Optimal Transportation under Controlled Stochastic Dynamics*

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Abstract

We consider an extension of the Monge-Kantorovitch optimal transportation problem. The mass is transported along a continuous semimartingale, and the cost of transportation depends on the drift and the diffusion coefficients of the continuous semimartingale. The optimal transportation problem minimizes the cost among all continuous semimartingales with given initial and terminal distribution. Our first main result is an extension of the Kantorovitch duality to this context. We also suggest a finite-difference scheme combined with the gradient projection algorithm to approximate the dual value. We prove the convergence of the scheme, and we derive a rate of convergence.

We finally provide an application in the context of financial mathematics, which originally motivated our extension of the Monge-Kantorovitch problem. Namely, we implement our scheme to approximate no-arbitrage bounds on the prices of exotic options given the implied volatility curve of some maturity.

Key words. Mass transportation, Kantorovitch duality, viscosity solutions, gradient projection algorithm.

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1 Introduction

The following stochastic mass transportation mechanism was introduced by Mikami and Thieullen [29] as an extension of the Monge-Kantorovitch optimal transportation

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problem. Let X be an $\mathbb{R}^d$-continuous semimartingale with decomposition

$$X_t = X_0 + \int_0^t \beta_s ds + W_t, \tag{1.1}$$

where W_t is a d-dimensional standard Brownian motion under the filtration $\mathbb{F}^X$ generated by X. The optimal mass transportation problem consists in minimizing the cost of transportation defined by some cost functional ℓ along all transportation plans with initial distribution μ_0 and final distribution μ_1:

$$V(\mu_0, \mu_1) := \inf \mathbb{E} \int_0^1 \ell(s, X_s, \beta_s) ds,$$

where the infimum is taken over all semimartingales given by (1.1) satisfying $\mathbb{P} \circ X_0^{-1} = \mu_0$ and $\mathbb{P} \circ X_1^{-1} = \mu_1$. Mikami and Thieullen [29] proved a strong duality result thus extending the classical Kantorovitch duality to this context.

Motivated by a problem in financial mathematics, our main objective is to extend [29] to a larger class of transportation plans defined by continuous semimartingales with absolutely continuous characteristics:

$$X_t = X_0 + \int_0^t \beta_s ds + \int_0^t \sigma_s dW_s,$$

with transportation cost depending on the drift and diffusion coefficients as well as the trajectory of X.

This problem is also intimately connected to the so-called Skorokhod Embedding Problem (SEP), see Obloj [31] for a review. Given a one-dimensional Brownian motion W and a centered $|x|$-integrable probability distribution μ_1 on $\mathbb{R}$, the SEP consists in searching for a stopping times τ such that $W_\tau \sim \mu_1$ and $(W_{t\wedge\tau})_{t\geq 0}$ is uniformly integrable. This problem is well-known to have infinitely many solutions. However, some solutions have been proved to satisfy some optimality with respect to some criterion (Azéma and Yor [1], Root [32] and Rost [33]). This problem can be formulated in our context by restricting the finite variation part to zero, i.e. transportation along a martingale. Indeed, given a solution τ of the SEP, the process $X_t := W_{\tau\wedge\frac{t}{1-t}}$ defines a continuous local martingale satisfying $X_1 \sim \mu_1$. Conversely every continuous local martingale can be represented as time-changed Brownian motion by the Dubins-Schwartz theorem (see e.g. Theorem 4.6, Chapiter 3 of Karatzas and Shreve [25]).

Our extension of Mikami and Thieullen is motivated by Hobson's [22] observation of the connection between the SEP and the problem of finding optimal no-arbitrage bounds for the prices of exotic options (e.g. variance options, lookback option etc.) given the observation of the implied volatility curve for some maturity T, i.e. $T-$maturity European options of all strikes. We refer to Hobson [23] for an overview on some specific applications of the SEP in the context of finance. As observed by Galichon, Henry-Labordère and Touzi [20], our formulation in terms of an optimal transportation problem allows for a systematic treatment of this problem.

Our first main result is to establish the Kantorovitch strong duality for our semimartingale optimal transportation problem. The dual value function consists in the minimization of $\mu_0(\lambda_0) - \mu_1(\lambda_1)$ over all continuous and bounded functions λ_1, where λ_0 is the initial value of a standard stochastic control problem with final cost λ_1. In the Markovian case, the function λ_0 can be characterized as the unique viscosity solution of the corresponding dynamics programming equation with terminal condition λ_1.

Our second main contribution is to exploit the dual formulation for the purpose of numerical approximation of the optimal cost of transportation. In the context of a bounded set of admissible characteristics of the semimartingale, we follow the approach initiated in Bonnans and Tan [11] by suggesting a numerical scheme which combines finite differences and the gradient projection algorithm. We prove convergence of the scheme, and we derive a rate of convergence.

The paper is organized as follows. Section 2 introduces the optimal mass transportation problem under controlled stochastic dynamics. In Section 3, we extend the Kantorovitch duality to our context by using the classical convex duality approach. The convex conjugate of the primal problem turns out to be the value function of a classical stochastic control problem with final condition given by the Lagrange multiplier lying in the space of bounded continuous functions. Then the dual formulation consists in maximizing this value over the class of all Lagrange multipliers. We also show, under some conditions, that the Lagrange multipliers can be restricted to the subclass of $C^\infty-$ functions with bounded derivative of any order. In the Markovian case, we characterize convex dual as the viscosity solution of a dynamic programming equation in the Markovian case in Section 4.

Section 5 introduces a numerical scheme to approximate the dual formulation in the Markovian case. We first use the probabilistic arguments to restrict the optimal control problem to a bounded domain of $\mathbb{R}^d$, then use the finite difference scheme to solve the control problem. The maximization is approximated by means of the gradient projection algorithm. We provide some general convergence results together with some control of the error. Finally, we provide an implementation of our algorithm in the context of an application in financial mathematics. Namely, we consider the problem of robust hedging *variance swap* derivatives given the prices of options of all strikes. The solution of the last problem is known explicitly and allows to test the accuracy of our algorithm.

Notation: Let E be a Polish space, we denote by $\mathbf{M}(E)$ the space of all Borel probability measures on E, equipped with the weak topology, which is also a Polish space. In particular, $\mathbf{M}(\mathbb{R}^d)$ is the space of all probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. S_d denotes the set of $d \times d$ positive symmetric matrices. Given $u = (a, b) \in S_d \times \mathbb{R}^d$, we define $|u|$ by its L^2-norm as an element in $\mathbb{R}^{d^2+d}$.

