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HOMOGENIZATION AND ASYMPTOTICS FOR SMALL TRANSACTION COSTS*

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Abstract. We consider the classical Merton problem of lifetime consumption-portfolio optimization with small proportional transaction costs. The first order term in the asymptotic expansion is explicitly calculated through a singular ergodic control problem which can be solved in closed form in the one-dimensional case. Unlike the existing literature, we consider a general utility function and general dynamics for the underlying assets. Our arguments are based on ideas from homogenization theory and use convergence tools from the theory of viscosity solutions. The multidimensional case is studied in our companion paper [D. Possamaï, H. M. Soner, and N. Touzi, *Homogenization and Asymptotics for Small Transaction Costs: The Multidimensional Case*, arXiv:1212.6275v2 [math.AP], preprint, 2012] using the same approach.

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1. Introduction.

The problem of investment and consumption in a market with transaction costs was first studied by Magill and Constantinides [26] and later by Constantinides [9]. Since then, starting with the classical paper of Davis and Norman [11], an impressive understanding of this problem has been achieved. In these papers and in [12, 35] the dynamic programming approach in one space dimension has been developed. The problem of proportional transaction costs is a special case of a singular stochastic control problem in which the state process can have controlled discontinuities. The related PDE for this class of optimal control problems is a quasi-variational inequality which contains a gradient constraint. Technically, the multidimensional setting presents intriguing free boundary problems, and the only regularity results to date are [37] and [34]. For the financial problem, we refer the reader to the recent book by Kabanov and Safarian [24]. It provides an excellent exposition to the later developments and the solutions in multidimensions.

It is well known that in practice the proportional transaction costs are small, and in the limiting case of zero costs, one recovers the classical problem of Merton [28]. Then, a natural approach to simplify the problem is to obtain an asymptotic expansion in terms of the small transaction costs. This was initiated in the pioneering paper of Constantinides [9]. The first proof in this direction was obtained in the appendix of [35]. Since then, several rigorous results [5, 20, 22, 32] and formal asymptotic results [1, 21, 38] have been obtained. The rigorous results have been restricted to one space dimension with the exception of the recent manuscript by Bichuch and Shreve [6].

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In this paper and its companion paper [31], we consider this classical problem of small proportional transaction costs and develop a unified approach to the problem of asymptotic analysis. We also relate the first order asymptotic expansion in ϵ to an ergodic singular control problem.

Although our formal derivation in section 3 and the analysis of [31] are multi-dimensional, to simplify the presentation, in this introduction we restrict ourselves to a single risky asset with a price process $\{S_t, t \geq 0\}$. We assume S_t is given by a time homogeneous stochastic differential equation together with $S_0 = s$ and volatility function $\sigma(\cdot)$. For an initial capital z, the value function of the Merton infinite horizon optimal consumption-portfolio problem (with zero transaction costs) is denoted by $v(s,z)$. On the other hand, the value function for the problem with transaction costs is a function of s and the pair (x,y) representing the wealth in the saving and in the stock accounts, respectively. Then, the total wealth is simply given by $z = x + y$. For a small proportional transaction cost $\epsilon^3 > 0$, we let $v^\epsilon(s,x,y)$ be the maximum expected discounted utility from consumption. It is clear that $v^\epsilon(s,x,y)$ converges to $v(s,x+y)$ as ϵ tends to zero. Our main analytical objective is to obtain an expansion for v^ϵ in the small parameter ϵ.

To achieve such an expansion, we assume that v is smooth and let

$$\eta(s,z) := -\frac{v_z(s,z)}{v_{zz}(s,z)} \tag{1.1}$$

be the corresponding risk tolerance. The solution of the Merton problem also provides us an optimal feedback portfolio strategy $\mathbf{y}(s,z)$ and an optimal feedback consumption function $\mathbf{c}(s,z)$. Then, the first term in the asymptotic expansion is given through an ergodic singular control problem defined for every fixed point (s,z) by

$$\bar{a}(s,z) := \inf_M J(s,z,M),$$

where M is a control process of bounded variation with variation norm $\|M\|$,

$$J(s,z,M) := \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \frac{|\sigma(s)\xi_t|^2}{2} + \|M\|_T\right],$$

and the controlled process ξ satisfies the dynamics driven by a Brownian motion B and parameterized by the fixed data (s,z):

$$d\xi_t = \alpha(s,z)dB_t + dM_t, \quad \text{where } \alpha := \sigma[\mathbf{y}(1-\mathbf{y}_z) - s\mathbf{y}_s].$$

The above problem is defined more generally in Remark 3.3 and solved explicitly in subsection 4.1 in terms of the zero transaction cost value function v.

Let $\{\hat{Z}_t^{s,z}, t \geq 0\}$ be the optimal wealth process using the feedback strategies $\mathbf{y}, \mathbf{c}$, and starting from the initial conditions $S_0 = s$ and $\hat{Z}_0^{s,z} = z$. Our main result is on the convergence of the function

$$\bar{u}^\epsilon(x,y) := \frac{v(s,x+y) - v^\epsilon(s,x,y)}{\epsilon^2}.$$

Main Theorem. *Let $\bar{a}$ be as above, and set $a := \eta v_z \bar{a}$. Then, as ϵ tends to zero,*

$$\bar{u}^\epsilon(x,y) \to u(s,z) := \mathbb{E}\left[\int_0^\infty e^{-\beta t} a(S_t, \hat{Z}_t^{s,z})dt\right], \quad \textit{locally uniformly.} \tag{1.2}$$

Naturally, the above result requires assumptions, and we refer the reader to Theorem 6.1 for a precise statement. Moreover, the definition and the convergence of u^ϵ are equivalent to the expansion

$$v^\epsilon(s,x,y) = v(z) - \epsilon^2 u(z) + \circ(\epsilon^2), \tag{1.3}$$

where as before $z = x+y$ and $\circ(\epsilon^k)$ is any function such that $\circ(\epsilon^k)/\epsilon^k$ converges to zero locally uniformly.

A *formal multidimensional derivation* of this result is provided in section 3. Our approach is similar to all formal studies starting from the initial paper by Whalley and Wilmott [38]. These formal calculations also provide the connection with another important class of asymptotic problems, namely homogenization problems. Indeed, the dynamic programming equation of the ergodic problem described above is the *corrector (or cell) equation* in the homogenization terminology. This identification allows us to construct a rigorous proof similar to the ones in homogenization. These assertions are formulated into a formal theorem at the end of section 3. The analysis of section 3 is very general and can easily extend to other similar problems. Moreover, the above ergodic problem is a singular one, and we show in [31] that its continuation region also describes the asymptotic shape of the no-trade region in the transaction cost problem.

The connection between homogenization and asymptotic problems in finance has already played an important role in several other problems. Fouque, Papanicolaou, and Sircar [18] use this approach for stochastic volatility models. We refer to the recent book [19] for information on this problem and also extensions to multidimensions. In the stochastic volatility context the homogenizing (or the so-called fast variable) is the volatility and is given exogenously. Indeed, for homogenization problems, the fast variable is almost always given. In the transaction cost problem, however, this is not the case, and the main difficulty is to identify the "fast" variable. A similar difficulty is also apparent in a problem with an illiquid financial market which becomes asymptotically liquid. The expansions for that problem were obtained in [30]. We use their techniques in an essential way.

The later sections of the paper are concerned with the rigorous proof. The main technique is the viscosity approach of Evans to homogenization [13, 14]. This powerful method combined with the relaxed limits of Barles and Perthame [2] provides the necessary tools. As is well known, this approach has the advantage of using only a simple local L^∞ bound which is described in section 5. In addition to [2, 13, 14], the rigorous proof utilizes several other techniques from the theory of viscosity solutions developed in the papers [2, 3, 9, 15, 16, 17, 25, 33, 36] for asymptotic analysis.

For the rigorous proof, we concentrate on the simpler one-dimensional setting. This simpler setting allows us to highlight the technique with the least possible technicalities. The more general multidimensional problem is considered in [31].

The paper is organized as follows. The problem is introduced in the next section, and the approach is formally introduced in section 3. In one dimension, the corrector equation is solved in the next section. We state the general assumptions in section 5 and prove the convergence result in section 6. In section 7 we discuss the assumptions. Finally, a short summary for the power utility is given in section 8.

2. The general setting. The structure we adopt is the one developed and studied in the recent book by Kabanov and Safarian [24]. We briefly recall it here.

We assume a financial market consisting of a nonrisky asset S^0 and d risky assets with price process $\{S_t = (S_t^1, \ldots, S_t^d), t \geq 0\}$ given by the stochastic differential equations

$$\frac{dS_t^0}{S_t^0} = r(S_t)dt, \quad \frac{dS_t^i}{S_t^i} = \mu^i(S_t)dt + \sum_{j=1}^{d} \sigma^{i,j}(S_t)dW_t^j, \quad 1 \le i \le d,$$

where $r : \mathbb{R}^d \to \mathbb{R}_+$ is the instantaneous interest rate and $\mu : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ are the coefficients of instantaneous mean return and volatility. We use the notation $\mathcal{M}_d(\mathbb{R})$ to denote $d \times d$ matrices with real entries. The standing assumption on the coefficients, i.e.,

$$r, \mu, \sigma \text{ are bounded and Lipschitz, and } (\sigma\sigma^T)^{-1} \text{ is bounded,}$$

will be in force throughout the paper (although not recalled in our statements). In particular, the above stochastic differential equation has a unique strong solution.

The portfolio of an investor is represented by the dollar value X invested in the nonrisky asset and the vector process $Y = (Y^1, \dots, Y^d)$ of the value of the positions in each risky asset. The portfolio position is allowed to change in continuous time by transfers from any asset to any other one. However, such transfers are subject to *proportional transaction costs*.

We continue by describing the portfolio rebalancing in the present setting. For all $i, j = 0, \dots, d$, let $L_t^{i,j}$ be the total amount of transfers (in dollars) from the ith to the jth asset cumulated up to time t. Naturally, the processes $\{L_t^{i,j}, t \ge 0\}$ are defined as càd-làg, nondecreasing, adapted processes with $L_{0^-} = 0$ and $L^{i,i} \equiv 0$. The proportional transaction cost induced by a transfer from the ith to the jth stock is given by $\epsilon^3 \lambda^{i,j}$, where $\epsilon > 0$ is a small parameter and

$$\lambda^{i,j} \ge 0, \quad \lambda^{i,i} = 0, \quad i, j = 0, \dots, d.$$

The scaling ϵ^3 is chosen to state the expansion results in a simpler way. We refer the reader to the recent book of Kabanov and Safarian [24] for a thorough discussion of the model.

The *solvency region* K_ϵ is defined as the set of all portfolio positions which can be transferred into portfolio positions with nonnegative entries through an appropriate portfolio rebalancing. We use the notation $\ell = (\ell^{i,j})_{i,j=0,\dots,d}$ to denote this appropriate instantaneous transfer of size $\ell^{i,j}$. We directly compute that the induced change in each entry, after subtracting the corresponding transaction costs, is given by the linear operator $\mathbf{R} : \mathcal{M}_{d+1}(\mathbb{R}_+) \to \mathbb{R}^{d+1}$,

$$\mathbf{R}^i(\ell) := \sum_{j=0}^{d} \left(\ell^{j,i} - (1 + \epsilon^3 \lambda^{i,j})\ell^{i,j}\right), \quad i = 0, \dots, d, \ \ \forall\ \ell \in \mathcal{M}_{d+1}(\mathbb{R}_+),$$

where $\ell^{i,j} > 0$ and $\ell^{j,i} > 0$ for some i, j would clearly be suboptimal. Then, K_ϵ is given by

$$K_\epsilon := \Big\{(x, y) \in \mathbb{R} \times \mathbb{R}^d : \ (x, y) + \mathbf{R}(\ell) \in \mathbb{R}_+^{1+d} \ \text{ for some } \ \ell \in \mathcal{M}_{d+1}(\mathbb{R}_+)\Big\}.$$

For later use, we denote by $(e_0, \dots, e_d)$ the canonical basis of $\mathbb{R}^{d+1}$ and set

$$\Lambda_{i,j}^\epsilon := e_i - e_j + \epsilon^3 \lambda^{i,j}\ e_i, \qquad i, j = 0, \dots, d.$$

In addition to the trading activity, the investor consumes at a rate determined by a nonnegative progressively measurable process $\{c_t, t \geq 0\}$. Here c_t represents the rate of consumption in terms of the nonrisky asset S^0. Such a pair $\nu := (c, L)$ is called a *consumption-investment strategy*. For any initial position $(X_{0^-}, Y_{0^-}) = (x, y) \in \mathbb{R} \times \mathbb{R}^d$, the portfolio position of the investor is given by the state equation

$$dX_t = \big(r(S_t)X_t - c_t\big)dt + \mathbf{R}^0(dL_t), \quad \text{and} \quad dY_t^i = Y_t^i \frac{dS_t^i}{S_t^i} + \mathbf{R}^i(dL_t), \quad i = 1, \dots, d.$$

The above solution depends on the initial condition (x, y), the control ν, and also on the initial condition of the stock process s. Let $(X, Y)^{\nu,s,x,y}$ be the solution of the above equation. Then, a consumption-investment strategy ν is said to be *admissible* for the initial position (s, x, y) if

$$(X, Y)_t^{\nu,s,x,y} \in K_\epsilon \qquad \forall\, t \geq 0, \quad \mathbb{P}\text{-a.s.}$$

The set of admissible strategies is denoted by $\Theta^\epsilon(s, x, y)$. For given initial positions $S_0 = s \in \mathbb{R}_+^d$, $X_{0^-} = x \in \mathbb{R}$, $Y_{0^-} = y \in \mathbb{R}^d$, the investment-consumption problem is the maximization problem

$$v^\epsilon(s, x, y) := \sup_{(c,L)\in\Theta^\epsilon(s,x,y)} \mathbb{E}\left[\int_0^\infty e^{-\beta t}\, U(c_t)dt\right],$$

where $U : (0, \infty) \mapsto \mathbb{R}$ is a utility function. We assume that U is C^2, increasing, and strictly concave, and we denote its convex conjugate by

$$\tilde{U}(\tilde{c}) := \sup_{c>0}\big\{U(c) - c\tilde{c}\big\}, \qquad \tilde{c} \in \mathbb{R}.$$

