$$0 \leq \liminf_{(t',s',\varepsilon)\to(t,s,0)} V^{\varepsilon,1,a}(t',s',a) \leq \limsup_{(t',s',\varepsilon)\to(t,s,0)} V^{\varepsilon,1,a}(t',s') \leq \frac{3}{2}\frac{e^{-\frac{1}{2}d_0(s,K,\tau)^2}}{K\sigma\sqrt{2\pi\tau}} + c_*(T-t)^{\frac{5\gamma}{2(1-\gamma)}},$$

i.e. the order of the expansion is at least $2/5$.

Proof. It is clear that $V^{\varepsilon} \geq v^{BS}$. To prove the reverse inequality, we start by following a technique similar to the one used in the proof of Theorem 6.1. Set

$$v^{\varepsilon,2} := v^{BS,\epsilon^a} + \epsilon v^{(1),\epsilon^a} + c_*(T-t)^{\beta+\frac{\nu-1}{2}}\epsilon^{2-3a-a(\nu+2\beta)} + c_*(T-t)^{\nu}\epsilon^{3-2a(3+\nu)}.$$

We proceed exactly as in Theorem 6.1 using Proposition 7.1. The result is

$$-v_t^{\varepsilon,2}(t,s) + \hat{H}^{\varepsilon}(t,s,v_{ss}^{\varepsilon,2}(t,s)) \geq 0, \text{ for } (t,s) \in [0,T) \times \mathbb{R}_+.$$

We now analyze the terminal condition. Since $2\beta+\nu > 1$, we have

$$v^{\varepsilon,2}(T,s) = v^{BS,\epsilon^a}(T,s).$$

Hence, $v^{\varepsilon,2}$ is a super-solution of (6.2) and therefore of (2.3). Then, by the comparison theorem for (2.3) (proved in [6]), we conclude that $V^{\varepsilon}(t,s) \leq v^{\varepsilon,2}(t,s)$.
Then by Proposition 7.2 and the conditions imposed on a, β and ν, we obtain easily the uniform convergence on compact sets of V^{ε} to v^{BS} by letting ε go to 0.
Now for the first order term, we would like to use our expansions and obtain a finite majorant for $V^{\varepsilon,1,c}$ with the largest possible c. It is easy to argue that $c = a$ is the best choice possible. This, in turn, imposes the following condition

$$a \leq \min\left\{\frac{1}{2}, \frac{2}{4+2\beta+\nu}, \frac{3}{7+2\nu}\right\} = \frac{2}{4+2\beta+\nu}.$$

Now it follows that, for all $\gamma > 0$ small enough, there are β and ν satisfying our conditions so that $\frac{2}{4+2\beta+\nu} = \frac{2}{5}(1-\gamma)$. It suffices then to take the lim inf and lim sup in the inequality to prove the result. $\square$

7.2 Numerical results

The digital option liquidity premium In this section, we provide numerical results for the case of the Digital option. As in the section 6.3 the PDE (2.3) is solved with finite difference method. We represent below the behaviour of the liquidity premium when the time to maturity t and the spot price vary

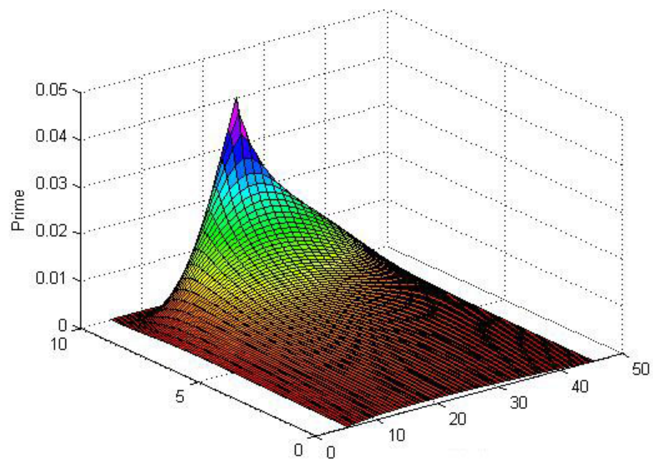

Figure 3: Digital liquidity premium - $T = 10$, $K = 25$, $\sigma = 0.5$, $\epsilon = 0.1$, $\ell = 1$

Qualitatively, the liquidity premium behaves as in the Call case. However, as expected the effects of illiquidity are even stronger for ATM options near maturity, since the Γ of a digital option explodes faster. Moreover, with our set of parameters, the first order correction to the price is at most 0.04 for a BS price of 0.21, which means that the hedge against liquidity risk is much more expensive in the case of a digital option, for a same level of liquidity in the market.