2 The semimartingale transportation problem

Let $\Omega := C([0, 1], \mathbb{R}^d)$ be the canonical space, X be the canonical process, i.e.

$$X_t(\omega) := \omega_t \quad \text{for all} \quad t \in [0, 1],$$

and $\mathbb{F} = (\mathcal{F}_t)_{1\le t\le 1}$ be the canonical filtration generated by X. We recall that $\mathcal{F}_t$ coincides with the Borel $\sigma-$field on Ω induced by the seminorm $|\omega|_{\infty,t} := \sup_{0\le s\le t}|\omega_s|$, $\omega \in \Omega$. See e.g. the discussions in Section 1.3, Chapter 1 of Stroock and Varadhan [34].

Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F}_1)$ under which the canonical process X is a $\mathbb{F}-$continuous semimartingale. Then, we have the unique continuous decomposition:

$$X_t = X_0 + B^{\mathbb{P}}_t + M^{\mathbb{P}}_t, \quad t \in [0,1], \quad \mathbb{P}-\text{a.s.} \tag{2.1}$$

where $B^{\mathbb{P}} = (B^{\mathbb{P}}_t)_{0\le t\le 1}$ is the finite variation part and $M^{\mathbb{P}} = (M^{\mathbb{P}}_t)_{0\le t\le 1}$ is the local martingale part satisfying $B_0 = M_0 = 0$. Denote by $A^{\mathbb{P}}_t := \langle M^{\mathbb{P}}\rangle_t$ the quadratic variation of $M^{\mathbb{P}}$ between 0 and t and $A^{\mathbb{P}} = (A^{\mathbb{P}}_t)_{0\le t\le 1}$. Then, following Jacod and Shiryaev [24], we say that the $\mathbb{P}-$continuous semimartingale X has characteristics $(A^{\mathbb{P}}, B^{\mathbb{P}})$.

In this paper, we further restrict to the case where the processes $A^{\mathbb{P}}$ and $B^{\mathbb{P}}$ are absolutely continuous in t w.r.t. Lebesgue measure, $\mathbb{P}-$a.s. Then there are $\mathbb{F}-$progressive processes $\nu^{\mathbb{P}} = (\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})$ (see e.g. Proposition I.3.13 of [24]) such that

$$A^{\mathbb{P}}_t = \int_0^t \alpha^{\mathbb{P}}_s ds, \quad B^{\mathbb{P}}_t = \int_0^t \beta^{\mathbb{P}}_s ds, \quad t\in[0,1] \quad \text{up to a } \mathbb{P}-\text{evanescent set.} \tag{2.2}$$

Remark 2.1. *By Doob's martingale representation theorem (see e.g. Theorem 4.2 in Chapter 3 of Karatzas and Shreve [25]), we can find a Brownian motion $W^{\mathbb{P}}$ (possibly in an enlarged space) such that X has an Itô representation:*

$$X_t = X_0 + \int_0^t \beta^{\mathbb{P}}_s ds + \int_0^t \sigma^{\mathbb{P}}_s dW^{\mathbb{P}}_s,$$

where $\sigma^{\mathbb{P}}_t = (\alpha^{\mathbb{P}}_t)^{1/2}$ (i.e. $\alpha^{\mathbb{P}}_t = \sigma^{\mathbb{P}}_t(\sigma^{\mathbb{P}}_t)^T$).

Remark 2.2. *With the unique processes $(A^{\mathbb{P}}, B^{\mathbb{P}})$, the progressively measurable processes $\nu^{\mathbb{P}} = (\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})$ may not be unique. However, they are unique in sense $d\mathbb{P}\times dt-$a.e.. Since the transportation cost defined below is a $d\mathbb{P}\times dt$ integral, then the choice of $\nu^{\mathbb{P}} = (\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})$ will not change the cost value and then is not essential.*

We next introduce the set U defining some restrictions on the admissible characteristics:

$$U \quad \text{closed and convex subset of} \quad S_d \times \mathbb{R}^d, \tag{2.3}$$

and we denote by $\mathcal{P}$ the set of probability measures $\mathbb{P}$ on Ω under which X has the decomposition (2.1), and satisfies (2.2) with characteristics $\nu^{\mathbb{P}}_t := (\alpha^{\mathbb{P}}_t, \beta^{\mathbb{P}}_t) \in U$, $d\mathbb{P}\times dt-$a.e.

Given two arbitrary probability measures μ_0 and μ_1 in $\mathbf{M}(\mathbb{R}^d)$, we also denote

$$\mathcal{P}(\mu_0) := \big\{\mathbb{P}\in\mathcal{P} \;:\; \mathbb{P}\circ X_0^{-1} = \mu_0\big\}, \tag{2.4}$$
$$\mathcal{P}(\mu_0,\mu_1) := \big\{\mathbb{P}\in\mathcal{P}(\mu_0) \;:\; \mathbb{P}\circ X_1^{-1} = \mu_1\big\}. \tag{2.5}$$

Remark 2.3. *In general, $\mathcal{P}(\mu_0, \mu_1)$ may be empty. However, in the one-dimensional case $d = 1$ and $U = \mathbb{R}^+ \times \mathbb{R}$, the initial distribution $\mu_0 = \delta_{x_0}$ for some constant $x_0 \in \mathbb{R}$, and the final distribution satisfies $\int_{\mathbb{R}} |x|\mu_1(dx) < \infty$, we now verify that $\mathcal{P}(\mu_0, \mu_1)$ is not empty. First, we can choose any constant in $\mathbb{R}$ for the drift part, hence we can suppose, without loss of generality, that $x_0 = 0$ and μ_1 is centered distributed, i.e. $\int_{\mathbb{R}} x\mu_1(dx) = 0$. Then, given a Brownian motion W, by Skorokhod embedding (see e.g. Section 3 of Obloj [31]), there is a stopping time τ such that $W_\tau \sim \mu_1$ and $(W_{t\wedge\tau})_{t\geq 0}$ is uniformly integrable. Therefore, $M = (M_t)_{0\leq t\leq 1}$ defined by $M_t := W_{\tau\wedge\frac{t}{1-t}}$ is a continuous martingale with marginal distribution $\mathbb{P} \circ M_1^{-1} = \mu_1$. Moreover, its quadratic variation $\langle M \rangle_t = \tau \wedge \frac{t}{1-t}$ is absolutely continuous in t w.r.t Lebesgue for every fixed ω, which can induce a probability on Ω belonging to $\mathcal{P}(\mu_0, \mu_1)$.*

The semimartingale X under $\mathbb{P}$ can be viewed as a vehicle of mass transportation, from the $\mathbb{P}-$distribution of X_0 to the $\mathbb{P}-$distribution of X_1. We then associate $\mathbb{P}$ with a transportation cost

$$J(\mathbb{P}) := \mathbb{E}^{\mathbb{P}} \int_0^1 L(t, X, \nu_t^{\mathbb{P}})dt, \tag{2.6}$$

where we denoted by $\mathbb{E}^{\mathbb{P}}$ the expectation under the probability measure $\mathbb{P}$. The above expectation is well defined on $\mathbb{R}^+ \cup \{+\infty\}$ in view of the subsequent Assumption 3.1 which states in particular that L is nonnegative.