Then $\tilde{U}$ is a C^2 convex function. It is well known that the value function is a viscosity solution of the corresponding dynamic programming equation. In one dimension, it was proved in [35]. In the generality that is considered in this paper, we refer to [24]. To state the equation, we first need to introduce some additional notation. We define a second order linear partial differential operator by

$$\mathcal{L} := \mu \cdot (\mathbf{D}_s + \mathbf{D}_y) + r\mathbf{D}_x + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^{\mathrm{T}}\left(\mathbf{D}_{yy} + \mathbf{D}_{ss} + 2\mathbf{D}_{sy}\right)\right], \tag{2.1}$$

where $^{\mathrm{T}}$ denotes the transpose and, for $i, j = 1, \dots, d$,

$$\mathbf{D}_x := x\frac{\partial}{\partial x}, \quad \mathbf{D}_s^i := s^i\frac{\partial}{\partial s^i}, \quad \mathbf{D}_y^i := y^i\frac{\partial}{\partial y^i},$$
$$\mathbf{D}_{ss}^{i,j} := s^i s^j\frac{\partial^2}{\partial s^i \partial s^j}, \quad \mathbf{D}_{yy}^{i,j} := y^i y^j\frac{\partial^2}{\partial y^i \partial y^j}, \quad \mathbf{D}_{sy}^{i,j} := s^i y^j\frac{\partial^2}{\partial s^i \partial y^j},$$

$\mathbf{D}_s = (\mathbf{D}_s^i)_{1\leq i\leq d}$, $\mathbf{D}_y = (\mathbf{D}_y^i)_{1\leq i\leq d}$, $\mathbf{D}_{yy} := (\mathbf{D}_{yy}^{i,j})_{1\leq i,j\leq d}$, $\mathbf{D}_{ss} := (\mathbf{D}_{ss}^{i,j})_{1\leq i,j\leq d}$, $\mathbf{D}_{sy} := (\mathbf{D}_{sy}^{i,j})_{1\leq i,j\leq d}$. Moreover, for a smooth scalar function $(s, x, y) \in \mathbb{R}_+^d \times \mathbb{R} \times \mathbb{R}^d \mapsto \varphi(x, y)$, we set

$$\varphi_x := \frac{\partial\varphi}{\partial x} \in \mathbb{R}, \qquad \varphi_y := \frac{\partial\varphi}{\partial y} \in \mathbb{R}^d.$$

THEOREM 2.1. *Assume that the value function v^ϵ is locally bounded. Then, v^ϵ is a viscosity solution of the dynamic programming equation in $\mathbb{R}^d_+ \times K_\epsilon$,*

$$\min_{0\le i,j\le d} \left\{ \beta v^\epsilon - \mathcal{L} v^\epsilon - \tilde U(v^\epsilon_x),\ \Lambda^\epsilon_{i,j}\cdot (v^\epsilon_x, v^\epsilon_y) \right\} = 0. \tag{2.2}$$

Moreover, v^ϵ is concave in (x, y) and converges to the Merton value function $v := v^0$ as $\epsilon > 0$ tends to zero.

Under further conditions the uniqueness in the above statement is proved in [24]. However, this is not needed in our subsequent analysis.

2.1. Merton problem. The limiting case of $\epsilon = 0$ corresponds to the classical Merton portfolio-investment problem in a frictionless financial market. In this limit, since the transfers from one asset to the other are costless, the value of the portfolio can be measured in terms of the nonrisky asset S^0. We then denote by $Z := X + Y^1 + \cdots + Y^d$ the total wealth obtained by the aggregation of the positions on all assets. In the present setting, we denote by $\theta^i := Y^i$ and $\theta := (\theta^1, \ldots, \theta^d)$ the vector process representing the positions on the risky assets. The wealth equation for the Merton problem is then given by

$$dZ_t = \big(r(S_t)Z_t - c_t\big)dt + \sum_{i=1}^d \theta^i_t \left(\frac{dS^i_t}{S^i_t} - r(S_t)dt\right). \tag{2.3}$$

An admissible consumption-investment strategy is now defined as a pair (c, θ) of progressively measurable processes with values in $\mathbb{R}_+$ and $\mathbb{R}^d$, respectively, and such that the corresponding wealth process is well defined and a.s. nonnegative for all times. The set of all admissible consumption-investment strategies is denoted by $\Theta(s, z)$.

The Merton optimal consumption-investment problem is defined by

$$v(s,z) := \sup_{(c,\theta)\in\Theta(s,z)} \mathbb{E}\left[\int_0^\infty e^{-\beta t}\, U(c_t)dt\right],\ \ s\in\mathbb{R}^d_+,\quad z\ge 0.$$

Throughout this paper, we assume that the Merton value function v is strictly concave in z and is a classical solution of the dynamic programming equation,

$$\beta v - rzv_z - \mathcal{L}^0 v - \tilde U(v_z) - \sup_{\theta\in\mathbb{R}^d}\left\{\theta\cdot\big((\mu - r\mathbf{1}_d)v_z + \sigma\sigma^{\mathrm T}\mathbf{D}_{sz}v\big) + \frac12|\sigma^{\mathrm T}\theta|^2 v_{zz}\right\} = 0,$$

where $\mathbf{1}_d := (1,\ldots,1)\in\mathbb{R}^d$, $\mathbf{D}_{sz} := \frac{\partial}{\partial z}\mathbf{D}_s$, and

$$\mathcal{L}^0 := \mu\cdot\mathbf{D}_s + \frac12\mathrm{Tr}\big[\sigma\sigma^{\mathrm T}\mathbf{D}_{ss}\big]. \tag{2.4}$$

The optimal controls are smooth functions $\mathbf{c}(s, z)$ and $\mathbf{y}(s, z)$ obtained as the maximizers of the Hamiltonian. Hence,

$$0 = \beta v - \mathcal{L}^0 v - \tilde U(v_z) - rzv_z - \mathbf{y}\cdot(\mu - r\mathbf{1}_d)v_z - \sigma\sigma^{\mathrm T}\mathbf{y}\cdot\mathbf{D}_{sz}v - \frac12|\sigma^{\mathrm T}\mathbf{y}|^2 v_{zz}, \tag{2.5}$$

the optimal consumption rate is given by

$$\mathbf{c}(s,z) := -\tilde U'\big(v_z(s,z)\big) = \big(U'\big)^{-1}\big(v_z(s,z)\big)\quad \text{for } s\in\mathbb{R}^d_+,\ \ z\ge 0, \tag{2.6}$$

and the optimal investment strategy $\mathbf{y}$ is obtained by solving the finite-dimensional maximization problem,

$$\max_{\theta\in\mathbb{R}^d}\left\{\frac{1}{2}|\sigma^{\mathrm{T}}\theta|^2 v_{zz}+\theta\cdot\big((\mu-r\mathbf{1}_d)v_z+\sigma\sigma^{\mathrm{T}}\mathbf{D}_{sz}v\big)\right\}.$$

Since v is strictly concave, the Merton optimal investment strategy $\mathbf{y}(s,z)$ satisfies

$$-v_{zz}(s,z)\,\sigma\sigma^{\mathrm{T}}(s)\mathbf{y}(s,z)=(\mu-r\mathbf{1}_d)(s)v_z(s,z)+\sigma\sigma^{\mathrm{T}}(s)\mathbf{D}_{sz}v(s,z). \tag{2.7}$$

3. Formal asymptotics. In this section, we provide the formal derivation of the expansion in any space dimension. In the subsequent sections, we prove this expansion rigorously for the one-dimensional case. Convergence proof in higher dimensions is carried out in a forthcoming paper [31]. In what follows we use the standard notation $O(\epsilon^k)$ to denote any function which is less than a locally bounded function times ϵ^k, and $\circ(\epsilon^k)$ is a function such that $\circ(\epsilon^k)/\epsilon^k$ converges to zero locally uniformly.

Based on previous results [38, 1, 21, 22, 35], we postulate the expansion

$$v^\epsilon(s,x,y)=v(s,z)-\epsilon^2u(s,z)-\epsilon^4w(s,z,\xi)+\circ(\epsilon^2), \tag{3.1}$$

where $(z,\xi)=(z,\xi_\epsilon)$ is a transformation of $(x,y)\in K_\epsilon$ given by

$$z=x+y^1+\cdots+y^d,\qquad \xi^i:=\xi^i_\epsilon(x,y)=\frac{y^i-\mathbf{y}^i(s,z)}{\epsilon},\qquad i=1,\ldots,d,$$

and $\mathbf{y}=\big(\mathbf{y}^1,\ldots,\mathbf{y}^d\big)$ is the Merton optimal investment strategy of (2.7). In the postulated expansion (3.1), we have also introduced two functions

$$u:\mathbb{R}^d_+\times\mathbb{R}_+\mapsto\mathbb{R}\quad\text{and}\quad w:\mathbb{R}^d_+\times\mathbb{R}_+\times\mathbb{R}^d\mapsto\mathbb{R}.$$

The main goal of this section is to formally derive equations for these two functions. A rigorous proof will also be provided in the subsequent sections, and the precise statement for this expansion is stated in section 6.

Notice that the expansion (3.1) is assumed to hold up to ϵ^2, i.e., the $\circ(\epsilon^2)$ term. Therefore, the reason for having a higher term like $\epsilon^4w(z,\xi)$ explicitly in the expansion may not be clear. However, this term contains the fast variable ξ, and its second derivative is of order ϵ^2, which will then contribute to the asymptotics since v^ϵ solves a second order PDE. This follows the intuition introduced in the pioneering work of Papanicolaou and Varadhan [29] in the theory of homogenization.

Since $(x,y)\in K_\epsilon\mapsto(z,\xi)\in\mathbb{R}_+\times\mathbb{R}^d$ is a one-to-one change of variables, in what follows for any function f of (s,x,y) we use the convention

$$\hat f(s,z,\xi):=f\big(s,z-\epsilon\xi-\mathbf{y}(s,z),\epsilon\xi+\mathbf{y}(s,z)\big). \tag{3.2}$$

The new variable ξ is the "fast" variable, and in the limit it homogenizes to yield the convergence of $\hat v^\epsilon(s,z,\xi)$ to the Merton function $v(s,z)$, which depends only on the (s,z)-variables. This is the main formal connection of this problem to the theory of homogenization. This variable was also used centrally by Goodman and Ostrov [21]. Indeed, their asymptotic results use the properties of the stochastic equation satisfied by $\epsilon\xi^\epsilon(X_t,Y_t)$.

First we directly differentiate the expansion (3.1) and compute the terms appearing in (2.2) in term of u and w. The directional derivatives are given by

$$\Lambda^\epsilon_{i,j}\cdot(v^\epsilon_x,v^\epsilon_y)=-\epsilon^4(e_i-e_j)\cdot(w_x(s,z,\xi),w_y(s,z,\xi))+\epsilon^3\lambda^{i,j}v_z+O(\epsilon^4).$$

We directly calculate that

$$(w_x, w_y)(s,z,\xi) = \Big(w_z - \frac{1}{\epsilon}\mathbf{y}_z \cdot w_\xi\Big)\mathbf{1}_{d+1} + \frac{1}{\epsilon}\,(0, w_\xi)\,. \tag{3.3}$$

To simplify the notation, we introduce

$$\hat{D}_\xi w(s,z,\xi) := (0, D_\xi w(s,z,\xi)) \in \mathbb{R}^{d+1}. \tag{3.4}$$

Then,

$$\Lambda^\epsilon_{i,j} \cdot (v^\epsilon_x, v^\epsilon_y) = \epsilon^3\big(\lambda^{i,j} v_z + (e_j - e_i)\cdot \hat{D}w\big) + O(\epsilon^4). \tag{3.5}$$

The elliptic equation in (2.2) requires a longer calculation, and we will later use the Merton identities (2.5), (2.6), and (2.7). First, by (2.5),

$$\begin{aligned}
I^\epsilon &:= \beta v^\epsilon - \mathcal{L}v^\epsilon - \tilde{U}(v^\epsilon_x)\\
&= (\mathbf{y} - y)\cdot\big[(\mu - r\mathbf{1}_d)v_z + \sigma\sigma^{\mathrm{T}}\mathbf{D}_{sz}v\big] + \frac{1}{2}\big(|\sigma^{\mathrm{T}}\mathbf{y}|^2 - |\sigma^{\mathrm{T}}y|^2\big)v_{zz}\\
&\quad + \Big(\tilde{U}(v_z) - \tilde{U}\big(v_z + \epsilon^2 u_z + O(\epsilon^3)\big)\Big)\\
&\quad - \epsilon^2\Big(\beta u - \mathcal{L}u\Big) + \frac{\epsilon^4}{2}\mathrm{Tr}[\sigma\sigma^{\mathrm{T}}\mathbf{D}_{yy}w] + O(\epsilon^3).
\end{aligned}$$

We use Taylor expansions on the terms involving $\tilde{U}$ and (2.6)–(2.7) in the first line to arrive at

$$\begin{aligned}
I^\epsilon &= \Big(-\sigma^{\mathrm{T}}(\mathbf{y}-y)\cdot\sigma^{\mathrm{T}}\mathbf{y} + \frac{1}{2}\big(|\sigma^{\mathrm{T}}\mathbf{y}|^2 - |\sigma^{\mathrm{T}}y|^2\big)\Big)v_{zz}\\
&\quad - \epsilon^2\Big(\beta u - \mathcal{L}u + \hat{c}u_z\Big) + \frac{\epsilon^4}{2}\mathrm{Tr}[\sigma\sigma^{\mathrm{T}}(\mathbf{D}_{yy} + \mathbf{D}_{ss} + \mathbf{D}_{sy})w] + O(\epsilon^3)\\
&= -\frac{1}{2}|\sigma^{\mathrm{T}}(\mathbf{y}-y)|^2 v_{zz} - \epsilon^2\Big(\beta u - \mathcal{L}u + \hat{c}u_z\Big)\\
&\quad + \frac{\epsilon^4}{2}\mathrm{Tr}[\sigma\sigma^{\mathrm{T}}(\mathbf{D}_{yy} + \mathbf{D}_{ss} + \mathbf{D}_{sy})w] + O(\epsilon^3)\\
&= \epsilon^2\Big(-\frac{1}{2}|\sigma^{\mathrm{T}}\xi|^2 v_{zz} - \beta u + \mathcal{L}u - \hat{c}u_z\Big) + \frac{\epsilon^4}{2}\mathrm{Tr}[\sigma\sigma^{\mathrm{T}}(\mathbf{D}_{yy} + \mathbf{D}_{ss} + \mathbf{D}_{sy})w] + O(\epsilon^3).
\end{aligned} \tag{3.6}$$