Numerical confirmation of the expansion order We represent below the liquidity premium for a fixed value of the spot when the parameter ε varies with a logarithmic scale.

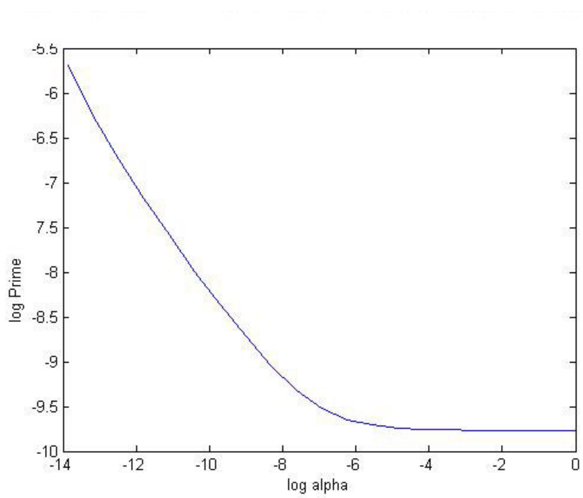

Figure 4: $\log\left(V^{\varepsilon} - v^{BS}\right)$ - $T = 1$, $K = 25$, $s = 15$, $\sigma = 0.5$, $\epsilon = 0.1$, $\ell = 1$

For small values of ε we observe the expected linear behaviour of $\log\left(V^{\varepsilon} - v^{BS}\right)$. The slope of the above curve is roughly equal to $1/2$ (the exact value here is 0.54), which is close to our minimal value of $2/5$. The numerical results suggest that the true expansion order lies in the interval $[2/5, 1/2]$.

It is also important to realize the financial implications of our results. We just have highlighted the fact that the first order effect exhibits a phase transition for discontinuous payoff, in the sense that derivative securities of the type of digital options induce a cost of illiquidity which vanishes at a significantly slower rate than the continuous payoff case. This means that derivative with discontinuous payoff are more rapidly affected by the illiquidity cost.

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References

[1] Barles, G. and Perthame, B (1987). Discontinuous solutions of deterministic optimal stopping problems. *Math. Modeling Numerical Analysis*, 21, 557–579.

[2] Barles, G. and Soner, H.M (1998). Option pricing with transaction costs and a nonlinear Black-Scholes equation, *Finance and Stochastics*, 2, 369–397.

[3] Çetin, U., Jarrow, R. and Protter, P. (2004). Liquidity risk and arbitrage pricing theory, *Finance and Stochastics*, 8, 311–341.

[4] Çetin, U., Jarrow, R., Protter, P. and Warachka, M. (2006) Pricing options in an extended Black-Scholes economy with illiquidity: theory and empirical evidence, *The Review of Financial Studies*, 19, 493–529.

[5] Çetin, U. and Rogers, L.C.G. (2006) Modelling liquidity effects in discrete time, *Math. Finance*, forthcoming.

[6] Cetin, U., Soner, H.M., and Touzi, N. (2007). Options hedging for small investors under liquidity costs, preprint.

[7] Cheridito, P., Soner, H.M. and Touzi, N. (2005a). The multi-dimensional super-replication problem under gamma constraints, *Annales de l'Institute Henri Poincaré (C) Non Linear Analysis*, 22 (5): 633-666.

[8] Cheridito, P., Soner, H.M. and Touzi, N. (2005b). Small time path behavior of double stochastic integrals and applications to stochastic control, *Annals of Applied Probability*, 15 (4): 2472-2495.

[9] Cheridito, P., Soner, H.M., Touzi, N., and Victoir, N. (2007). Second Order Backward Stochastic Differential Equations and Fully Non-Linear Parabolic PDEs, *Communications on Pure and Applied Mathematics*, 60 (7): 1081-1110.

[10] Crandall, M.G., Ishii, H., and Lions, P.L. (1992). User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc.* 27(1), 1–67.