Our main interest is on the following optimal mass transportation problem, given two probability measures μ_0, $\mu_1 \in \mathbf{M}(\mathbb{R}^d)$:

$$V(\mu_0, \mu_1) := \inf_{\mathbb{P}\in\mathcal{P}(\mu_0,\mu_1)} J(\mathbb{P}), \tag{2.7}$$

with the convention $\inf \emptyset = \infty$.

3 The duality theorem

The main objective of this section is to prove a duality result for problem (2.7) which extends the classical Kantorovitch duality in optimal transportation theory.

This will be achieved by classical convex duality techniques which require to verify that the function V is convex and lower semicontinuous. For general theory on duality analysis in Banach spaces, we refer to Bonnans and Shapiro [10] and Ekeland and Temam [17]. In our context, the value function of the optimal transportation problem is defined on the Polish space of measures on $\mathbb{R}^d$, and our main reference is Deuschel and Stroock [16].

3.1 The main duality result

We first formulate the assumptions needed for our duality result.

Assumption 3.1. *The function $(t, \mathbf{x}, u) \in [0, 1] \times \Omega \times U \mapsto L(t, \mathbf{x}, u) \in \mathbb{R}^+$ is nonnegative, continuous in $(t, \mathbf{x}, u)$, and convex in u.*

Notice that we do not impose any progressive measurability for the dependence of L on the trajectory $\mathbf{x}$. However, by immediate conditioning, we may reduce the problem so that such a progressive measurability is satisfied.

The next condition controls the dependence of the cost functional on the time variable.

Assumption 3.2. *The function L is uniformly continuous in t in sense that*

$$\Delta_t L(\varepsilon) := \sup \frac{|L(s,\mathbf{x},u) - L(t,\mathbf{x},u)|}{1 + L(t,\mathbf{x},u)} \longrightarrow 0, \quad \text{as} \quad \varepsilon \to 0,$$

where the supremum is taken over all $0 \le s,t \le 1$ such that $|t-s| \le \varepsilon$ and all $\mathbf{x} \in \Omega$, $u \in U$.

We finally need the following coercivity condition on the cost functional.

Assumption 3.3. *There are constants $p > 1$ and $C_0 > 0$ such that*

$$|u|^p \le C_0(1 + L(t,\mathbf{x},u)) < \infty \quad \text{for every} \quad (t,\mathbf{x},u) \in [0,1] \times \Omega \times U.$$

Remark 3.1. *In the particular case $U = \{I_d\} \times \mathbb{R}^d$, the last condition coincides with Assumption A.1 of Mikami and Thieullen [29]. Moreover, whenever U is bounded, Assumption 3.3 is a direct consequence of Assumption 3.1.*

Let $C_b(\mathbb{R}^d)$ denote the set of all bounded continuous functions on $\mathbb{R}^d$ and

$$\mu(\phi) := \int_{\mathbb{R}^d} \phi(x)\mu(dx) \quad \text{for all} \quad \mu \in \mathbf{M}(\mathbb{R}^d) \quad \text{and} \quad \phi \in \mathcal{L}^1(\mu).$$

We define the dual formulation of (2.7) by

$$\mathcal{V}(\mu_0, \mu_1) := \sup_{\lambda_1 \in C_b(\mathbb{R}^d)} \big\{\mu_0(\lambda_0) - \mu_1(\lambda_1)\big\}, \tag{3.1}$$

where

$$\lambda_0(x) := \inf_{\mathbb{P} \in \mathcal{P}(\delta_x)} \mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s, X, \nu_s^{\mathbb{P}})ds + \lambda_1(X_1)\Big], \tag{3.2}$$

with $\mathcal{P}(\delta_x)$ defined in (2.4). We notice that $\mu_0(\lambda_0)$ is well defined since λ_0 is bounded from below and measurable by the following Lemma.

Lemma 3.1. *Let Assumptions 3.1 and 3.2 hold true, and assume that λ_0 is locally bounded. Then, λ_0 is measurable w.r.t. the Borel $\sigma-$field on $\mathbb{R}^d$ completed by μ_0, and*

$$\mu_0(\lambda_0) \quad = \quad \inf_{\mathbb{P} \in \mathcal{P}(\mu_0)} \mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s, X, \nu_s^{\mathbb{P}})ds + \lambda_1(X_1)\Big].$$

The proof of Lemma 3.1 is based on a measurable selection argument, and is reported at the end of Section 4.2. We now state the main duality result.

Theorem 3.1. *Let Assumptions 3.1, 3.2 and 3.3 hold, and suppose that λ_0 is locally bounded for all $\lambda_1 \in C_b(\mathbb{R}^d)$. Then:*

$$V(\mu_0,\mu_1) = \mathcal{V}(\mu_0,\mu_1) \quad \textit{for all} \quad \mu_0,\ \mu_1 \in \mathbf{M}(\mathbb{R}^d),$$

and existence holds for the problem $V(\mu_0,\mu_1)$.

The proof of this result is reported in the subsequent subsections.

We finally state a duality result in the space $C_b^\infty(\mathbb{R}^d)$ of all functions with bounded derivatives of any order:

$$\overline{\mathcal{V}}(\mu_0,\mu_1) \quad := \quad \sup_{\lambda_1 \in C_b^\infty(\mathbb{R}^d)} \big\{\mu_0(\lambda_0) - \mu_1(\lambda_1)\big\}. \tag{3.3}$$

Assumption 3.4. *The function L is uniformly continuous in $\mathbf{x}$ in sense that*

$$\Delta_x L(\varepsilon) := \sup \frac{|L(t,\mathbf{x}^1,u) - L(t,\mathbf{x}^2,u)|}{1 + L(t,\mathbf{x}^2,u)} \longrightarrow 0, \quad \textit{as} \quad \varepsilon \to 0,$$

where the supremum is taken over all $0 \le t \le 1$, $u \in U$ and all $\mathbf{x}^1, \mathbf{x}^2 \in \Omega$ such that $|\mathbf{x}^1 - \mathbf{x}^2|_\infty \le \varepsilon$.