Finally, from (3.3), we see that

$$\partial_y w = w_z\mathbf{1}_d + \frac{1}{\epsilon}\big(I_d - \mathbf{1}_d\mathbf{y}_z^{\mathrm{T}}\big)w_\xi.$$

Therefore,

$$\begin{aligned}
\partial_{yy}w &= \Big(w_{zz} - \frac{1}{\epsilon}(\mathbf{y}_{zz}\cdot w_\xi + \mathbf{y}_z\cdot w_{z\xi})\Big)\mathbf{1}_d\mathbf{1}_d^{\mathrm{T}} + \frac{1}{\epsilon}\big(w_{z\xi}\mathbf{1}_d^{\mathrm{T}} + \mathbf{1}_d w_{z\xi}^{\mathrm{T}}\big)\\
&\quad + \frac{1}{\epsilon^2}\big(I_d - \mathbf{1}_d\mathbf{y}_z^{\mathrm{T}}\big)w_{\xi\xi}\big(I_d - \mathbf{y}_z\mathbf{1}_d^{\mathrm{T}}\big).
\end{aligned}$$

We substitute this in (3.6) and use the fact that $y = \mathbf{y} + O(\epsilon)$. This yields

$$I^\epsilon = \epsilon^2\Big(-\frac{1}{2}|\sigma^{\mathrm{T}}\xi|^2 v_{zz} + \frac{1}{2}\mathrm{Tr}\big[\alpha\alpha^{\mathrm{T}}w_{\xi\xi}\big] - \mathcal{A}u\Big) + O(\epsilon^3), \tag{3.7}$$

where $\alpha(s,z)$ is given by

$$\alpha(s,z) = \big\{\big(I_d - \mathbf{y}_z \mathbf{1}_d^{\mathrm{T}}\big)\mathrm{diag}[\mathbf{y}] - \mathbf{y}_s^{\mathrm{T}}\mathrm{diag}[s]\big\}(s,z)\sigma(s), \tag{3.8}$$

$\mathrm{diag}[\mathbf{y}]$ denotes the diagonal matrix with ith diagonal entry $\mathbf{y}^i$, and

$$\mathcal{A}u = \beta u - \mathcal{L}^0 u - \big(rz + \mathbf{y}\cdot(\mu - r\mathbf{1}_d) - \mathbf{c}\big)u_z - \frac{1}{2}|\sigma^{\mathrm{T}}\mathbf{y}|^2\, u_{zz} - \sigma\sigma^{\mathrm{T}}\mathbf{y}\cdot\mathbf{D}_{sz}u. \tag{3.9}$$

Recall that $\mathcal{L}^0$ is the infinitesimal generator of the stock price process. Observe that the above operator is the infinitesimal generator of the pair process $(S,\hat{Z})$, where $\hat{Z}$ is the optimal wealth process in the Merton zero-transaction cost problem corresponding to the optimal feedback controls $(\mathbf{c},\mathbf{y})$. In particular, the dynamic programming equation (2.5) for the Merton problem may be expressed as

$$\mathcal{A}v(s,z) = U(\mathbf{c}(s,z)). \tag{3.10}$$

We have now obtained expressions for all the terms in the dynamic programming equation (2.2). We substitute (3.5) and (3.7) into (2.2). Notice that since $\epsilon > 0$, for any A, B, $\max\{\epsilon^2 A, \epsilon^3 B\} = 0$ is equivalent to $\max\{A,B\} = 0$. Hence, w and u satisfy

$$\begin{aligned}\max_{0\le i,j\le d}\max\Big\{&\frac{1}{2}\big|\sigma^{\mathrm{T}}(s)\xi\big|^2 v_{zz}(s,z) - \frac{1}{2}\mathrm{Tr}\big[\alpha\alpha^{\mathrm{T}}(s,z)w_{\xi\xi}(s,z,\xi)\big] + a(s,z)\,,\\ &-\lambda^{i,j}v_z(s,z) + (e_i - e_j)\cdot\hat{D}_\xi w(z,\xi)\Big\} = 0,\end{aligned} \tag{$*$}$$

where $\hat{D}_\xi = (0, D_\xi w)$ is as in (3.4) and a is given by

$$a(s,z) := \mathcal{A}u(s,z), \quad s\in\mathbb{R}_+^d, \quad z>0.$$

In $(*)$, the pair (s,z) is simply a parameter, and the independent variable is ξ. Also the value of the function $w(s,z,0)$ is irrelevant in (3.1) as it only contributes to the ϵ^4 term. Therefore, to obtain a unique w, we set its value at the origin to zero. We continue by presenting (3.10) and $(*)$ in a form that is compatible with the power case. So first we divide the above equation by v_z and then introduce the new variable

$$\rho = \xi/\eta(s,z),$$

where η is the risk tolerance coefficient defined by (1.1). We also set

$$\bar{w}(s,z,\rho) := \frac{w(s,z,\eta(s,z)\rho)}{\eta(s,z)v_z(s,z)}, \quad \bar{a}(s,z) := \frac{a(s,z)}{\eta(s,z)v_z(s,z)}, \quad \bar{\alpha}(s,z) := \frac{\alpha(s,z)}{\eta(s,z)}.$$

Then, the corrector equations in this context are the following pair of equations.

Definition 3.1 (corrector equations). *For a given point* $(s,z)\in\mathbb{R}_+^d\times\ R_+$, the first corrector equation *is for the unknown pair* $(\bar{a}(s,z),\bar{w}(s,z,\cdot))\in\mathbb{R}\times C^2(\mathbb{R}^d)$,

$$\begin{aligned}\max_{0\le i,j\le d}\max\Big\{&-\frac{|\sigma^{\mathrm{T}}(s)\rho|^2}{2} - \frac{1}{2}\,Tr\big[\bar{\alpha}\bar{\alpha}^{\mathrm{T}}(s,z)\bar{w}_{\rho\rho}(s,z,\rho)\big] + \bar{a}(s,z)\,,\\ &-\lambda^{i,j} + (e_i - e_j)\cdot\hat{D}_\rho\bar{w}(s,z,\rho)\Big\} = 0 \quad \forall\ \rho\in\mathbb{R}^d,\end{aligned} \tag{3.11}$$

together with the normalization $\bar{w}(s,z,0) = 0$.

The second corrector equation *uses the constant term $\bar{a}(s,z)$ from the first corrector equation and is a simple linear equation for the function $u : \mathbb{R}^d_+ \times \mathbb{R}^+ \mapsto \mathbb{R}^1$,*

$$\mathcal{A}u(s,z) = a(s,z) = v_z(s,z)\eta(s,z)\bar{a}(s,z) \qquad \forall\ s \in \mathbb{R}^d_+,\ z \in \mathbb{R}^+. \tag{3.12}$$

We say that the pair (u,w) is the solution of the corrector equations *for a given utility function or, equivalently, for a given Merton value function.*

We summarize our formal calculations in the following.

Formal Expansion Theorem. *The value function has the expansion* (3.1)*, where (u,w) is the unique solution of the corrector equations.*

Remark 3.1. The function u introduced in (1.2) is a solution of the second corrector equation (3.12), provided that it is finite. Then, assuming that uniqueness holds for the linear PDE (3.12) in a convenient class, it follows that u is given by the stochastic representation (1.2).

Remark 3.2. Usually a second order equation like (3.12) in $(0,\infty)$ needs to be completed by a boundary condition at the origin. However, as we have already remarked, the operator $\mathcal{A}$ is the infinitesimal generator of the optimal wealth process in the Merton problem. Then, under the Inada conditions satisfied by the utility function U, we expect that this process does not reach the origin. Hence, we need only appropriate growth conditions near the origin and at infinity to ensure uniqueness.

Remark 3.3. The first corrector equation has the following stochastic representation as the dynamic programming equation of an ergodic control problem. For this representation we fix (s,z) and let $\{M^{i,j}_t, t \geq 0\}$ be nondecreasing control processes for each $i,j = 0,\ldots,d$. Let ρ be the controlled process defined by

$$\rho^i_t = \rho^i_0 + \sum_{j=1}^d \bar{\alpha}^{i,j}(s,z)B^j_t + \sum_{j=0}^d \left(M^{j,i}_t - M^{i,j}_t\right)$$

for some arbitrary initial condition ρ_0 and a d-dimensional standard Brownian motion B. Then, the ergodic control problem is

$$\bar{a}(s,z) := \inf_M\ J(s,z,M),$$

where

$$J(s,z,M) := \limsup_{T\to\infty}\ \frac{1}{T}\ \mathbb{E}\left[\frac{1}{2}\int_0^T \left|\sigma^{\mathrm{T}}(s)\rho_t\right|^2 dt + \sum_{i,j=0}^d \lambda^{i,j} M^{i,j}_T\right].$$

In the scalar case, this problem is closely related to the classical finite fuel problem introduced by Benes, Shepp, and Witsenhausen [4]. We refer to the paper by Menaldi, Robin, and Taksar [27] for the present multidimensional setting.

The function $\bar{w}$ is the so-called potential function in ergodic control. We refer the reader to the book and the manuscript of Borkar [7, 8] for information on the dynamic programming approach for the ergodic control problems.

Remark 3.4. The calculation leading to (3.7) is used several times in the paper. Therefore, for future reference, we summarize it once again. Let v, z, and ξ be as above. For *any* smooth functions

$$\phi : \mathbb{R}^d_+ \times \mathbb{R}_+ \mapsto \mathbb{R}, \quad \varpi : \mathbb{R}^d_+ \times \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R},$$

and $\epsilon \in (0,1]$, set

$$\Psi^\epsilon(s,x,y) := v(s,z) - \epsilon^2\phi(s,z) - \epsilon^4\varpi(s,z,\xi).$$

In the above calculations, we obtained an expansion for the second order nonlinear operator

$$\begin{aligned}\mathcal{J}(\Psi^\epsilon) &:= \beta\Psi^\epsilon - \mathcal{L}\Psi^\epsilon - \tilde{U}(\Psi^\epsilon_x)\\ &= \epsilon^2\Big(- \frac{v_{zz}}{2}|\sigma^{\mathrm{T}}\xi|^2 + \frac{1}{2}\mathrm{Tr}\big[\alpha\alpha^{\mathrm{T}}\varpi_{\xi\xi}\big] - \mathcal{A}\phi + \mathcal{R}^\epsilon\Big),\end{aligned} \tag{3.13}$$

where α, $\mathcal{A}$ are as before and $\mathcal{R}^\epsilon(s,x,y)$ is the remainder term. Moreover, $\mathcal{R}^\epsilon$ is locally bounded by an ϵ times a constant depending only on the values of the Merton function v, ϕ, and ϖ. Indeed, a more detailed description and an estimate will be proved in one space dimension in section 6.

4. Corrector equation in one dimension. In this section, we solve the first corrector equation explicitly in the one-dimensional case. Then, we provide some estimates for the remainder introduced in Remark 3.4.

4.1. Closed-form solution of the first corrector equation. Recall that $w = \eta v_z \bar{w}$, $a = \eta v_z \bar{a}$, and the solution of the corrector equations is a pair $(\bar{w}, \bar{a})$ satisfying

$$\max\left\{ -\frac{1}{2}\sigma^2\rho^2 - \frac{1}{2}\bar{\alpha}^2\bar{w}_{\rho\rho} + \bar{a}, -\lambda^{1,0} + \bar{w}_\rho, -\lambda^{0,1} - \bar{w}_\rho \right\} = 0, \quad \bar{w}(s,z,0) = 0, \tag{4.1}$$

where $\bar{\alpha} = \alpha/\eta$ and $\alpha(s,z)$ is given in (3.8). We also recall that the variables (s,z) are fixed parameters in this equation. Therefore, throughout this section, we suppress the dependencies of σ, α, and $\bar{w}$ on these variables.

In order to compute the solution explicitly in terms of η, we postulate a solution of the form

$$\bar{w}(\rho) = \begin{cases} k_4\rho^4 + k_2\rho^2 + k_1\rho, & \rho_1 \le \rho \le \rho_0,\\ \bar{w}(\rho_1) - \lambda^{0,1}(\rho - \rho_1), & \rho \le \rho_1,\\ \bar{w}(\rho_0) + \lambda^{1,0}(\rho - \rho_0), & \rho \ge \rho_0.\end{cases} \tag{4.2}$$

We first determine k_4 and k_2 by imposing that the fourth order polynomial solves the second order equation in (ρ_0, ρ_1). A direct calculation yields

$$k_4 = \frac{-\sigma^2}{12\bar{\alpha}^2} \quad \text{and} \quad k_2 = \frac{\bar{a}}{\bar{\alpha}^2}.$$

We now impose the smooth pasting condition, namely the assumption that $\bar{w}$ is C^2 at the points ρ_0 and ρ_1. Then, the continuity of the second derivatives yields

$$\rho_0^2 = \rho_1^2 = \frac{2\bar{a}}{\sigma^2} \text{ implying that } \bar{a} \ge 0 \ \text{ and } \ \rho_0 = -\rho_1 = \Big(\frac{2\bar{a}}{\sigma^2}\Big)^{1/2}. \tag{4.3}$$

The continuity of the first derivatives of $\bar{w}$ at the points ρ_0 and ρ_1 yields

$$\begin{aligned}4k_4(\rho_0)^3 + 2k_2\rho_0 + k_1 &= -\lambda^{0,1},\\ 4k_4(\rho_1)^3 + 2k_2\rho_1 + k_1 &= \lambda^{1,0}.\end{aligned}$$

Since $\rho_0 = -\rho_1$, we determine the value of k_1 by summing the two equations to get

$$k_1 = \frac{\lambda^{1,0} - \lambda^{0,1}}{2}.$$

Finally, we obtain the value of $\bar{a}$ by further substituting the values of k_4, k_2, and $\rho_0 = -\rho_1$. The result is

$$\bar{a} = \frac{\sigma^2}{2}\rho_0^2 \quad \text{and} \quad \rho_0 = \left(\frac{3\bar{\alpha}^2}{4\sigma^2}(\lambda^{1,0} + \lambda^{0,1})\right)^{1/3}. \tag{4.4}$$

All coefficients of our candidate are now uniquely determined. Moreover, we verify that the gradient constraint

$$-\lambda^{1,0} \leq \bar{w}_\rho \leq \lambda^{0,1} \tag{4.5}$$

holds true for all $\rho \in \mathbb{R}$. Hence, $\bar{w}$ constructed above is a solution of the corrector equation. One may also prove that it is the unique solution. However, in the subsequent analysis we simply use the function $\bar{w}$ defined in (4.2) with the constants determined above. Therefore, we do not study the question of uniqueness of the corrector equation.