[11] Fleming, W.H., and Soner, H.M. (1989). Asymptotic expansions for Markov processes with Levy generators. *Applied Mathematics and Optimization* , 19(3), 203–223.

[12] Fleming, W.H., and Soner, H.M. (1993). *Controlled Markov Processes and Viscosity Solutions*. Applications of Mathematics 25. Springer-Verlag, New York.

[13] Fleming, W.H., and Souganidis, P.E. (1986). Asymptotic series and the method of vanishing viscosity, *Indiana University Mathemtics Journal* , 35(2), 425–447.

[14] Lehoczky, J., Sethi, S.P., Soner, H.M., and Taksar, M.I. (1991). An asymptotic analysis of hierarchical control of manufacturing systems under uncertainity. *Mathematics of Operations Research*, 16(3), 596–608.

[15] Sethi, S., Soner, H.M., Zhang, Q., and Jiang, J. (1992). Turnpike Sets and Their Analysis in Stochastic Production Planning Problems. *Mathematics of Operations Research*, 17(4), 932–950.

[16] Soner, H.M. (1993). Singular perturbations in manufacturing. *SIAM J. Control and Opt.* 31(1), 132–146.

[17] Soner, H.M., and Touzi, N. (2000). Super-replication under gamma constraints. *SIAM J. Control and Opt.* 39(1), 73–96.

[18] Soner, H.M., and Touzi, N. (2002). Stochastic target problems, dynamic programming and viscosity solutions, *SIAM J. Control and Opt.* 41, 404–424.

[19] Soner, H.M., and Touzi, N. (2002). Dynamic programming for stochastic target problems and geometric flows, *J. European Math. Soc.*, **4**, 201–236.

[20] Soner, H.M., and Touzi, N. (2007). The dynamic programming equation for second order stochastic target problems, preprint.

A Technical Proofs

Proof. [Proof of Proposition 6.1] We start by proving the inequality for $v_{ss}^{BS,\alpha}$. By dominated convergence, it is clear that $sv_{ss}^{BS,\alpha}$ goes to 0 when s approaches 0 or $+\infty$. Hence for $\alpha \neq 0$, it also converges to 0 when τ tends to 0. Thus $sv_{ss}^{BS,\alpha}$ is less than a constant C_α independent of s and τ. However, when α tends to zero, we obtain the classical expression of the Γ of a call option

$$v_{ss}^{BS}(t,s) = \frac{e^{-\frac{1}{2}d_1(s,K,\tau)^2}}{s\sigma\sqrt{2\pi\tau}}, \tag{A.1}$$

which is known to explode only when $s = K$ and $\tau \to 0$. Therefore, to understand the dependence in α of C_α, we only have to study the behaviour of $sv_{ss}^{BS,\alpha}$ when $s = K$ and when both α and τ go to 0.

Let us therefore take $\alpha = \epsilon^a$ and $\tau = \varepsilon^b$ with a and b strictly positive numbers. For all $\beta \in [1/2, 1]$ we have

$$\tau^{1-\beta}\alpha^{2\beta-1}sv_{ss}^{BS,\varepsilon^a} = \frac{\epsilon^{(b/2-a)(1-2\beta)}}{\sigma\sqrt{2\pi}}\int_{-1}^{1}\phi(u)e^{-\frac{1}{2}\left(\frac{\sigma\varepsilon^{b/2}}{2}-\frac{\varepsilon^{-b/2}}{\sigma}\log\left(1+\frac{\varepsilon^a u}{K}\right)\right)^2}du$$

Therefore, if $a < b/2$ (i.e. if τ goes to 0 faster than α) the quantity above always goes to 0 when $\varepsilon \to 0$ due to the exponential term. If $a \geq b/2$, the exponential term goes to 1, but since $\beta \in [1/2, 1]$ the above expression has always a finite limit. Hence the inequality for $sv_{ss}^{BS,\alpha}$.