Theorem 3.2. *Under the conditions of Theorem 3.1 together with Assumption 3.4, we have $\mathcal{V} = \overline{\mathcal{V}}$ on $\mathbf{M}(\mathbb{R}^d) \times \mathbf{M}(\mathbb{R}^d)$.*

The proof of the last result follows exactly the same arguments as those of Mikami and Thieullen [29] in the proof of their Theorem 2.1. We report it in Section 3.6 for completeness.

3.2 An enlarged space

In preparation of the proof of Theorem 3.1, we introduce the enlarged canonical space

$$\overline{\Omega} := C\big([0,1], \mathbb{R}^d \times \mathbb{R}^{d^2} \times \mathbb{R}^d\big) \tag{3.4}$$

following the technique used by Haussmann [21] in the proof of his Proposition 3.1.

On $\overline{\Omega}$, we denote the canonical filtration by $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{0 \le t \le 1}$, and the canonical process by (X, A, B), where X, B are $d-$dimensional processes and A is a d^2-dimensional process.

We consider a probability measure $\overline{\mathbb{P}}$ on $\overline{\Omega}$ such that X is an $\overline{\mathbb{F}}-$semimartingale characterized by (A, B), and moreover, (A, B) is $\overline{\mathbb{P}}-$a.s. absolutely continuous w.r.t. t and $\nu_s \in U$, $d\overline{\mathbb{P}} \times dt$ a.e., where $\nu = (\alpha, \beta)$ is defined by:

$$\alpha_t := \limsup_{n \to \infty} n \left(A_t - A_{t-\frac{1}{n}}\right), \quad \text{and} \quad \beta_t := \limsup_{n \to \infty} n \left(B_t - B_{t-\frac{1}{n}}\right). \tag{3.5}$$

We also denote by $\overline{\mathcal{P}}$ the set of all the probability measures $\overline{\mathbb{P}}$ on $(\overline{\Omega}, \overline{\mathcal{F}}_1)$ satisfying the above conditions, and

$$\overline{\mathcal{P}}(\mu_0) := \big\{\overline{\mathbb{P}} \in \overline{\mathcal{P}} : \overline{\mathbb{P}} \circ X_0^{-1} = \mu_0\big\}, \quad \overline{\mathcal{P}}(\mu_0,\mu_1) := \big\{\overline{\mathbb{P}} \in \overline{\mathcal{P}}(\mu_0) : \overline{\mathbb{P}} \circ X_1^{-1} = \mu_1\big\}.$$

Finally, we denote

$$\overline{J}(\overline{\mathbb{P}}) \quad := \quad \mathbb{E}^{\overline{\mathbb{P}}} \int_0^1 L(t, X, \nu_t) dt.$$

Lemma 3.2. *The function $\overline{J}$ is lower semicontinuous on $\overline{\mathcal{P}}$.*

Proof. We follow the lines of arguments for proving the inequality (3.17) of Mikami [30]. Let $(\overline{\mathbb{P}}^n)_{n\geq 1}$ be a sequence of probability measures in $\overline{\mathcal{P}}$ which converges weakly to some $\overline{\mathbb{P}}^0 \in \overline{\mathcal{P}}$.

First, by Assumption 3.2, for every $s \in [0,1)$, $\varepsilon \in (0, 1-s)$, $t \in [s, s+\varepsilon]$, $\mathbf{x} \in \Omega$ and $\mathbb{R}^{d^2+d}-$valued process η,

$$L(s,\mathbf{x},\eta_t) \leq L(t,\mathbf{x},\eta_t) + \Delta_t L(\varepsilon)(1+L(t,\mathbf{x},\eta_t)) = \Delta_t L(\varepsilon) + (1+\Delta_t L(\varepsilon))L(t,\mathbf{x},\eta_t).$$

It follows from the convexity of L in u that

$$L\Big(s,\mathbf{x},\frac{1}{\varepsilon}\int_s^{s+\varepsilon}\eta_t dt\Big) \leq \frac{1}{\varepsilon}\int_s^{s+\varepsilon} L\big(s,\mathbf{x},\eta_t\big)dt \leq \Delta_t L(\varepsilon) + \frac{1+\Delta_t L(\varepsilon)}{\varepsilon}\int_s^{s+\varepsilon} L(t,\mathbf{x},\eta_t)dt.$$

Integrating both side on s from 0 to $1-\varepsilon$, we get

$$\begin{aligned}\int_0^{1-\varepsilon} L\Big(s,\mathbf{x},\frac{1}{\varepsilon}\int_s^{s+\varepsilon}\eta_t dt\Big)ds &\leq (1-\varepsilon)\Delta_t L(\varepsilon) + \frac{1+\Delta_t L(\varepsilon)}{\varepsilon}\int_0^{1-\varepsilon}\int_s^{s+\varepsilon} L(t,\mathbf{x},\eta_t)dtds \\ &\leq (1-\varepsilon)\Delta_t L(\varepsilon) + (1+\Delta_t L(\varepsilon))\int_0^1 L(s,\mathbf{x},\eta_s)ds\end{aligned}$$

by integration by parts formula. Therefore,

$$\int_0^1 L(s,\mathbf{x},\eta_s)ds \geq \frac{1}{1+\Delta_t L(\varepsilon)}\int_0^{1-\varepsilon} L\Big(s,\mathbf{x},\frac{1}{\varepsilon}\int_s^{s+\varepsilon}\eta_t dt\Big)ds - \Delta_t L(\varepsilon).$$

Then replacing $\mathbf{x}$ by X, η by ν defined in (3.5), taking expectation under $\overline{\mathbb{P}}^n$, by the definition of ν_t as well as the absolute continuity of (A,B) in t, it follows that

$$\begin{aligned}\overline{J}(\overline{\mathbb{P}}^n) &= \mathbb{E}^{\overline{\mathbb{P}}^n}\int_0^1 L(s,X,\nu_s)ds \\ &\geq \frac{1}{1+\Delta_t L(\varepsilon)}\mathbb{E}^{\overline{\mathbb{P}}^n}\Big[\int_0^{1-\varepsilon} L\Big(s,X,\frac{1}{\varepsilon}\int_s^{s+\varepsilon}\nu_t dt\Big)ds\Big] - \Delta_t L(\varepsilon) \\ &= \frac{1}{1+\Delta_t L(\varepsilon)}\mathbb{E}^{\overline{\mathbb{P}}^n}\Big[\int_0^{1-\varepsilon} L\Big(s,X,\frac{1}{\varepsilon}(A_{s+\varepsilon}-A_s),\frac{1}{\varepsilon}(B_{s+\varepsilon}-B_s)\Big)ds\Big] - \Delta_t L(\varepsilon).\end{aligned}$$