Remark 4.1. In the homothetic case with constant coefficients r, μ, and σ, one can explicitly calculate all the functions; see section 8. Here we only report that, in that case, all functions are independent of the s-variable and $\rho_0, \bar{a}(z)$ are constants. Therefore, $a(z)$ is a positive constant times the Merton value function.

Remark 4.2. Pointwise estimates on the derivatives of w will be used in the subsequent sections. So we record them here for future reference. Indeed, by (4.5) and the fact that $w(\cdot, 0) = 0$,

$$|w(s,z,\xi)| \leq \overline{\lambda}\, v_z(s,z)|\xi|, \quad |w_\xi(s,z,\xi)| \leq \overline{\lambda}\, v_z(s,z), \quad \text{where } \overline{\lambda} := \lambda^{0,1} \vee \lambda^{1,0}. \tag{4.6}$$

Moreover, under the smoothness assumption on v, we obtain the following pointwise estimates:

$$\big(|w| + |w_s| + |w_{ss}| + |w_z| + |w_{zz}|\big)(z,\xi) \leq C(s,z)(1 + |\xi|), \tag{4.7}$$

$$\big(|w_\xi| + |w_{z\xi}| + |w_{s\xi}|\big)(s,z) \leq C(s,z) \quad \text{and} \quad |w_{\xi\xi}| \leq \big(C\mathbf{1}_{[\xi_0,\xi_1]}\big)(s,z), \tag{4.8}$$

where C is an appropriate continuous function in $\mathbb{R}_+^2$, depending on the Merton value function and its derivatives.

4.2. Remainder estimate. In this subsection, we estimate the remainder term in Remark 3.4. So, let Ψ^ϵ be as in Remark 3.4 with ϖ satisfying the same estimates (4.7)–(4.8) as w. We have seen in (3.13) that

$$\begin{aligned}
\mathcal{J}(\Psi^\epsilon)(s,x,y) &:= \big(\beta\Psi^\epsilon - \mathcal{L}\Psi^\epsilon - \tilde{U}(\Psi_x^\epsilon)\big)(s,x,y) \\
&= \epsilon^2\left[-\frac{1}{2}v_{zz}(s,z)\xi^2 + \frac{1}{2}\alpha^2(s,z)\varpi_{\xi\xi}(s,z,\xi) - \mathcal{A}\phi(s,z) + \mathcal{R}^\epsilon(s,z,\xi)\right],
\end{aligned}$$

where α, $\mathcal{A}$ are defined in (3.8)–(3.9) and $\mathcal{R}^\epsilon$ is the remainder. By a direct (tedious) calculation, the remainder term can be obtained explicitly. In view of our previous bounds (4.7)–(4.8) on the derivatives of w, we obtain the estimate

$$\begin{aligned}
\big|\mathcal{R}^\epsilon(s,z,\xi)\big| \leq{}& \epsilon\left(|\xi||\mu - r||\phi_z| + \frac{1}{2}\sigma^2(\epsilon\xi^2 + 2|\xi||\mathbf{y}|)|\phi_{zz}| + \sigma^2|\xi||\phi_{sz}|\right)(s,z) \\
&+ \epsilon C(s,z)\big(1 + \epsilon|\xi| + \epsilon^2|\xi|^2 + \epsilon^3|\xi|^3\big) \\
&+ \epsilon^{-2}\big|\tilde{U}(\psi_x^\epsilon) - \tilde{U}(v_z) - (\psi_x^\epsilon - v_z)\tilde{U}'(v_z)\big|
\end{aligned}$$

for some continuous function $C(s,z)$. Since $\tilde{U}$ is C^1 and convex,

$$\begin{aligned}
&|\mathcal{R}^\epsilon(s,z,\xi)| \\
&\quad \le \epsilon \left(|\xi||\mu - r||\phi_z| + \frac{1}{2}\sigma^2(\epsilon\xi^2 + 2|\xi||\mathbf{y}|)|\phi_{zz}| + \sigma^2|\xi||\phi_{sz}| \right)(s,z) \\
&\qquad + \epsilon C(s,z)\big(1 + \epsilon|\xi| + \epsilon^2|\xi|^2 + \epsilon^3|\xi|^3\big) \\
&\qquad + (|\phi_z| + \epsilon^2|\phi_z| + \epsilon\mathbf{y}_z|\varpi_\xi|)\big|\tilde{U}'(v_z) + \epsilon^2|\phi_z| + \epsilon^4|\varpi_z| + \epsilon^3\mathbf{y}_z|\varpi_\xi| - \tilde{U}'(v_z)\big|.
\end{aligned}$$

Suppose that ϖ satisfies the same estimates (4.7)–(4.8) as w. Then,

$$\begin{aligned}
\big|\mathcal{R}^\epsilon(s,z,\xi)\big| &\le \epsilon \left(|\xi||\mu - r||\phi_z| + \frac{1}{2}\sigma^2(\epsilon\xi^2 + 2|\xi||\mathbf{y}|)|\phi_{zz}| + \sigma^2|\xi||\phi_{sz}| \right)(s,z) \\
&\quad + \epsilon C(s,z)\big(1 + \epsilon|\xi| + \epsilon^2|\xi|^2 + \epsilon^3|\xi|^3\big) \\
&\quad + \epsilon^2\big(|\phi_z| + \epsilon C(s,z)(1+\epsilon|\xi|)\big)^2\tilde{U}''\big(v_z + \epsilon^2|\phi_z| + \epsilon^3 C(s,z)(1+\epsilon|\xi|)\big).
\end{aligned}$$

5. Assumptions. The main objective of this paper is to characterize the limit of the following sequence:

$$\bar{u}^\epsilon(s,x,y) := \frac{v(s,z) - v^\epsilon(s,x,y)}{\epsilon^2}, \quad s \ge 0,\ (x,y) \in K_\epsilon.$$

Our proof follows the general methodology developed by Barles and Perthame [2] in the context of viscosity solutions. Hence, we first define relaxed semilimits by

$$u^*(\zeta) := \limsup_{(\epsilon,\zeta')\to(0,\zeta)} \bar{u}^\epsilon(\zeta'), u_*(\zeta) := \liminf_{(\epsilon,\zeta')\to(0,\zeta)} \bar{u}^\epsilon(\zeta').$$

Then, we show under appropriate conditions that they are viscosity subsolution and supersolution, respectively, of the second corrector equation (3.12).

We shall now formulate some conditions which guarantee that

(i) the relaxed semilimits are finite,
(ii) the second corrector equation (3.12) verifies comparison for viscosity solutions.

We may then conclude that $u^* \le u_*$. Since the opposite inequality is obvious, this shows that $u = u^* = u_*$ is the unique solution of the second corrector equation (3.12).

In this short subsection, for the convenience of the reader, we collect all the assumptions needed for the convergence proof, including the ones that were already used.

We first focus on the finiteness of the relaxed semilimits u_* and u^*. A local lower bound is easy to obtain in view of the obvious inequality $v^\epsilon(s,x,y) \le v(s,x+y)$ which implies that $\bar{u}^\epsilon \ge 0$. Our first assumption complements this with a local upper bound.

Assumption 5.1 (uniform local bound). The family of functions $\bar{u}^\epsilon$ is locally uniformly bounded from above.

The above assumption states that for any $(s_0,x_0,y_0) \in \mathbb{R}_+ \times \mathbb{R}^2$ with $x_0+y_0>0$, there exist $r_0 = r_0(s_0,x_0,y_0) > 0$ and $\epsilon_0 = \epsilon_0(s_0,x_0,y_0) > 0$ so that

$$b(s_0,x_0,y_0) := \sup\{\ u^\epsilon(s,x,y)\ :\ (s,x,y) \in B_{r_0}(s_0,x_0,y_0),\ \epsilon \in (0,\epsilon_0]\ \} < \infty, \tag{5.1}$$

where $B_{r_0}(s_0,x_0,y_0)$ denotes the open ball with radius r_0, centered at (s_0,x_0,y_0).

This assumption is verified in section 7 under some conditions on v and its derivatives by constructing an appropriate subsolution to the dynamic programming equation (2.2). However, the subsolution does not need to have the exact ϵ^2 behavior as needed in other approaches to this problem starting from [35, 22]. Indeed, in these earlier approaches, both the sub- and the supersolution must be sharp enough to have the exact limiting behavior in the leading ϵ^2 term. For the above estimate, however, this term needs to be only locally bounded.

The next assumption is a regularity condition on the Merton problem.

Assumption 5.2 (smoothness). The Merton value function v and the Merton optimal investment strategy $\mathbf{y}$ are twice continuously differentiable in the open domain $(0,\infty)^2$ and $v_z(s,z) > 0$ for all $s, z > 0$. Moreover, there exist $c_1 \geq c_0 > 0$ such that

$$c_0 z \leq [\mathbf{y}(1-\mathbf{y}_z) - s\mathbf{y}](s,z) \leq c_1 z \qquad \forall\ s, z \in \mathbb{R}_+. \tag{5.2}$$

In particular, together with our standing assumption on the volatility function σ, the above assumption implies that the diffusion coefficient $\alpha(s,z)$ in the first corrector equation is nondegenerate away from the origin. For later use we record that there exist two constants $0 < \alpha_* \leq \alpha^*$ so that

$$0 < \alpha_* \leq \frac{\alpha(s,z)}{z} \leq \alpha^* \qquad \forall\ s, z \in \mathbb{R}_+. \tag{5.3}$$

We will not attempt to verify the above hypothesis. However, in the power utility case, the value function is always smooth, and the condition (5.2) can be directly checked as the optimal investment policy $\mathbf{y}$ is explicitly available.

We next assume that the second corrector equation (3.12) has comparison. Recall the function u introduced in (1.2), let b be as in (5.1), and set

$$B(s,z) := b\big(s, z - \mathbf{y}(z), \mathbf{y}(z)\big), \qquad s, z \in \mathbb{R}_+. \tag{5.4}$$

Assumption 5.3 (comparison). For any upper-semicontinuous (resp., lower-semicontinuous) viscosity subsolution (resp., supersolution) u_1 (resp., u_2) of (3.12) in $(0,\infty)^2$ satisfying the growth condition $|u_i| \leq B$ on $(0,\infty)^2$, $i = 1, 2$, we have $u_1 \leq u \leq u_2$ in $(0,\infty)^2$.

In the above comparison, notice that the growth of the supersolution and the subsolution is controlled by the function B which is defined in (5.4) by means of the local bound function b. In particular, B controls the growth both at infinity and near the origin. This observation is further detailed in Remark 7.1 below.

We observe, however, that, as discussed earlier, the operator $\mathcal{A}$ is the infinitesimal generator of the optimal wealth process in the limiting Merton problem. In view of our Assumption 5.2, we implicitly assume that this process does not reach the origin with probability one.

We finally formulate a natural assumption which was verified in [35, Remark 11.3], in the context of power utility functions. This assumption will be used for the proof of the subsolution property. To state this assumption, we first introduce the *no-transaction region* defined by

$$\mathcal{N}^\epsilon := \left\{(s,x,y) \in K_\epsilon : \Lambda^\epsilon_{0,1} \cdot Dv^\epsilon(s,x,y) > 0 \text{ and } \Lambda^\epsilon_{1,0} \cdot Dv^\epsilon(s,x,y) > 0\right\}. \tag{5.5}$$

By the dynamic programming equation (2.2), the value function v^ϵ is a viscosity solution of

$$\beta v^\epsilon - \mathcal{L}v^\epsilon - \tilde{U}(v^\epsilon_x) = 0 \ \text{ on } \ \mathcal{N}^\epsilon.$$

Assumption 5.4 (no transaction region). The no-transaction region $\mathcal{N}^\epsilon$ contains the Merton line $\mathcal{M} := \{(s, z - \mathbf{y}(z), \mathbf{y}(z)) \ : \ s, z \in \mathbb{R}_+\}$.

Remark 5.1. In our companion paper [31], the expansion result in the d-dimensional context is proved without Assumption 5.4. However, this induces an important additional technical effort. Therefore, for the sake of simplicity, we refrained from including this improvement in the present one-dimensional paper.

6. Convergence in one dimension. For the convergence proof, we introduce the following "corrected" version of $\bar{u}^\epsilon$:

$$u^\epsilon(s,x,y) := \bar{u}^\epsilon(s,x,y) - \epsilon^2 w(s,z,\xi), \quad s \geq 0,\ (x,y) \in K_\epsilon.$$

Notice that both families $\bar{u}^\epsilon$ and u^ϵ have the same relaxed semilimits u^* and u_*.

THEOREM 6.1. *Under Assumptions* 5.1–5.4 *the sequence* $\{u^\epsilon\}_{\epsilon>0}$ *converges locally uniformly to the function* u *defined in* (1.2).

Proof. In subsections 6.1–6.3, we will show that the semilimits u_* and u^* are viscosity supersolution and subsolution, respectively, of (3.12). Then, by the comparison assumption, Assumption 5.3, we conclude that $u^* \leq u \leq u_*$. Since the opposite inequality is obvious, this implies that $u^* = u_* = u$. The local uniform convergence follows immediately from this and the definitions. $\square$

6.1. First properties. In this subsection, we use only the assumptions on the smoothness of the limiting Merton problem and the local boundedness of $\{u^\epsilon\}_\epsilon$. We first recall that

$$\overline{\lambda} := \lambda^{0,1} \vee \lambda^{1,0}.$$

LEMMA 6.1. (i) *For all* $\epsilon, s > 0$, $(x,y) \in K_\epsilon$, $u^\epsilon(s,x,y) \geq -\epsilon\overline{\lambda} v_z(s,z)|y - \mathbf{y}(s,z)|$. *In particular,* $u^* \geq 0$.