A change of variable and direct calculations imply that, for all $\nu \in [0, 1]$, we have

$$\tau^{\frac{1-\nu}{2}}\alpha^{1+\nu}sv_{ss}^{(1),\alpha}(t,s) = \frac{\alpha^{1+\nu}\tau^{-\frac{1+\nu}{2}}}{8\ell\pi s}\int_0^1\int_{(-1,1)^2}\frac{\phi(x)\phi(y)\widetilde{h}_\alpha(\tau,\tau v,s,K,x,y)}{\sqrt{v}(2-v)^{3/2}}dxdydv, \tag{A.2}$$

where

$$\begin{aligned}\frac{\widetilde{h}_\alpha(\tau,v,s,K,x,y)}{h_\alpha(\tau,v,s,K,x,y)} &= 2 + \left(2\delta(\tau,\tau v,s,K) - \frac{\log\left(1+\frac{\alpha x}{K}\right)+\log\left(1+\frac{\alpha y}{K}\right)}{\sigma\sqrt{\tau(2-v)}}\right)^2 \\ &\quad + \left(2\delta(\tau,\tau v,s,K) - \frac{\log\left(1+\frac{\alpha x}{K}\right)+\log\left(1+\frac{\alpha y}{K}\right)}{\sigma\sqrt{\tau(2-v)}}\right)\sigma\sqrt{\tau(2-v)}.\end{aligned}$$

Using the same arguments as in the proof of the previous inequality, we can show again that the only problem corresponds to the case where $s = K$ and α and τ go to 0. Using the same notations, we have

$$
\begin{aligned}
h_{\varepsilon^a}(\varepsilon^b, \varepsilon^b v, s, s, x, y) = \exp &\left(-\frac{\sigma^2 \varepsilon^b (1-2v)^2}{4(2-v)} + \frac{(1-2v)\left(\log\left(1+\frac{\varepsilon^a x}{K}\right) + \log\left(1+\frac{\varepsilon^a y}{K}\right)\right)}{2(2-v)} \right) \\
&\times \exp\left(-\frac{\varepsilon^{-b}}{\sigma^2(2-v)} \log\left(1+\frac{\varepsilon^a x}{K}\right) \log\left(1+\frac{\varepsilon^a y}{K}\right) \right) \\
&\times \exp\left(-\frac{\varepsilon^{-b}\left(\log\left(1+\frac{\alpha x}{K}\right) - \log\left(1+\frac{\alpha y}{K}\right)\right)^2}{2\sigma^2 v(2-v)} \right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\widetilde{h}_{\varepsilon^a}(\varepsilon^b, v, s, s, x, y)}{h_{\varepsilon^a}(\varepsilon^b, v, s, s, x, y)} = 2 &+ \left(\frac{\sigma \varepsilon^{\frac{b}{2}}(1-2v)}{\sqrt{2-v}} + \varepsilon^{-b} \frac{\log\left(1+\frac{\varepsilon^a x}{K}\right) + \log\left(1+\frac{\varepsilon^a y}{K}\right)}{\sigma\sqrt{2-v}} \right)^2 \\
&- \left(\frac{\sigma \varepsilon^{\frac{b}{2}}(1-2v)}{\sqrt{2-v}} + \varepsilon^{-b} \frac{\log\left(1+\frac{\varepsilon^a x}{K}\right) + \log\left(1+\frac{\varepsilon^a y}{K}\right)}{\sigma\sqrt{2-v}} \right) \sigma\sqrt{2-v}\,\varepsilon^{\frac{b}{2}}.
\end{aligned}
$$

Therefore, if $a < b/2$, $\widetilde{h}_{\varepsilon^a}$ always goes to 0. Otherwise, the integral has a finite limite but since $\nu \in [0,1]$ and $a \geq b/2$, the expression in (A.2) has a finite limit. This proves the second inequality. □

Proof. [Proof of Proposition 6.2] The first result is straightforward and only uses the fact that the function ϕ is symmetric, which allows us to get rid off the odd terms in the expansion. For the second one, we directly calculate that