By Skorokhod's theorem (see e.g. Theorem 3.3 of Billingsley [8]), we may consider a probability space $(\Omega',\mathbb{F}',\mathbb{P}')$ together with a sequence of processes $(X^n,A^n,B^n)_{n\geq 0}$ on it such that (X^n,A^n,B^n) under $\mathbb{P}'$ has the same distribution as (X,A,B) under $\overline{\mathbb{P}}^n$ and $(X^n,A^n,B^n) \longrightarrow (X^0,A^0,B^0)$ for a.e. $\omega' \in \Omega'$ as $n \to \infty$ under norm $|\cdot|_\infty$. Then by Fatou's lemma, we get that

$$\liminf_{n\to\infty}\overline{J}(\overline{\mathbb{P}}^n) \geq \frac{1}{1+\Delta_t L(\varepsilon)}\mathbb{E}^{\overline{\mathbb{P}}^0}\left[\int_0^{1-\varepsilon} L\Big(s,X,\frac{1}{\varepsilon}\int_s^{s+\varepsilon}\nu_t dt\Big)ds\right] - \Delta_t L(\varepsilon).$$

Note that by the absolute continuity assumption of (A,B) in t under $\overline{\mathbb{P}}^0$,

$$\frac{1}{\varepsilon}\int_s^{s+\varepsilon}\nu_t(\omega)dt \longrightarrow \nu_s(\omega), \quad \text{as } \varepsilon \to 0, \quad \text{for } d\overline{\mathbb{P}}^0 \times dt-\text{a.e. } (\omega,s) \in \Omega \times [0,1),$$

and $\Delta_t L(\varepsilon) \to 0$ as $\varepsilon \to 0$ from Assumption 3.2, we then finish the proof by sending ε to zero and using Fatou's Lemma. □

Remark 3.2. *In the Markovian case $L(t, \mathbf{x}, u) = \ell(t, \mathbf{x}(t), u)$, for some deterministic function ℓ, we observe that Assumption 3.2 is stronger than Assumption A2 in Mikami [30]. However, we can easily adapt this proof by introducing the trajectory set $\{\mathbf{x} : \sup_{0\le t,s\le 1, |t-s|\le\varepsilon} |\mathbf{x}(t)-\mathbf{x}(s)| \le \delta\}$ and then letting $\varepsilon,\ \delta \to 0$ as in the proof of inequality (3.17) in [30].*

Our next objective is to establish a one-to-one connection between the cost functional J defined on the set $\mathcal{P}(\mu_0, \mu_1)$ of probability measures on Ω and the cost functional $\overline{J}$ defined on the corresponding set $\overline{\mathcal{P}}(\mu_0, \mu_1)$ on the enlarged space $\overline{\Omega}$.

Proposition 3.1. (i) *For any probability measure $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$, there exists a probability $\overline{\mathbb{P}} \in \overline{\mathcal{P}}(\mu_0, \mu_1)$ such that $J(\mathbb{P}) = \overline{J}(\overline{\mathbb{P}})$.*
(ii) *Conversely, let $\overline{\mathbb{P}} \in \overline{\mathcal{P}}(\mu_0, \mu_1)$ be such that $\mathbb{E}^{\overline{\mathbb{P}}} \int_0^1 |\beta_s| ds < \infty$. Then, under Assumption 3.1, there exists a probability measure $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$ such that $J(\mathbb{P}) \le \overline{J}(\overline{\mathbb{P}})$.*

Proof. (i) Given $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$, define the processes $A^{\mathbb{P}}$, $B^{\mathbb{P}}$ from decomposition (2.1), and observe that the mapping $\omega \in \Omega \mapsto (X_t(\omega), A^{\mathbb{P}}_t(\omega), B^{\mathbb{P}}_t(\omega)) \in \mathbb{R}^{2d+d^2}$ is measurable for every $t \in [0, 1]$. Then the mapping $\omega \in \Omega \mapsto (X(\omega), A^{\mathbb{P}}(\omega), B^{\mathbb{P}}(\omega)) \in \overline{\Omega}$ is also measurable, see e.g. discussions in Chapter 2 of Billingsley [7] at Page 57.

Let $\overline{\mathbb{P}}$ be the probability measure on $(\overline{\Omega}, \overline{\mathcal{F}}_1)$ induced by $(\mathbb{P}, (X, A^{\mathbb{P}}(X), B^{\mathbb{P}}(X)))$. In the enlarged space $(\overline{\Omega}, \overline{\mathcal{F}}_1, \overline{\mathbb{P}})$, the canonical process X is clearly a continuous semimartingale characterized by $(A^{\mathbb{P}}(X), B^{\mathbb{P}}(X))$. Moreover, $(A^{\mathbb{P}}(X), B^{\mathbb{P}}(X)) = (A, B)$, $\overline{\mathbb{P}}$−a.s., where (X, A, B) are canonical processes in $\overline{\Omega}$. It follows that, on the enlarged space $(\overline{\Omega}, \overline{\mathbb{F}}, \overline{\mathbb{P}})$, X is a continuous semimartingale characterized by (A, B). Also (A, B) is clearly $\overline{\mathbb{P}}$−a.s. absolutely continuous in t, with $\nu^{\mathbb{P}}(X)_t = \nu_t$, $d\overline{\mathbb{P}} \times dt$−a.e., where ν is defined in (3.5). Then $\overline{\mathbb{P}}$ is the required probability in $\overline{\mathcal{P}}(\mu_0, \mu_1)$ and satisfies $\overline{J}(\overline{\mathbb{P}}) = J(\mathbb{P})$.
(ii) Let us first consider the enlarged space $\overline{\Omega}$, denote by $\overline{\mathbb{F}}^X = (\overline{\mathcal{F}}^X_t)_{0\le t\le 1}$ the filtration generated by process X. Then for every $\overline{\mathbb{P}} \in \overline{\mathcal{P}}(\mu_0, \mu_1)$, $(\overline{\Omega}, \overline{\mathbb{F}}^X, \overline{\mathbb{P}}, X)$ is still a continuous semimartingale, by the stability property of semimartingales. It follows from Theorem A.1 in Appendix that the decomposition of X under filtration $\overline{\mathbb{F}}^X = (\overline{\mathcal{F}}^X_t)_{0\le t\le 1}$ can be written as