(ii) *If, in addition, Assumption* 5.1 *holds, then*

$$0 \leq u_*(s,x,y) \leq u^*(s,x,y) < \infty \quad \forall\ s,x,y > 0.$$

Proof. Since statement (ii) is a direct consequence, we focus on (i). From the obvious inequality $v^\epsilon(s,x,y) \leq v(s,x+y)$, it follows that $u^\epsilon(s,x,y) \geq -\epsilon^2 w(s,z,\xi)$, so that the required result follows from the bound (4.5) on w_ξ together with $w(\cdot,0) = 0$. $\square$

We next show that the relaxed semilimits u^* and u_* depend on the pair (x,y) only through the aggregate variable $z = x + y$.

LEMMA 6.2. *Let Assumptions* 5.1 *and* 5.2 *hold true. Then,* u^* *and* u_* *are functions of* (s,z) *only. Moreover, for all* $s, z \geq 0$,

$$u_*(s,z) = \liminf_{(\epsilon,s',z')\to(0,s,z)} u^\epsilon\big(s', z' - \mathbf{y}(z'), \mathbf{y}(z')\big),$$

and

$$u^*(s,z) = \limsup_{(\epsilon,s',z')\to(0,s,z)} u^\epsilon\big(s', z' - \mathbf{y}(z'), \mathbf{y}(z')\big).$$

Proof. This result is a consequence of the gradient constraints in the dynamic programming equation (2.2),

$$\Lambda^\epsilon_{1,0} \cdot (v^\epsilon_x, v^\epsilon_y) \geq 0 \ \text{ and } \ \Lambda^\epsilon_{0,1} \cdot (v^\epsilon_x, v^\epsilon_y) \geq 0 \ \text{ in the viscosity sense.}$$

1. We change variables and use the above inequalities to obtain

$$\big(1 + \lambda^{1,0}\epsilon^3(1 - \mathbf{y}_z)\big)\hat{v}^\epsilon_\xi \geq -\lambda^{1,0}\epsilon^4\hat{v}^\epsilon_z, \big(1 + \lambda^{0,1}\epsilon^3\mathbf{y}_z\big)\hat{v}^\epsilon_\xi \leq \lambda^{0,1}\epsilon^4\hat{v}^\epsilon_z \tag{6.1}$$

in the viscosity sense. Since v^ϵ is concave in (x,y), the partial gradients v^ϵ_x and v^ϵ_y exist almost everywhere. By the smoothness of the Merton optimal investment strategy $\mathbf{y}$, this implies that the partial gradient $\hat{v}^\epsilon_z$ also exists almost everywhere. Then, by the definition of u^ϵ, we conclude that the partial gradients $\hat{u}^\epsilon_z$ and $\hat{u}^\epsilon_\xi$ exist almost everywhere. In view of condition (5.2) in Assumption 5.2, we conclude from (6.1) and the fact that $\hat{v}^\epsilon_z \geq 0$ that

$$\left|\hat{v}^\epsilon_\xi\right| \leq \overline{\lambda}\epsilon^4 \hat{v}^\epsilon_z. \tag{6.2}$$

We now claim that

$$\begin{aligned}\hat{v}^\epsilon_z(s,z,\xi) \;&\leq\; \gamma^\epsilon(s,x,y)\\ &:= v_z(s,z-\epsilon) + \epsilon\big(u^\epsilon(s,x-\epsilon,y) + u^\epsilon(s,x,y-\epsilon)\big)\\ &\quad+\; \epsilon^3\overline{\lambda} v_z(s,z)\left(1 + |\mathbf{y}_z(s,z)| + |\xi| + \frac{|\mathbf{y}(s,z)-\mathbf{y}(s,z-\epsilon)|}{\epsilon}\right).\end{aligned} \tag{6.3}$$

We postpone the justification of this claim to the next step and continue with the proof. Then, it follows from (6.2), (6.3) together with Assumption 5.2 and (4.5) that

$$\begin{aligned}\left|\hat{u}^\epsilon_\xi(s,z,\xi)\right| &\leq \epsilon^2\bar{\lambda}\left(v_z(s,z) + \hat{v}^\epsilon_z(s,z,\xi)\right)\\ &\leq \epsilon^2\bar{\lambda}\left(v_z(s,z) + \gamma^\epsilon(s,z,\xi)\right).\end{aligned} \tag{6.4}$$

Hence,

$$(e_1 - e_0)\cdot(u^\epsilon_x, u^\epsilon_y) = -\frac{1}{\epsilon}\hat{u}^\epsilon_\xi \leq \epsilon\bar{\lambda}\left(v_z(s,z) + \gamma^\epsilon(s,z,\xi)\right).$$

By the local boundedness of $\{u^\epsilon\}_\epsilon$, for any (s,x,y), there are an open neighborhood of (s,x,y) and a constant K, both independent of ϵ, such that the maps

$$t \mapsto u^\epsilon(s,x-t,y+t) + \epsilon K t \quad\text{and}\quad t\mapsto -u^\epsilon(s,x-t,y+t)+\epsilon K t$$

are nondecreasing for all $\epsilon > 0$. Then, it follows from the definition of the relaxed semilimits that $\hat{u}^*$ and $\hat{u}_*$ are independent of the ξ-variable.

2. We now prove (6.3). For $\epsilon > 0$ and $(x,y), (x-\epsilon,y), (x,y-\epsilon) \in K_\epsilon$, we denote as usual $z = x+y$ and $\xi = (y - \mathbf{y}(s,z))/\epsilon$. By the concavity of v^ϵ in the pair (x,y) and the concavity of the Merton function v in z it follows that

$$\begin{aligned}v^\epsilon_x(s,x,y) &\leq \frac{1}{\epsilon}\big(v^\epsilon(s,x,y) - v^\epsilon(s,x-\epsilon,y)\big)\\ &\leq \frac{1}{\epsilon}\big(v(s,z) - v(s,z-\epsilon)\big) + \frac{1}{\epsilon}\big(v(s,z-\epsilon) - v^\epsilon(s,x-\epsilon,y)\big)\\ &\leq v_z(s,z-\epsilon) + \frac{1}{\epsilon}\big(v(s,z-\epsilon) - v^\epsilon(s,x-\epsilon,y)\big).\end{aligned}$$

By the definition of u^ϵ,

$$v^\epsilon_x(s,x,y) \leq v_z(s,z-\epsilon) + \epsilon\big(u^\epsilon(s,x-\epsilon,y) + \epsilon^2 w(s,z-\epsilon,\xi_\epsilon)\big),$$

where $\xi_\epsilon := (y - \mathbf{y}(s,z-\epsilon))/\epsilon = \xi + (\mathbf{y}(s,z) - \mathbf{y}(s,z-\epsilon))/\epsilon$. We use the bound (4.6) on w to arrive at

$$v^\epsilon_x(s,x,y) \leq v_z(s,z-\epsilon) + \epsilon u^\epsilon(s,x-\epsilon,y) + \epsilon^3\overline{\lambda} v_z(s,z)\left(1 + |\xi| + \frac{|\mathbf{y}(s,z)-\mathbf{y}(s,z-\epsilon)|}{\epsilon}\right).$$

By exactly the same argument, we also conclude that

$$\begin{aligned} v^\epsilon_y(s,x,y) &\le v_z(s,z-\epsilon)+\epsilon u^\epsilon(s,x,y-\epsilon)\\ &\quad + \epsilon^3\overline{\lambda} v_z(s,z)\left(1+|\xi|+\frac{|-\epsilon+\mathbf{y}(s,z)-\mathbf{y}(s,z-\epsilon)|}{\epsilon}\right). \end{aligned}$$

Then, using the bounds on $\mathbf{y}_z$ from Assumption 5.2,

$$\begin{aligned} \hat{v}^\epsilon_z(s,z,\xi) &= \partial_z v^\epsilon\big(s,z-\epsilon\xi-\mathbf{y}(s,z),\epsilon\xi+\mathbf{y}(s,z)\big)\\ &= \big(1-\mathbf{y}_z(s,z)\big)v^\epsilon_x(s,x,y)+\mathbf{y}_z(s,z)v^\epsilon_y(s,x,y)\\ &\le v_z(s,z-\epsilon)+\epsilon\big(u^\epsilon(s,x-\epsilon,y)+u^\epsilon(s,x,y-\epsilon)\big)\\ &\quad + \epsilon^3\overline{\lambda} v_z(s,z)\left(1+|\mathbf{y}_z(s,z)|+|\xi|+\frac{|\mathbf{y}(s,z)-\mathbf{y}(s,z-\epsilon)|}{\epsilon}\right). \end{aligned}$$

3. The final statement in the lemma follows from (6.4), the expression of γ^ϵ in (6.3), and Assumption 5.1. ∎

6.2. Viscosity subsolution property. In this section, we prove the following proposition.

PROPOSITION 6.1. *Under Assumptions* 5.1 *and* 5.2*, the function u^* is a viscosity subsolution of the second corrector equation* (3.12)*.*

Proof. Let $(s_0,z_0,\varphi)\in(0,\infty)^2\times C^2(\mathbb{R}^2_+)$ be such that

$$0=(u^*-\varphi)(s_0,z_0)>(u^*-\varphi)(s,z)\quad \forall\, s,z\ge 0,\quad (s,z)\ne(s_0,z_0). \tag{6.5}$$

Our objective in the following steps is to prove that

$$\mathcal{A}\varphi(s_0,z_0)-a(s_0,z_0)\le 0. \tag{6.6}$$

1. By the definition of u^* and Lemma 6.2, there exists a sequence (s^ϵ,z^ϵ) so that

$$(s^\epsilon,z^\epsilon)\to(s_0,z_0)\quad\text{and}\quad \hat{u}^\epsilon(s^\epsilon,z^\epsilon,0)\to u^*(s_0,z_0)\ \text{as}\ \ \epsilon\downarrow 0,$$

where we used the notation (3.2). Then, it is clear that

$$\ell^\epsilon_* := \hat{u}^\epsilon(s^\epsilon,z^\epsilon,0)-\varphi(s^\epsilon,z^\epsilon)\to 0 \tag{6.7}$$

and

$$(x^\epsilon,y^\epsilon)=\big(z^\epsilon-\mathbf{y}(s^\epsilon,z^\epsilon),\mathbf{y}(s^\epsilon,z^\epsilon)\big)\longrightarrow(x_0,y_0):=\big(z_0-\mathbf{y}(s_0,z_0),\mathbf{y}(s_0,z_0)\big).$$

Since (u^ϵ) is locally bounded from above (Assumption 5.1), there are $r_0:=r_0(s_0,x_0,y_0)>0$ and $\epsilon_0:=\epsilon_0(s_0,x_0,y_0)>0$ so that

$$\begin{aligned} b_* &:= \sup\{u^\epsilon(s,x,y)\ :\ (s,x,y)\in B_0,\epsilon\in(0,\epsilon_0]\}<\infty,\\ &\quad \text{where}\quad B_0:=B_{r_0}(s_0,x_0,y_0) \end{aligned} \tag{6.8}$$

is the open ball centered at (s_0,x_0,y_0) with radius r_0. We may choose $r_0\le z_0/2$ so that B_0 does not intersect the line $z=0$. For $\epsilon,\delta\in(0,1]$, set

$$\hat{\psi}^{\epsilon,\delta}(s,z,\xi):=v(s,z)-\epsilon^2\ell^\epsilon_*-\epsilon^2\varphi(s,z)-\epsilon^4(1+\delta)w(s,z,\xi)-\epsilon^2\hat{\phi}^\epsilon(s,z,\xi),$$

where, following our standard notation (3.2), $\hat{\phi}^\epsilon$ is determined from the function

$$\phi^\epsilon(s,x,y) := C\left[(s-s^\epsilon)^4 + (x+y-z^\epsilon)^4 + (y-\mathbf{y}(s,x+y))^4\right],$$

and $C>0$ is a large constant that is chosen so that for all sufficiently small $\epsilon>0$,

$$\phi^\epsilon \geq 1 + b_* - \varphi \quad \text{on} \quad B_0 \setminus B_1 \text{ with } B_1 := B_{r_0/2}(s_0,x_0,y_0). \tag{6.9}$$

The constant C chosen above may depend on many things, including the test function φ, s_0, z_0, δ, but not on ϵ. The convergence of (s^ϵ, z^ϵ) to (s_0,z_0) determines how small ϵ should be for (6.9) to hold.