$$
\begin{aligned}
v^{(1),\alpha} = &\int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2 - \frac{\alpha^2(x-y)^2}{4K^2\sigma^2 v(1-\frac{v}{2\tau})} + o(\alpha^2)}}{8\pi\ell\sqrt{v(2\tau-v)}} dxdydv \\
&+ \alpha \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2 - \frac{\alpha^2(x-y)^2}{4K^2\sigma^2 v(1-\frac{v}{2\tau})} + o(\alpha^2)}\delta}{8\pi\ell K\sigma\sqrt{v}(2\tau-v)} (x+y) dxdydv \\
&+ \alpha^2 \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2 - \frac{\alpha^2(x-y)^2}{4K^2\sigma^2 v(1-\frac{v}{2\tau})} + o(\alpha^2)} \left(2(x+y)^2\delta^2 + \sigma\sqrt{2\tau-v}(x^2+y^2)\delta - 2xy\right)}{16\pi\ell K^2\sigma^2\sqrt{v}(2\tau-v)^{3/2}} dxdydv \\
&+ o\left(\alpha^2 \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2 - \frac{\alpha^2(x-y)^2}{4K^2\sigma^2 v(1-\frac{v}{2\tau})} + o(\alpha^2)}}{8\pi\ell\sqrt{v(2\tau-v)}} dxdydv \right),
\end{aligned}
$$

where we suppressed the arguments of the functions $v^{(1),\alpha}$ and δ for notational simplicity. Note that all the above integrals are well-defined and finite. Then using dominated convergence and the fact that ϕ is symmetric, it is easy to show that

$$
\begin{aligned}
v^{(1),\alpha} &= \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2 - \frac{\alpha^2 (x-y)^2}{4K^2\sigma^2 v(1-\frac{v}{2\tau})} + o(\alpha^2)}}{\sqrt{8\pi\ell v(2\tau - v)}} dxdydv \\
&+ \alpha \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2}\delta}{8\pi\ell K\sigma\sqrt{v}(2\tau - v)} (x+y) dxdydv \\
&+ \alpha^2 \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2}\left(2(x+y)^2\delta^2 + \sigma\sqrt{2\tau - v}(x^2+y^2)\delta - 2xy\right)}{16\pi\ell K^2\sigma^2\sqrt{v}(2\tau - v)^{3/2}} dxdydv + o\left(\alpha^2\right) \\
&= \int_0^\tau \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y) \frac{e^{-\delta^2 - \frac{\alpha^2 (x-y)^2}{4K^2\sigma^2 v(1-\frac{v}{2\tau})} + o(\alpha^2)}}{8\pi\ell\sqrt{v(2\tau - v)}} dxdydv + o(\alpha).
\end{aligned}
$$

Now the first term in the expansion above goes clearly to $v^{(1)}$ as α tends to 0. Then we have

$$
v^{(1),\alpha} - v^{(1)} = \int_0^\tau \int_{-1}^1 \int_{-1}^1 \frac{e^{-\delta(\tau,v,s,K)^2}\phi(x)\phi(y)}{8\pi\ell\sqrt{v(2\tau - v)}} \left(e^{-\frac{\alpha^2(x-y)^2}{4K^2\sigma^2 v(1-\frac{v}{2\tau})} + o(\alpha^2)} - 1 \right) dxdydv + o(\alpha).
$$

Using the change of variable $u = \frac{\alpha|x-y|}{2K\sigma\sqrt{v}}$, the first term above can be rewritten as

$$
\frac{\alpha}{8\pi\ell K\sigma} \int_{\frac{\alpha|x-y|}{2K\sigma\sqrt{\tau}}}^{+\infty} \int_{-1}^1 \int_{-1}^1 \frac{e^{-\delta(\tau, \frac{\alpha^2(x-y)^2}{4K^2\sigma^2u^2}, s, K)^2}\phi(x)\phi(y)|x-y|}{\sqrt{2\tau - \frac{\alpha^2(x-y)^2}{4K^2\sigma^2u^2}}} \frac{e^{-\frac{u^2}{1-\frac{\alpha^2(x-y)^2}{8\tau K^2\sigma^2u^2}} + o(\alpha^2)} - 1}{u^2} dxdydu.
$$

A simple application of the dominated convergence and Fubini theorems shows that the above integral (without the α factor) has a finite limit as α approaches 0 and is given by

$$
\frac{e^{-\frac{1}{2}d_0(s,K,\tau)^2}}{8\pi\ell K\sigma\sqrt{2\tau}} \int_{-1}^1 \int_{-1}^1 \phi(x)\phi(y)|x-y| dxdy \int_0^{+\infty} \frac{e^{-u^2} - 1}{u^2} du.
$$

Since the last integral is equal to $\sqrt{\pi}$, we obtain the second expansion. □