$$X_t = X_0 + \bar{B}(X)_t + \bar{M}(X)_t = X_0 + \int_0^t \bar{\beta}_s ds + \bar{M}(X)_t,$$

with $\overline{A}(X)_t := \langle \bar{M}(X) \rangle_t = \int_0^t \bar{\alpha}_s ds$, $\bar{\beta}_s = \mathbb{E}^{\overline{\mathbb{P}}}\left[\beta_s \middle| \overline{\mathcal{F}}^X_s\right]$ and $\bar{\alpha}_s = \alpha_s$, $d\overline{\mathbb{P}} \times dt$−a.e. Moreover, by the convexity property (2.3) of set U, it follows that $(\bar{\alpha}, \bar{\beta}) \in U$, $d\overline{\mathbb{P}} \times dt$−a.e. Finally, since $\overline{\mathcal{F}}^X_t = \mathcal{F}_t \otimes \{\emptyset, C([0,1], \mathbb{R}^{d^2} \times \mathbb{R}^d)\}$, $\overline{\mathbb{P}}$ then induces a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_1)$ by

$$\mathbb{P}[E] := \overline{\mathbb{P}}\big[E \times C([0,1], \mathbb{R}^{d^2} \times \mathbb{R}^d)\big], \quad \forall E \in \mathcal{F}_1.$$

Clearly, $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$ and $J(\mathbb{P}) \le \overline{J}(\overline{\mathbb{P}})$ by the convexity of L in b of Assumption 3.1 and Jensen's inequality. □

Remark 3.3. *Let $\overline{\mathbb{P}} \in \overline{\mathcal{P}}$ be such that $\overline{J}(\overline{\mathbb{P}}) < \infty$, then from the coercivity property of L in u in Assumption 3.3, it follows immediately that*

$$\mathbb{E}^{\overline{\mathbb{P}}} \int_0^1 |\beta_s| ds < \infty.$$

3.3 Lower semicontinuity and existence

By the correspondence between J and $\overline{J}$ (Proposition 3.1) and the lower semicontinuity of $\overline{J}$ (Lemma 3.2), we now obtain the corresponding property for V under the crucial Assumption 3.3, which guarantees the tightness of any minimizing sequence of our problem $V(\mu_0, \mu_1)$.

Lemma 3.3. *Under Assumptions 3.1, 3.2 and 3.3, the map*

$$(\mu_0, \mu_1) \in \mathbf{M}(\mathbb{R}^d) \times \mathbf{M}(\mathbb{R}^d) \quad \longmapsto \quad V(\mu_0, \mu_1) \in \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$$

is lower semicontinuous.

Proof. Let (μ_0^n) and (μ_1^n) be two sequences in $\mathbf{M}(\mathbb{R}^d)$ converging weakly to $\mu_0, \mu_1 \in \mathbf{M}(\mathbb{R}^d)$, respectively, and let us prove that

$$\liminf_{n\to\infty} V(\mu_0^n, \mu_1^n) \geq V(\mu_0, \mu_1).$$

We focus on the case $\liminf_{n\to\infty} V(\mu_0^n, \mu_1^n) < \infty$ as the result is trivial in the alternative case. Then, after possibly extracting a subsequence, we can assume that $(V(\mu_0^n, \mu_1^n))_{n\geq 1}$ is bounded, and there is a sequence $(\mathbb{P}_n)_{n\geq 1}$ such that $\mathbb{P}_n \in \mathcal{P}(\mu_0^n, \mu_1^n)$ for all $n \geq 1$ and

$$\sup_{n\geq 1} J(\mathbb{P}_n) < \infty, \quad 0 \leq J(\mathbb{P}_n) - V(\mu_0^n, \mu_1^n) \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{3.6}$$

By Assumption 3.3 it follows that $\sup_{n\geq 1} \mathbb{E}^{\mathbb{P}_n} \int_0^1 |\nu_s^{\mathbb{P}_n}|^p ds < \infty$. Then, it follows from Theorem 3 of Zheng [36] that the sequence $(\overline{\mathbb{P}}_n)_{n\geq 1}$, of probability measures induced by $(\mathbb{P}_n, X, A^{\mathbb{P}_n}, B^{\mathbb{P}_n})$ on $(\overline{\Omega}, \overline{\mathcal{F}}_1)$, is tight. Moreover, under any one of their limit laws $\overline{\mathbb{P}}$, the canonical process X is a semimartingale characterized by (A, B) such that (A, B) are still absolutely continuous in t. Moreover, $\nu \in U$, $d\overline{\mathbb{P}} \times dt-$a.e. since $\frac{1}{t-s}(A_t - A_s, B_t - B_s) \in U$, $d\overline{\mathbb{P}}-$a.s. for every $t, s \in [0,1]$, hence $\overline{\mathbb{P}} \in \overline{\mathcal{P}}(\mu_0, \mu_1)$. We then deduce from (3.6), Proposition 3.1, and Lemma 3.2 that:

$$\liminf_{n\to\infty} V(\mu_0^n, \mu_1^n) = \liminf_{n\to\infty} J(\mathbb{P}_n) = \liminf_{n\to\infty} \overline{J}(\overline{\mathbb{P}}_n) \geq \overline{J}(\overline{\mathbb{P}}).$$

By Remark 3.3 and Proposition 3.1, we may find $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$ such that $\overline{J}(\overline{\mathbb{P}}) \geq J(\mathbb{P})$. Hence $\liminf_{n\to\infty} V(\mu_0^n, \mu_1^n) \geq J(\mathbb{P}) \geq V(\mu_0, \mu_1)$, completing the proof. □

Proposition 3.2. *Let Assumptions 3.1, 3.2 and 3.3 hold true. Then for every μ_0, $\mu_1 \in \mathbf{M}(\mathbb{R}^d)$ such that $V(\mu_0, \mu_1) < \infty$, existence holds for the minimization problem $V(\mu_0, \mu_1)$. Moreover, the set of minimizers $\{\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1) : J(\mathbb{P}) = V(\mu_0, \mu_1)\}$ is a compact set of probability measures on Ω.*

Proof. We just let $(\mu_0^n, \mu_1^n) = (\mu_0, \mu_1)$ in the proof of Lemma 3.3, then the required existence result is proved by following the same arguments. □