2. We first show that, for all sufficiently small $\epsilon>0$, $\delta>0$, the difference $(v^\epsilon - \psi^{\epsilon,\delta})$ or, equivalently,

$$\begin{aligned} I^{\epsilon,\delta}(s,x,y) &:= \frac{v^\epsilon(s,x,y) - \psi^{\epsilon,\delta}(s,x,y)}{\epsilon^2} \\ &= -u^\epsilon(s,x,y) + \varphi(s,z) + \ell_*^\epsilon + \phi^\epsilon(s,x,y) + \epsilon^2\delta w(s,z,\xi), \end{aligned}$$

has a local minimizer in B_0. Indeed, by the definition of u^ϵ, $\psi^{\epsilon,\delta}$, and ℓ_*^ϵ, (6.9), (6.8), and the fact that $w\geq 0$, for any $(s,x,y)\in\partial B_0$,

$$I^{\epsilon,\delta}(s,x,y) \geq -u^\epsilon(s,x,y) + \ell_*^\epsilon + 1 + b_* + \epsilon^2\delta w(s,z,\xi) \geq 1 + \ell_*^\epsilon > 0$$

for sufficiently small ϵ in view of (6.7). Since $I^{\epsilon,\delta}(s^\epsilon,x^\epsilon,y^\epsilon)=0$, we conclude that $I^{\epsilon,\delta}$ has a local minimizer $(\tilde{s}^\epsilon,\tilde{x}^\epsilon,\tilde{y}^\epsilon)$ in B_0 with $\tilde{z}^\epsilon := \tilde{x}^\epsilon + \tilde{y}^\epsilon$, $\tilde{\xi}^\epsilon := (\tilde{y}^\epsilon - \mathbf{y}(\tilde{s}^\epsilon,\tilde{z}^\epsilon))/\epsilon$ satisfying

$$\min_{(s,z,\xi)\in B_1}(\hat{v}^\epsilon - \hat{\psi}^{\epsilon,\delta}) = (\hat{v}^\epsilon - \hat{\psi}^{\epsilon,\delta})(\tilde{z}_\epsilon, \tilde{\xi}_\epsilon) \leq 0, \quad |\tilde{s}^\epsilon - s_0| + |\tilde{z}^\epsilon - z_0| < r_0 \quad |\xi_\epsilon| < r_1/\epsilon$$

for some constant r_1. Since v^ϵ is a viscosity supersolution of the dynamic programming equation (2.2), we conclude that

$$\left(\beta v^\epsilon - \mathcal{L}\psi^{\epsilon,\delta} - \tilde{U}\left(\psi_x^{\epsilon,\delta}\right)\right)(\tilde{s}^\epsilon,\tilde{x}^\epsilon,\tilde{y}^\epsilon) \geq 0, \tag{6.10}$$

and

$$\begin{aligned} \Lambda_{1,0}^\epsilon \cdot \left(\psi_x^{\epsilon,\delta},\psi_y^{\epsilon,\delta}\right)(\tilde{s}^\epsilon,\tilde{x}^\epsilon,\tilde{y}^\epsilon) &= \left(\psi_x^{\epsilon,\delta} - (1-\lambda^{1,0}\epsilon^3)\psi_y^{\epsilon,\delta}\right)(\tilde{s}^\epsilon,\tilde{x}^\epsilon,\tilde{y}^\epsilon) \geq 0, \\ \Lambda_{0,1}^\epsilon \cdot \left(\psi_x^{\epsilon,\delta},\psi_y^{\epsilon,\delta}\right)(\tilde{s}^\epsilon,\tilde{x}^\epsilon,\tilde{y}^\epsilon) &= \left(\psi_y^{\epsilon,\delta} - (1-\lambda^{0,1}\epsilon^3)\psi_x^{\epsilon,\delta}\right)(\tilde{s}^\epsilon,\tilde{x}^\epsilon,\tilde{y}^\epsilon) \geq 0. \end{aligned}$$

By a direct calculation using the boundedness of $(\tilde{s}^\epsilon,\tilde{z}^\epsilon,\epsilon\tilde{\xi}^\epsilon)$, we rewrite the last gradient inequalities as follows:

$$-4\epsilon^2(\epsilon\tilde{\xi}^\epsilon)^3 + \epsilon^3 v_z(\tilde{s}^\epsilon,\tilde{z}^\epsilon)\left[\lambda^{1,0} - (1+\delta)\overline{w}_\rho(\tilde{s}^\epsilon,\tilde{z}^\epsilon,\tilde{\rho}^\epsilon)\right] + \circ(\epsilon^3) \geq 0, \tag{6.11}$$

$$4\epsilon^2(\epsilon\tilde{\xi}^\epsilon)^3 + \epsilon^3 v_z(\tilde{s}^\epsilon,\tilde{z}^\epsilon)\left[\lambda^{0,1} + (1+\delta)\overline{w}_\rho(\tilde{s}^\epsilon,\tilde{z}^\epsilon,\tilde{\rho}^\epsilon)\right] + \circ(\epsilon^3) \geq 0, \tag{6.12}$$

where $\tilde{\rho}^\epsilon := \tilde{\xi}^\epsilon/\eta(\tilde{s}^\epsilon,\tilde{z}^\epsilon)$.

3. Let $\rho_0(s,z)$ be as in (4.3). In this step, we show that

$$|\tilde{\rho}^\epsilon| < \rho_0(\tilde{s}^\epsilon,\tilde{z}^\epsilon) \quad \forall \text{ sufficiently small } \epsilon\in(0,1]. \tag{6.13}$$

Indeed, assume that $\tilde{\rho}^{\epsilon_n} \le -\rho_0(\tilde{s}^{\epsilon_n}, \tilde{z}^{\epsilon_n}) = \rho_1(\tilde{s}^{\epsilon_n}, \tilde{z}^{\epsilon_n})$ for some sequence $\epsilon_n \in (0,1]$ with $\epsilon_n \to 0$. Then, $\overline{w}_\rho(\tilde{s}^{\epsilon_n}, \tilde{z}^{\epsilon_n}, \tilde{\rho}^{\epsilon_n}) = -\lambda^{0,1}$, and it follows from inequality (6.12), together with the fact that $\tilde{\rho}^{\epsilon_n} \le \rho_1(\tilde{s}^{\epsilon_n}, \tilde{z}^{\epsilon_n}) \le 0$, that

$$0 \le 4\epsilon_n^2(\epsilon_n\tilde{\xi}^{\epsilon_n})^3 - \epsilon_n^3 v_z(\tilde{s}^{\epsilon_n}, \tilde{z}^{\epsilon_n})\delta\lambda^{0,1} + \circ(\epsilon_n^3) \le -{\epsilon_n}^3 v_z(\tilde{s}^{\epsilon_n}, \tilde{z}^{\epsilon_n})\delta\lambda^{0,1} + \circ({\epsilon_n}^3).$$

Since $\delta > 0$, this cannot happen for large n. Similarly, if $\tilde{\rho}^{\epsilon_n} \ge \rho_0(\tilde{s}^{\epsilon_n}, \tilde{z}^{\epsilon_n})$ for some sequence $\epsilon_n \to 0$, we have $\overline{w}_\rho(\tilde{s}^{\epsilon_n}, \tilde{z}^{\epsilon_n}, \tilde{\rho}^{\epsilon_n}) = \lambda^{1,0}$, and it follows from inequality (6.11), together with the fact that $\tilde{\rho}^{\epsilon_n} \ge \rho_0(\tilde{s}^{\epsilon_n}, \tilde{z}^{\epsilon_n}) \ge 0$, that

$$0 \le -4\epsilon_n^2(\epsilon_n\tilde{\xi}^{\epsilon_n})^3 + \epsilon_n^3 v_z(\tilde{s}^{\epsilon_n}, \tilde{z}^{\epsilon_n})(-\delta\lambda^{1,0}) + \circ({\epsilon_n}^3) \le -\epsilon_n^3 v_z(\tilde{s}^{\epsilon_n}, \tilde{z}^{\epsilon_n})\delta\lambda^{1,0} + \circ(\epsilon_n^3),$$

which leads again to a contradiction for large n, completing the proof of (6.13).

4. Since $(\tilde{s}^\epsilon, \tilde{z}^\epsilon)$ is bounded and $(s,z) \mapsto \rho_0(s,z)$ is continuous, we conclude from (6.13) that the sequence $(\tilde{\xi}^\epsilon)_\epsilon$ is bounded. Hence, there exists a sequence $\epsilon_n \to 0$ so that

$$(s_n, z_n, \xi_n) := (\tilde{s}^{\epsilon_n}, \tilde{z}^{\epsilon_n}, \tilde{\xi}^{\epsilon_n}) \longrightarrow (\hat{s}, \hat{z}, \hat{\xi}) = (s_0, z_0, \hat{\xi})$$

for some $\hat{\xi} \in \mathbb{R}$. The fact that the limit of (s_n, z_n) is equal to (s_0, z_0) follows from standard arguments using the strict minimum property of (s_0, z_0) in (6.5). We now take the limit in (6.10) along the sequence ϵ_n. Since the function $\psi^{\epsilon,\delta}$ has the form as in Remark 3.4, we do not repeat the computations given in section 3, and, given the remainder estimate of section 4.2, we directly conclude that

$$\begin{aligned} 0 &\le \lim_{\epsilon_n \to 0} \epsilon_n^{-2}\Big(\beta v^{\epsilon_n} - \mathcal{L}\psi^{\epsilon_n,\delta} - \tilde{U}\big(\psi_x^{\epsilon_n,\delta}\big)\Big)(s_n, z_n, \xi_n) \\ &= \frac{1}{2}(\eta\sigma^2)(s_0, z_0)\hat{\xi}^2 + \frac{1}{2}(1+\delta)\alpha^2(s_0, z_0)w_{\xi\xi}(s_0, z_0, \hat{\xi}) - \mathcal{A}\varphi(s_0, z_0). \end{aligned} \tag{6.14}$$

In the above, we also used the fact that all derivatives of ϕ^ϵ vanish at the origin as ϵ tends to zero.

5. In step 3, we have proved that $|\rho_\epsilon| \le \rho_0(z_\epsilon)$. Hence, $|\hat{\xi}| \le (\eta\rho_0)(s_0, z_0)$. Since $w = \eta v_z \bar{w}$, $a = \eta v_z \bar{a}$, the first corrector equation (3.11) implies that

$$a(s_0, z_0) = \frac{1}{2}(\sigma^2\eta)(s_0, z_0)\hat{\xi}^2 + \frac{1}{2}\alpha^2(s_0, z_0)w_{\xi\xi}(s_0, z_0, \hat{\xi}).$$

We use the above identity in (6.14). The result is

$$\begin{aligned} \mathcal{A}\varphi(s_0, z_0) &\le \frac{1}{2}(\sigma^2\eta)(s_0, z_0)\hat{\xi}^2 + \frac{1}{2}(1+\delta)\alpha^2(s_0, z_0)w_{\xi\xi}(s_0, z_0, \hat{\xi}) \\ &= a(s_0, z_0) + \frac{1}{2}\delta\alpha^2(s_0, z_0)w_{\xi\xi}(s_0, z_0, \hat{\xi}). \end{aligned}$$

Finally, we let δ go to zero. However, $\hat{\xi} = \hat{\xi}^\delta$ depends on δ, and care must be taken. But since $|\xi_n| \le (\eta\rho_0)(s_n, z_n)$, it follows that $\hat{\xi}^\delta$ is uniformly bounded in δ. Hence the second term in the above equation goes to zero with δ, and we obtain the desired inequality (6.6). ∎

6.3. Viscosity supersolution property. In this section, we prove the following proposition.

PROPOSITION 6.2. *Let Assumptions* 5.1, 5.2, *and* 5.4 *hold true. Then, the function u_* is a viscosity supersolution of the second corrector equation* (3.12).

As remarked earlier, the above result holds true without Assumption 5.4, as proved in our forthcoming paper [31]. However, in this paper we utilize it to provide a somehow shorter proof. We first need the following consequence of Assumption 5.4 and the convexity of v^ϵ. Similar arguments are also used in [35].

LEMMA 6.3. *Assume the hypothesis of Proposition* 6.2. *Let* (x, y) *be an arbitrary element of* K_ϵ. *Then,*

(i) *for* $y \geq \mathbf{y}(s,z)$ *(or, equivalently,* $\xi \geq 0$*), we have* $\Lambda^\epsilon_{0,1} \cdot (v^\epsilon_x(s,x,y), v^\epsilon_y(s,x,y)) > 0$,

(ii) *for* $y \leq \mathbf{y}(s,z)$ *(or, equivalently,* $\xi \leq 0$*), we have* $\Lambda^\epsilon_{1,0} \cdot (v^\epsilon_x(s,x,y), v^\epsilon_y(s,x,y)) > 0$.

Proof. For $z \in \mathbb{R}_+$ set

$$\mathbf{y}^\epsilon_+(s,z) := \sup\big\{y : (z-y,y) \in K_\epsilon, \quad \text{and} \quad \Lambda^\epsilon_{0,1} \cdot (v^\epsilon_x, v^\epsilon_y)(s, z-y, y) = 0\big\}.$$

In view of the form of K_ϵ, we have $y \geq -z/(\epsilon^3 \lambda^{0,1})$, and by convention the above supremum is equal to this lower bound if the set is empty. By the concavity of v^ϵ, we conclude that

$$\Lambda^\epsilon_{0,1} \cdot (v^\epsilon_x, v^\epsilon_y)(s,x,y) \begin{cases} = 0 & \forall\ y \leq \mathbf{y}^\epsilon_+(s,z), \\ > 0 & \forall\ y > \mathbf{y}^\epsilon_+(s,z). \end{cases}$$

Let $\mathcal{N}^\epsilon$ be as in (5.5). Therefore it is included in the set $\{(s,x,y)\ :\ y > \mathbf{y}^\epsilon_+(s,z)\}$. Since Assumption 5.4 states that the Merton line $\{(s,x,y) :\ y = \mathbf{y}(s,z)\}$ is included in $\mathcal{N}^\epsilon$, we conclude that $\mathbf{y}(s,z) > \mathbf{y}^\epsilon_+(s,z)$. This proves statement (i). The other assertion is proved similarly. $\square$

Proof of Proposition 6.2. Let $(s_0, z_0, \varphi) \in (0,\infty)^2 \times C^2(\mathbb{R}_+)$ be such that

$$0 = (u_* - \varphi)(s_0, z_0) < (u_* - \varphi)(s,z) \quad \forall\ s, z \geq 0, \quad (s,z) \neq (s_0,z_0). \tag{6.15}$$

We proceed to prove that

$$\mathcal{A}\varphi(s_0,z_0) - a(s_0,z_0) \geq 0. \tag{6.16}$$

1. By the definition of u_* and Lemma 6.2, there exists a sequence (s^ϵ, z^ϵ) such that

$$(s^\epsilon, z^\epsilon) \to (s_0, z_0) \quad \text{and} \quad \hat{u}^\epsilon(s^\epsilon, z^\epsilon, 0) \to u_*(s_0,z_0) \quad \text{as} \ \ \epsilon \downarrow 0,$$

where we used the notation (3.2). Then, it is clear that

$$\ell^*_\epsilon := \hat{u}^\epsilon(s^\epsilon, z^\epsilon, 0) - \varphi(s^\epsilon, z^\epsilon) \longrightarrow 0$$

and

$$(x^\epsilon, y^\epsilon) = \big(z^\epsilon - \mathbf{y}(s^\epsilon, z^\epsilon), \mathbf{y}(s^\epsilon, z^\epsilon)\big) \longrightarrow (x_0, y_0) := \big(z_0 - \mathbf{y}(s_0,z_0), \mathbf{y}(s_0,z_0)\big).$$

Since $u^\epsilon(s,x,y) \geq -\epsilon^2 w(s,z,\xi) \geq -\epsilon C(s,z)|y - \mathbf{y}(s,z)|$, for some continuous function C, there are $r_0 := r_0(s_0,x_0,y_0) > 0$ and $\epsilon_0 := \epsilon_0(s_0,x_0,y_0) > 0$ so that

$$b^* := \inf_{(s,x,y)\in B_0} u^\epsilon(s,x,y) > -\infty, \quad \text{where } B_0 := B_{r_0}(s_0,x_0,y_0).$$