3.4 Convexity

Lemma 3.4. *Let Assumptions 3.1 and 3.3 hold, then the map* $(\mu_0, \mu_1) \mapsto V(\mu_0, \mu_1)$ *is convex.*

Proof. Given μ_0^1, μ_0^2, μ_1^1, $\mu_1^2 \in \mathbf{M}(\mathbb{R}^d)$ and $\mu_0 = \theta\mu_0^1 + (1-\theta)\mu_0^2$, $\mu_1 = \theta\mu_1^1 + (1-\theta)\mu_1^2$ with $\theta \in (0,1)$, we shall prove that

$$V(\mu_0, \mu_1) \le \theta V(\mu_0^1, \mu_1^1) + (1-\theta) V(\mu_0^2, \mu_1^2).$$

It is enough to show that for both $\mathbb{P}_i \in \mathcal{P}(\mu_0^i, \mu_1^i)$ such that $J(\mathbb{P}_i) < \infty$, $i = 1, 2$, we can find $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$ satisfying

$$J(\mathbb{P}) \le \theta J(\mathbb{P}_1) + (1-\theta) J(\mathbb{P}_2). \tag{3.7}$$

As in Lemma 3.3, let us consider the enlarged space $\overline{\Omega}$, on which the probability measures $\overline{\mathbb{P}}_i$ are induced by $(\mathbb{P}_i, X, A^{\mathbb{P}_i}, B^{\mathbb{P}_i})$, i=1,2. By Proposition 3.1, $(\overline{\mathbb{P}}_i)_{i=1,2}$ are probability measures under which X is a $\overline{\mathbb{F}}-$semimartingale characterized by the same process (A, B), which is absolutely continuous in t, such that $J(\mathbb{P}_i) = \overline{J}(\overline{\mathbb{P}}_i)$, $\ i = 1, 2$.

By Corollary III.2.8 of Jacod and Shiryaev [24], $\overline{\mathbb{P}} := \theta\overline{\mathbb{P}}_1 + (1-\theta)\overline{\mathbb{P}}_2$ is also a probability measure under which X is an $\overline{\mathbb{F}}-$semimartingale characterized by (A, B). Clearly, $\nu \in U$, $d\overline{\mathbb{P}} \times dt-$a.e. since it is true $d\overline{\mathbb{P}}_i \times dt-$a.e. for $i = 1, 2$. Thus $\overline{\mathbb{P}} \in \mathcal{P}(\mu_0, \mu_1)$ and it satisfies that

$$\overline{J}(\overline{\mathbb{P}}) = \theta \overline{J}(\overline{\mathbb{P}}_1) + (1-\theta)\overline{J}(\overline{\mathbb{P}}_2) = \theta J(\mathbb{P}_1) + (1-\theta) J(\mathbb{P}_2) < \infty.$$

Finally, by Remark 3.3 and Proposition 3.1, we can construct $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$ such that $J(\mathbb{P}) \le \overline{J}(\overline{\mathbb{P}})$, and it follows that inequality (3.7) holds true. $\square$

3.5 Proof of the duality result

If $V(\mu_0, \mu_1)$ is infinite for every $\mu_1 \in \mathbf{M}(\mathbb{R}^d)$, then $J(\mathbb{P}) = \infty$ for all $\mathbb{P} \in \mathcal{P}(\mu_0)$. It follows from (3.1) and Lemma 3.1 that

$$V(\mu_0, \mu_1) = \mathcal{V}(\mu_0, \mu_1) = \infty.$$

Now, suppose that $V(\mu_0, \cdot)$ is not always infinite. Let $\overline{\mathbf{M}}(\mathbb{R}^d)$ be the space of all finite signed measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, with topology defined in section 2.2 of Deuschel and Stroock [16]. As indicated in section 3.2 of [16], the topology inherited by $\mathbf{M}(\mathbb{R}^d)$ as a subset of $\overline{\mathbf{M}}(\mathbb{R}^d)$ is its weak topology. We then extend $V(\mu_0, \cdot)$ to $\overline{\mathbf{M}}(\mathbb{R}^d) \supset \mathbf{M}(\mathbb{R}^d)$ by setting $V(\mu_0, \mu_1) = \infty$ when $\mu_1 \in \overline{\mathbf{M}}(\mathbb{R}^d) \setminus \mathbf{M}(\mathbb{R}^d)$, thus $\mu_1 \mapsto V(\mu_0, \mu_1)$ is a convex and lower semicontinuous function defined on $\overline{\mathbf{M}}(\mathbb{R}^d)$. Then, the duality result $V = \mathcal{V}$ follows from Theorem 2.2.15 and Lemma 3.2.3 in [16], together with the fact that for $\lambda_1 \in C_b(\mathbb{R}^d)$:

$$\begin{aligned}
\sup_{\mu_1 \in \mathbf{M}(\mathbb{R}^d)} \big\{ \mu_1(-\lambda_1) - V(\mu_0, \mu_1) \big\} &= - \inf_{\substack{\mu_1 \in \mathbf{M}(\mathbb{R}^d) \\ \mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)}} \mathbb{E}^{\mathbb{P}} \Big[\int_0^1 L(s, X, \nu_s^{\mathbb{P}}) ds + \lambda_1(X_1) \Big] \\
&= - \inf_{\mathbb{P} \in \mathcal{P}(\mu_0)} \mathbb{E}^{\mathbb{P}} \Big[\int_0^1 L(s, X, \nu_s^{\mathbb{P}}) ds + \lambda_1(X_1) \Big] \\
&= - \mu_0(\lambda_0).
\end{aligned}$$

□

3.6 Proof of Theorem 3.2

Let $\psi \in C_c^\infty([-1,1]^d, \mathbb{R}^+)$ be such that $\int_{\mathbb{R}^d} \psi(x)dx = 1$, and define $\psi_\varepsilon(x) := \varepsilon^{-d}\psi(x/\varepsilon)$. We claim that

$$\overline{\mathcal{V}}(\mu_0,\mu_1) \quad \geq \quad \frac{\mathcal{V}(\psi_\varepsilon * \mu_0, \psi_\varepsilon * \mu_1)}{1+\Delta_x L(\varepsilon)} - \Delta_x L(\varepsilon). \tag{3.8}$$

Since the inequality $\mathcal{V} \geq \overline{\mathcal{V}}$ is obvious, the required result is then obtained by sending $\varepsilon \to 0$, and using Assumption 3.4 together with Lemma 3.3.