We also choose r_0 sufficiently small so that B_0 does not intersect the line $z = 0$. For $\epsilon \in (0,1]$ and $\delta > 0$, define

$$\hat{\psi}^{\epsilon,\delta}(s,z,\xi) := v(s,z) - \epsilon^2 \ell^*_\epsilon - \epsilon^2 \varphi(s,z) - \epsilon^4 (1-\delta) w(s,z,\xi) + \epsilon^2 \hat{\phi}^\epsilon(s,z,\xi),$$

where, following our notation convention (3.2), the function $\hat{\phi}^\epsilon$ is obtained from the function ϕ^ϵ defined by

$$\phi^\epsilon(s,x,y) := C\big[(s-s^\epsilon)^4 + (x+y-z^\epsilon)^4 + (y-\mathbf{y}(s,x+y))^4\big],$$

and, similarly to the proof of the supersolution property, $C>0$ is a constant chosen so that

$$-b^* + \ell^*_\epsilon + \big(\varphi - \phi^\epsilon\big)(s,x,y) < 0 \quad \text{on } \partial B_0. \tag{6.17}$$

2. Set

$$\begin{aligned} I^{\epsilon,\delta}(s,z,\xi) &:= \epsilon^{-2}\big(v^\epsilon - \psi^{\epsilon,\delta}\big)(s,x,y) \\ &= -u^\epsilon(s,x,y) + \varphi(s,z) + \ell^*_\epsilon - \phi^\epsilon(s,x,y) - \epsilon^2\delta w(s,z,\xi). \end{aligned}$$

Since $w(s,z,0)=0$, we have $I^{\epsilon,\delta}(s^\epsilon,z^\epsilon,0)=0$. On the other hand, it follows from (6.17) that

$$I^{\epsilon,\delta}(s,z,\xi) \le -b^* + \ell^*_\epsilon + \big(\varphi-\phi^\epsilon\big)(s,x,y) - \epsilon^2\delta w(s,z,\xi) < 0 \quad \text{on } \partial B_0.$$

Then, the difference $v^\epsilon - \psi^{\epsilon,\delta}$ has an interior maximizer $(\tilde{s}^\epsilon, \tilde{z}^\epsilon, \tilde{\xi}^\epsilon)$ in B_0,

$$\max_{B_0}\big(v^\epsilon - \psi^{\lambda,\epsilon}\big) = (v^\epsilon - \psi^{\lambda,\epsilon})(\tilde{s}^\epsilon, \tilde{x}_\epsilon, \tilde{y}^\epsilon), \tag{6.18}$$

and $|\tilde{s}^\epsilon - s_0| + |\tilde{z}_\epsilon - z_0| + |\epsilon\tilde{\xi}_\epsilon| \le r_1$ for some constant r_1. By the subsolution property of v^ϵ, at $(\tilde{s}^\epsilon, \tilde{x_\epsilon}, \tilde{y_\epsilon})$,

$$\min\big\{\beta v^\epsilon - \mathcal{L}\psi^{\epsilon,\delta} - \tilde{U}\big(\psi^{\epsilon,\delta}_x\big), \Lambda^\epsilon_{0,1}\cdot(\psi^{\epsilon,\delta}_x, \psi^{\epsilon,\delta}_y), \Lambda^\epsilon_{1,0}\cdot(\psi^{\epsilon,\delta}_x, \psi^{\epsilon,\delta}_y)\big\} \le 0. \tag{6.19}$$

3. In this step, we show that for all sufficiently small $\epsilon>0$,

$$\Lambda^\epsilon_{0,1}\cdot(\psi^{\epsilon,\delta}_x, \psi^{\epsilon,\delta}_y)(\tilde{s}^\epsilon, \tilde{x_\epsilon}, \tilde{y_\epsilon}) > 0 \quad \text{and} \quad \Lambda^\epsilon_{1,0}\cdot(\psi^{\epsilon,\delta}_x, \psi^{\epsilon,\delta}_y)(\tilde{s}^\epsilon, \tilde{x_\epsilon}, \tilde{y_\epsilon}) > 0. \tag{6.20}$$

By Lemma 6.3, it suffices to prove that

$$\begin{aligned} D^{0,1} &:= \Lambda^\epsilon_{0,1}\cdot(\psi^{\epsilon,\delta}_x, \psi^{\epsilon,\delta}_y)(\tilde{s}^\epsilon, \tilde{x_\epsilon}, \tilde{y_\epsilon}) > 0 \quad \text{for} \quad \tilde{\xi} < 0, \\ D^{1,0} &:= \Lambda^\epsilon_{1,0}\cdot(\psi^{\epsilon,\delta}_x, \psi^{\epsilon,\delta}_y)(\tilde{s}^\epsilon, \tilde{x_\epsilon}, \tilde{y_\epsilon}) > 0 \quad \text{for} \quad \tilde{\xi} > 0. \end{aligned} \tag{6.21}$$

We directly compute that

$$\begin{aligned} \psi^{\epsilon,\delta}_x &= v_z - \epsilon^2\varphi_z - \epsilon^4(1-\delta)\left(w_z - \frac{\mathbf{y}_z}{\epsilon}w_\xi\right) + 4\epsilon^2 C\big((z-z^\epsilon)^3 - \mathbf{y}_z(y-\mathbf{y})^3\big), \\ \psi^{\epsilon,\delta}_y &= v_z - \epsilon^2\varphi_z - \epsilon^4(1-\delta)\left(w_z + \frac{1-\mathbf{y}_z}{\epsilon}w_\xi\right) + 4\epsilon^2 C\big((z-z^\epsilon)^3 + (1-\mathbf{y}_z)(y-\mathbf{y})^3\big). \end{aligned}$$

Then, it follows from the estimates (6.18) that

$$\begin{aligned} D^{0,1} &= \epsilon^3\big((1-\delta)w_\xi + \lambda^{0,1}v_z\big)(\tilde{s}^\epsilon, \tilde{z}^\epsilon, \tilde{\xi}^\epsilon) - 4C\epsilon^2(\epsilon\tilde{\xi}^\epsilon)^3 + \circ(\epsilon^3), \\ D^{1,0} &= \epsilon^3\big(-(1-\delta)w_\xi + \lambda^{1,0}v_z\big)(\tilde{s}^\epsilon, \tilde{z}^\epsilon, \tilde{\xi}^\epsilon) + 4C\epsilon^2(\epsilon\tilde{\xi}^\epsilon)^3 + \circ(\epsilon^3). \end{aligned}$$

Since w solves (4.1), $w_\xi + \lambda^{0,1}v_z \ge 0$ and $-w_\xi + \lambda^{1,0}v_z \ge 0$. Then,

$$\begin{aligned} D^{0,1} &\ge -\epsilon^3\delta v_z(\tilde{s}^\epsilon, \tilde{z}^\epsilon) - 4C\epsilon^2(\epsilon\tilde{\xi}^\epsilon)^3 + \circ(\epsilon^3) \\ &\ge -\epsilon^3\delta v_z(\tilde{s}^\epsilon, \tilde{z}^\epsilon) + \circ(\epsilon^3) \quad \text{for} \quad \tilde{\xi} \le 0, \end{aligned}$$

and

$$\begin{aligned} D^{1,0} &\geq \epsilon^3 \delta v_z(\tilde{s}^\epsilon, \tilde{z}^\epsilon) + 4C\epsilon^2(\epsilon\tilde{\xi}^\epsilon)^3 + \circ(\epsilon^3) \\ &\geq \epsilon^3 \delta v_z(\tilde{s}^\epsilon, \tilde{z}^\epsilon) + \circ(\epsilon^3) \quad \text{for} \quad \tilde{\xi} \geq 0. \end{aligned}$$

Since $v_z > 0$, (6.21) holds for all sufficiently small $\epsilon > 0$.

4. In this step, we prove that $\tilde{\xi}_\epsilon$ is bounded in $\epsilon \in (0, 1]$. Indeed, in view of (6.19) and (6.20),

$$\begin{aligned} 0 &\geq \Big(\beta v^\epsilon - \mathcal{L}\psi^{\epsilon,\delta} - \tilde{U}\big(\psi_x^{\epsilon,\delta}\big)\Big)(\tilde{s}^\epsilon, \tilde{x_\epsilon}, \tilde{y_\epsilon}) \\ &= \epsilon^2 \Bigg[\frac{(-\sigma^2 v_{zz})(s^\epsilon, \tilde{z}_\epsilon)}{2}|\xi_\epsilon|^2 + \frac{1-\delta}{2}\alpha^2(\tilde{s}^\epsilon, \tilde{z}_\epsilon) w_{\xi\xi}(\tilde{z}_\epsilon, \tilde{\xi}_\epsilon) \\ &\qquad - \mathcal{A}u(\tilde{s}^\epsilon, \tilde{z}_\epsilon) + \mathcal{R}^\epsilon(\tilde{s}^\epsilon, \tilde{x}_\epsilon, \tilde{y}_\epsilon)\Bigg], \end{aligned} \tag{6.22}$$

where we used the fact that the function $\psi^{\epsilon,\delta}$ is exactly in the form assumed in Remark 3.4. Then, by the remainder estimate of section 4.2, we deduce that

$$|\mathcal{R}^\epsilon(\tilde{s}^\epsilon, \tilde{x}_\epsilon, \tilde{y}_\epsilon)| \leq C(\tilde{s}^\epsilon, \tilde{z}_\epsilon)\left[\epsilon + \epsilon|\tilde{\xi}_\epsilon| + \epsilon^2|\tilde{\xi}_\epsilon|^2\right]. \tag{6.23}$$

In section 4, the function w is explicitly constructed. Since w is linear in ξ for large values of ξ, there is a continuous function $\hat{C}(s, z)$ such that

$$0 \leq w_{\xi\xi}(s, z, \xi) \leq \hat{C}(s, z) \quad \forall\ (s, z, \xi) \in \mathbb{R}^2_+ \times \mathbb{R}^1.$$

Then, since $(\tilde{s}^\epsilon, \tilde{z}_\epsilon)$ is uniformly bounded in $\epsilon \in (0, 1]$, there are constants $C, \tilde{C} > 0$ such that

$$0 \geq \epsilon^2 \tilde{C}\left[\tilde{\xi}_\epsilon^2 - C\left(1 + \epsilon|\tilde{\xi}_\epsilon| + \epsilon^2|\tilde{\xi}_\epsilon|^2\right)\right].$$

Hence $(\tilde{\xi}_\epsilon)_\epsilon$ is also uniformly bounded in $\epsilon \in (0, 1]$ by a constant depending only on the test functions.

5. Since $(z_\epsilon, \xi_\epsilon)_{\epsilon\in(0,1]}$ is bounded, there exists a sequence $(\epsilon_n)_n$ such that

$$\epsilon_n \downarrow 0 \quad \text{and} \quad (z_n, \xi_n) := \big(z_{\epsilon_n}, \xi_{\epsilon_n}\big) \longrightarrow (\hat{z}, \hat{\xi}) = (z_0, \hat{\xi}) \in (0, \infty) \times \mathbb{R},$$

where the fact that $\hat{z} = z_0$ follows from the strict maximum property in (6.15) and classical arguments from the theory of viscosity solutions. We finally conclude from (6.22) and (6.23) that

$$\begin{aligned} 0 &\geq -\frac{1}{2}(\sigma^2 v_{zz})(s_0, z_0)\hat{\xi}^2 - \mathcal{A}\varphi(s_0, z_0) - \mathcal{A}\phi(0) + \frac{1}{2}(1-\delta)\alpha^2(s_0, z_0) w_{\xi\xi}(s_0, z_0, \hat{\xi}) \\ &= -\mathcal{A}\varphi(s_0, z_0) - \frac{1}{2}(\sigma^2 v_{zz})(s_0, z_0)\hat{\xi}^2 + \frac{1}{2}(1-\delta)\alpha^2(s_0, z_0) w_{\xi\xi}(s_0, z_0, \hat{\xi}), \end{aligned}$$

since $\mathcal{A}\phi(0) = 0$. Now, in view of the first corrector equation (3.11),

$$0 \geq -\mathcal{A}\varphi(s_0, z_0) + a(s_0, z_0) + \frac{1}{2}\delta\alpha^2(s_0, z_0) w_{\xi\xi}(s_0, z_0, \hat{\xi}).$$

Finally, we conclude that $\mathcal{A}\varphi(s_0, z_0) - a(s_0, z_0) \geq 0$ by sending δ to zero. $\square$

7. Verifying Assumption 5.1. In this section, we verify Assumption 5.1. This is done by constructing an appropriate subsolution of the dynamic programming equation (2.2). Clearly, this construction requires assumptions, and here we present only one possible set of assumptions. To simplify the presentation, we suppose that the coefficients are independent of the s-variable. Next, we assume that there exist constants $0 < k_* \le k^*$ so that the limit Merton value function satisfies

$$0 < k_* z \le \eta(z) \le k^* z. \tag{7.1}$$

Let $\mathbf{c}$ be the optimal Merton consumption policy given as in (2.6). We assume that

$$U(\mathbf{c}(z)) \ge k_* z v'(z) \tag{7.2}$$

for some constant $k_* > 0$. Notice that all the above assumptions hold in the power utility case. First, using (5.3) and the explicit representation of a, one may directly verify that there is a constant $a^* > 0$ so that

$$a(z) \le a^* z v'(z).$$

Then, the definition of $\mathcal{A}$ and the above assumptions imply that

$$\mathcal{A}v(z) = U(\mathbf{c}(z)) \ge k_* z v'(z) \ge \frac{k_*}{a^*} a(z) = \frac{k_*}{a^*} \mathcal{A}u(z). \tag{7.3}$$

Let u be the function defined in (1.2). Since v is assumed to be smooth, we may apply Itô's formula in a standard way to conclude from the last inequality that

$$0 \le u(z) \le \frac{a^*}{k_*}\, v(z). \tag{7.4}$$

Moreover, since we assume that coefficients are independent of the s-variable, (2.7) is equivalent to $\mathbf{y}(z) = \eta(z)(\mu - r)/\sigma^2$. Hence, (5.3) implies that

$$-v''(z) \le \eta(z)\; v''' \le -2v''(z). \tag{7.5}$$

We now use these observations to construct a subsolution of the dynamic programming equation of the form

$$V^\epsilon(x, y) := v(z) - K\epsilon^2 v(z) + \epsilon^4 \tilde{W}(z, \xi), \tag{7.6}$$

with a sufficiently large constant $K \ge a^*/k_*$ and a slightly modified corrector,

$$\tilde{W}(z, \xi) := z v'(z) \tilde{w}(\xi/z),$$

where the function $\tilde{w}(z)$ and the constant $\tilde{a} > 0$ are the unique solution of $\tilde{w}(0) = 0$ and

$$\max\left\{ -\frac{k_*\sigma^2}{2}\rho^2 - \frac{(\alpha^* k^*)^2}{2}\tilde{w}_{\rho\rho} + \tilde{a};\; -2\lambda^{1,0} + \tilde{w}_\rho;\; -2\lambda^{0,1} - \tilde{w}_\rho \right\}. \tag{7.7}$$

The solution of the above equation is explicitly available through the general solution obtained earlier in section 4.1.

The fact that V^ϵ is a subsolution of (2.2) follows from tedious but otherwise direct calculations. To streamline these calculations, we first state an estimate that follows from the explicit form of $\tilde{W}$.

LEMMA 7.1. *There is a constant $k^*>0$ such that*

$$z\left|\tilde W_{\xi\xi}(z,\xi)\right| \le k^* v'(z),$$
$$\left|\tilde W_{z}(z,\xi)\right| \le k^* v'(z)\left(1+\frac{|\xi|}{z}\right),$$
$$z\left|\partial_x\tilde W(z,\xi)\right| + z\left|\partial_y\tilde W(z,\xi)\right| \le k^* z v'(z)\left(\frac{1}{\epsilon}+\frac{|\xi|}{z}\right),$$
$$z^2\left|\partial_{yy}\tilde W(z,\xi) - \frac{(1-\mathbf{y}'(z))^2}{\epsilon^2}\tilde W_{\xi\xi}(z,\xi)\right| \le k^* z v'(z)\left(\frac{1}{\epsilon}+\frac{|\xi|}{z}\right).$$

Proof. These estimates follow directly from straightforward differentiation and the estimates (7.1), (7.5). ∎

LEMMA 7.2 (lower bound). *Assume* (7.1), (7.2), *and* (5.2). *Then, for sufficiently large $K>0$, V^ϵ defined in* (7.6) *is a subsolution of* (2.2) *in $\mathbb{R}^2_+$. Moreover,*

$$\bar u^\epsilon(x,y) \le K v(z) + \epsilon^2 \tilde W(z,\xi)$$

on $\mathbb{R}^2_+$, and Assumption 5.1 *holds.*

Proof. We need to show that at any point $(x,y)\in\mathbb{R}^2_+$ one of the three terms in (2.2) is nonpositive. Since $(x,y)\in\mathbb{R}^2_+$, by Assumption 5.2, we have

$$|\xi| = \frac{|y-\mathbf{y}(z)|}{\epsilon} \le \frac{z}{\epsilon} \quad\Rightarrow\quad \Xi := \frac{\xi}{z} \in \frac{1}{\epsilon}\,[-1,1].$$

Let $\rho_0>0$ be the threshold in (7.7). We analyze several cases separately.

Case 1. $\rho_0 \le \Xi \le 1/\epsilon$.

In this case, $\tilde W_\xi(z,\xi) = 2\lambda^{1,0}v'(z)$. We use Lemma 7.1 and (5.2) to arrive at

$$\begin{aligned}\Lambda^\epsilon_{1,0}\cdot(V^\epsilon_x,V^\epsilon_y) &= \frac{1}{\epsilon}\hat V^\epsilon_\xi + \epsilon^2\lambda^{1,0}(1-\mathbf{y}')\hat V^\epsilon_\xi + \epsilon^3\lambda^{1,0}\hat V^\epsilon_z\\ &= \epsilon^3\left[(1-\epsilon^3\lambda^{1,0}(1-\mathbf{y}'))\tilde W_\xi + (1-C\epsilon^2)v' - \lambda^{1,0}\epsilon^4\tilde W_z\right]\\ &\le \epsilon^3\lambda^{1,0}v'\left(-1+k^*\epsilon^3\right)\le 0,\end{aligned}$$

provided that ϵ is sufficiently small.

Case 2. $-1/\epsilon \le \Xi \le -\rho_0$.

A similar calculation shows that $\Lambda^\epsilon_{0,1}\cdot(V^\epsilon_x,V^\epsilon_y)\le 0$ for all sufficiently small ϵ.

Case 3. $|\Xi|\le\rho_0$. We now use Remark 3.4 to conclude that

$$\mathcal{J}(V^\epsilon) = \epsilon^2\left[-\frac{\sigma^2 v''(z)}{2}\xi^2 + \frac{\alpha^2(z)}{2}\tilde W_{\xi\xi}(z,\xi) - K\mathcal{A}v(z) + \mathcal{R}^\epsilon(z,\xi)\right].$$

We first use (7.1), (5.2), (7.7), (7.3) and set $\rho := \xi/z$. The result is

$$\begin{aligned}\mathcal{I} &:= \frac{\mathcal{J}(V^\epsilon)}{\epsilon^2}\\ &\le \epsilon^2 v'(z)\eta(z)\left[\frac{k_*\sigma^2}{2}\rho^2 + \frac{(\alpha^*k^*)^2}{2}\tilde w_{\rho\rho}(\rho) - K(k_*)^2\right] + \epsilon^2\mathcal{R}^\epsilon(z,\xi)\\ &= \epsilon^2 v'(z)\eta(z)\left[\tilde a - K(k_*)^2\right] + \epsilon^2\mathcal{R}^\epsilon(z,\xi).\end{aligned}$$

If K is sufficiently large, then $K(k_*)^2$ is larger than $\tilde{a}$, and by (7.1), the above estimate implies that

$$\mathcal{I} \leq -zv'(z) + \mathcal{R}^\epsilon(z,\xi).$$

We now estimate $\mathcal{R}^\epsilon$ by recalling the results of subsection 4.2. We split this into three terms coming from the value function v, the corrector $\tilde{W}$, and from the utility function:

$$|\mathcal{R}^\epsilon| := \mathcal{R}^\epsilon_v + \mathcal{R}^\epsilon_w + \mathcal{R}^\epsilon_U.$$

We estimate each one using Lemma 7.1. Then,

$$\begin{aligned}\mathcal{R}^\epsilon_v &\leq K\left[\epsilon\Xi(\mu-r)zv'(z) + \frac{\sigma^2}{2}\left(\epsilon^2\Xi^2 + 2\epsilon\Xi\left(\frac{\mathbf{y}}{z}\right)\right)z^2v''(z)\right]\\ &\leq \epsilon K k^* z v'(z).\end{aligned}$$

Also,

$$\begin{aligned}\mathcal{R}^\epsilon_w \leq \epsilon^2 &\left[\beta\tilde{W} - rz\left(\left(1-\left(\frac{\mathbf{y}}{z}\right)\right)+\epsilon\Xi\right)\tilde{W}_x - \mu z\left(\epsilon\Xi + \left(\frac{\mathbf{y}}{z}\right)\right)\tilde{W}_y\right.\\ &\quad - \frac{\sigma^2}{2}z^2\left(\epsilon\Xi + \left(\frac{\mathbf{y}}{z}\right)\right)^2\left(\tilde{W}_{yy} - \tilde{W}_{\xi\xi}\frac{(1-\mathbf{y}_z)^2}{\epsilon^2}\right)\\ &\quad \left. + \frac{\sigma^2}{2}z^2\tilde{W}_{\xi\xi}\frac{(1-\mathbf{y}_z)^2}{\epsilon^2}\left(\epsilon^2\Xi^2 + 2\epsilon\Xi\left(\frac{\mathbf{y}}{z}\right)\right)\right]\\ \leq k^* &zv'(z).\end{aligned}$$

Finally,

$$\begin{aligned}\mathcal{R}^\epsilon_U &= \tilde{U}(v') - \tilde{U}(V^\epsilon_x)\\ &\leq \tilde{U}(v') - \tilde{U}(v'[1-\epsilon^2K + k^*\epsilon^4]) \leq 0.\end{aligned}$$

Hence, there is k^* such that

$$|\mathcal{R}^\epsilon| \leq \epsilon k^* z v'(z).$$

Hence, if K is sufficiently large, V^ϵ is a subsolution of (2.2) for all small ϵ.

Boundary $y = 0$.

Then, again by (5.2), for all sufficiently small $\epsilon > 0$,

$$\Xi = \frac{y - \mathbf{y}(z)}{\epsilon} = \frac{-\mathbf{y}(z)}{\epsilon} < -\rho_0.$$

Hence, by the second case and Lemma 6.3,

$$\Lambda^\epsilon_{1,0}\cdot(V^\epsilon_x, V^\epsilon_y)(x,0) \leq 0 = \Lambda^\epsilon_{1,0}\cdot(v^\epsilon_x, v^\epsilon_y)(x,0) \quad \forall\ x > 0.$$

Boundary $x = 0$.

By a similar analysis, we can show that

$$\Lambda^\epsilon_{0,1}\cdot(V^\epsilon_x, V^\epsilon_y)(0,y) \leq 0 = \Lambda^\epsilon_{0,1}\cdot(v^\epsilon_x, v^\epsilon_y)(0,y) \quad \forall\ y > 0.$$

Then, on $\mathbb{R}^2_+$, V^ϵ is a subsolution of (2.2), while v^ϵ is a solution. Also on the boundary of $\mathbb{R}^2_+$, V^ϵ is again a subsolution of an oblique Neumann condition, and v^ϵ is a supersolution. Then, by comparison (or by a verification argument), we conclude that $v^\epsilon \geq \phi$ on $\mathbb{R}^2_+$. This proves the lower bound on u^ϵ on the positive orthant. ∎

Remark 7.1. In view of Lemma 7.2, it follows that the local upper bounding function B, defined in (5.4), is bounded by the function $Kv(z)$. In particular, this implies that the growth of u_* and u^*, both at infinity and at the origin, is the same as that of the zero-transaction cost Merton value function v. By introducing the logarithmic variable, we observe that the behavior near the origin transforms into a growth condition at minus infinity.

8. Homothetic case. In this short section, we consider the classical constant relative risk aversion utility function

$$U(c) := \frac{c^{1-\gamma}}{1-\gamma}, \qquad c > 0, \tag{8.1}$$

for some $\gamma > 0$ with $\gamma = 1$ corresponding to the logarithmic utility. Our objective is to reproduce the results of Janecek and Shreve [22] by directly applying our explicit expansion result of Theorem 6.1. Also, these calculations show how one may use our results to obtain the asymptotic formulae for problems with power utility that have explicitly known Merton value functions, such as factor models.

In the context of the power utility (8.1), the Merton value function is explicitly given by

$$v(z) = \frac{1}{(1-\gamma)}\,\frac{z^{1-\gamma}}{v_M^\gamma},$$

with the Merton constant

$$v_M = \frac{\beta - r(1-\gamma)}{\gamma} - \frac{1}{2}\frac{(\mu-r)^2}{\gamma^2\sigma^2}(1-\gamma).$$

Hence, the risk tolerance function and the optimal strategies are given by

$$\eta(z) = \frac{z}{\gamma}, \qquad \mathbf{y}(z) = \frac{\mu - r}{\gamma\sigma^2}\, z := \pi_M z, \qquad \mathbf{c}(z) = v_M z.$$

In particular, since $\mathbf{y}$ and $\mathbf{c}$ are linear in z, the comparison assumption, Assumption 5.3, is immediately verified to hold true. Indeed, by introducing the logarithmic variable $z' = \ln z$, the second corrector equation (3.12) becomes linear with constant coefficients on $(-\infty, \infty)$. The growth condition as discussed in Remark 7.1 transforms into an exponential sublinear growth. It is well known that this condition is sufficient to prove comparison. The corresponding probabilistic argument refers to the integrability of exponential sublinear growth with respect to the Gaussian density.

Moreover, since the conditions of section 7 are satisfied in the present context, it follows that Assumption 5.1 holds true in our power utility case, provided that $\pi_M \in (0,1)$. Finally, by Remark 11.3 in Shreve and Soner [35], the last condition also implies the validity of Assumption 5.4. We have then verified the following.

LEMMA 8.1. *Assume $\pi_M \in (0,1)$. Then, Assumptions* 5.1–5.4 *hold true in the context of the power utility function* (8.1).

Since the diffusion coefficient $\alpha(z) = \sigma\mathbf{y}(z)[1 - \mathbf{y}_z(z)]$, it follows that

$$\bar{\alpha} = \frac{\alpha(z)}{\eta(z)} = \gamma\sigma\pi_M(1-\pi_M).$$

The constants in the solution of the corrector equation are given by

$$\rho_0 = \left(\frac{3\bar{\alpha}^2}{4\sigma^2} \left(\lambda^{1,0} + \lambda^{0,1}\right) \right)^{1/3},$$

$$a(z) = \eta(z) v'(z) \bar{a} = \frac{\sigma^2(1-\gamma)}{2\gamma} \rho_0^2 \, v(z).$$

Since

$$\mathcal{A}v(z) = U(\mathbf{c}(z)) = \frac{1}{1-\gamma} \left(v_M z\right)^{1-\gamma} = v_M v(z),$$

the unique solution $u(z)$ of the second corrector equation

$$\mathcal{A}u(z) = a(z) = \frac{\sigma^2(1-\gamma)}{2\gamma} \rho_0^2 \, v(z)$$

is given by

$$u(z) = \frac{\sigma^2(1-\gamma)}{2\gamma} \rho_0^2 v_M^{-1} \, v(z) = u_0 z^{1-\gamma},$$

where

$$u_0 := \left(\pi_M(1-\pi_M)\right)^{4/3} v_M^{-(1+\gamma)}.$$

Finally, we summarize the expansion result in the following.

LEMMA 8.2. *For the power utility function U in* (8.1),

$$v^\epsilon(x,y) = v(z) - \epsilon^2 u_0 z^{1-\gamma} + O(\epsilon^3).$$

The width of the transaction region for the first correction equation $2\xi_0 = 2\eta(z)\rho_0$ is given by

$$2\xi_0 = \left(\frac{6}{\gamma} (\lambda^{0,1} + \lambda^{1,0}) \right)^{1/3} \left(\pi_M(1-\pi_M)\right)^{2/3}.$$

The above formulae with $\lambda^{i,j} = 1$ are exactly the same as equation (3.13) in Janecek and Shreve [22] .

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