In the rest of this proof, we denote $\delta := \Delta_x L(\varepsilon)$. To prove (3.8), we first observe from Assumption 3.4 that:

$$L(s,\mathbf{x},u) \geq \frac{L(s,\mathbf{x}+z,u)}{1+\delta} - \delta, \quad \text{for all} \quad z \in \mathbb{R} \;\; \text{satisfying} \;\; |z| \leq \varepsilon.$$

Here, $\mathbf{x}+z := (\mathbf{x}(t)+z)_{0\leq t\leq 1} \in \Omega$. For an arbitrary $\lambda_1 \in C_b(\mathbb{R}^d)$, we denote $\lambda_1^\varepsilon := (1+\delta)^{-1}\lambda_1 * \psi_\varepsilon \in C_b^\infty$, then for every $\mathbb{P} \in \mathcal{P}(\mu_0)$:

$$\begin{aligned}
& \mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s,X,\nu_s^{\mathbb{P}})ds + \lambda_1^\varepsilon(X_1)\Big] \\
= \quad & \int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s,X,\nu_s^{\mathbb{P}})ds + \frac{\lambda_1(X_1+z)}{1+\delta}\Big]\psi_\varepsilon(z)dz \\
\geq \quad & -\delta + \int_{\mathbb{R}^d} \frac{\psi_\varepsilon(z)}{1+\delta}\mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s,X+z,\nu_s^{\mathbb{P}})ds + \lambda_1(X_1+z)\Big]dz.
\end{aligned}$$

Let Z be a r.v. independent of X with distribution defined by the density function ψ_ε under $\mathbb{P}$. Then the probability $\overline{\mathbb{P}}_\varepsilon$ on $\overline{\Omega}$ induced by $(\mathbb{P}, X+Z := (X_t+Z)_{0\leq t\leq 1}, A^{\mathbb{P}}, B^{\mathbb{P}})$ is in $\overline{\mathcal{P}}(\psi_\varepsilon * \mu_0)$, and

$$\begin{aligned}
& \mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s,X,\nu_s^{\mathbb{P}})ds + \lambda_1^\varepsilon(X_1)\Big] \\
\geq \quad & -\delta + \frac{1}{1+\delta}\mathbb{E}^{\mathbb{P}}\Big[\int_0^1 L(s,X+Z,\nu_s^{\mathbb{P}})ds + \lambda_1(X_1+Z)\Big] \\
= \quad & -\delta + \frac{1}{1+\delta}\mathbb{E}^{\overline{\mathbb{P}}_\varepsilon}\Big[\int_0^1 L(s,X,\nu_s)ds + \lambda_1(X_1)\Big] \\
\geq \quad & -\delta + \frac{1}{1+\delta}\inf_{\tilde{\mathbb{P}} \in \mathcal{P}(\psi_\varepsilon * \mu_0)} \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\int_0^1 L(s,X,\nu_s^{\tilde{\mathbb{P}}})ds + \lambda_1(X_1)\Big],
\end{aligned}$$

where the last inequality follows from Proposition 3.1.

Notice that $\mu_1(\lambda_1^\varepsilon) = (1+\delta)^{-1}(\psi_\varepsilon * \mu_1)(\lambda_1)$ by Fubini's theorem. Then, by the arbitrariness of $\lambda_1 \in C_b(\mathbb{R}^d)$ and $\mathbb{P} \in \mathcal{P}(\mu_0)$, the last inequality implies (3.8). □

4 PDE characterization of the dual formulation

In the rest of the paper, we assume that

$$L(t,\mathbf{x},u) \quad = \quad \ell(t,\mathbf{x}(t),u),$$

where the deterministic function $\ell : (t, x, u) \in [0, 1] \times \mathbb{R}^d \times U \mapsto \ell(t, x, u) \in \mathbb{R}^+$ is non-negative and convex in u. Then, the function λ_0 in (3.2) is reduced to the value function of a standard Markovian stochastic control problem:

$$\lambda_0(x) \quad = \quad \inf_{\mathbb{P}\in\mathcal{P}(\delta_x)} \mathbb{E}^{\mathbb{P}}\Big[\int_0^1 \ell(s, X_s, \nu_s^{\mathbb{P}})ds + \lambda_1(X_1)\Big]. \tag{4.1}$$

Our main objective is to characterize λ_0 by means of the corresponding dynamic programming equations.

We consider the probability measures $\mathbb{P}$ on the canonical space $(\Omega, \mathcal{F}_1)$, under which the canonical process X is a semimartingale on $[t, 1]$, characterized by $\int_t^\cdot \nu_s^{\mathbb{P}} ds$ for some progressively measurable process $\nu^{\mathbb{P}}$. As discussed in Remark 2.2, $\nu^{\mathbb{P}}$ is unique in sense of $d\mathbb{P}\times dt-$a.e. To simplify the notation, we suppose that U contains the original point 0. Let

$$\mathcal{P}_{t,x} \quad := \quad \Big\{\mathbb{P} \in \mathcal{P} : \mathbb{P}[X_s = x,\ 0 \le s \le t] = 1\Big\}. \tag{4.2}$$

We notice that under probability $\mathbb{P} \in \mathcal{P}_{t,x}$, X is a semimartingale with $\nu^{\mathbb{P}} = 0$, $d\mathbb{P} \times dt-$a.e. on $\Omega \times [0, t]$. The dynamic value function is defined for any $\lambda_1 \in C_b(\mathbb{R}^d)$ by:

$$\lambda(t, x) \quad := \quad \inf_{\mathbb{P}\in\mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}}\Big[\int_t^1 \ell(s, X_s, \nu_s^{\mathbb{P}})ds + \lambda_1(X_1)\Big]. \tag{4.3}$$

As in the previous sections, we also introduce the corresponding probability measures on enlarged space $(\overline{\Omega}, \overline{\mathcal{F}}_1)$. For all $(t, x, a, b) \in [0, 1] \times \mathbb{R}^d \times \mathbb{R}^{d^2} \times \mathbb{R}^d$, let

$$\overline{\mathcal{P}}_{t,x,a,b} \quad := \quad \Big\{\overline{\mathbb{P}} \in \overline{\mathcal{P}} : \overline{\mathbb{P}}\big[(X_s, A_s, B_s) = (x, a, b), 0 \le s \le t\big] = 1\Big\}. \tag{4.4}$$

By similar arguments as in Proposition 3.1, we have under Assumption 3.1 that

$$\lambda(t, x) = \inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{t,x,a,b}} \mathbb{E}^{\overline{\mathbb{P}}}\left[\int_t^1 \ell(s, X_s, \nu_s)ds + \lambda_1(X_1)\right] \quad \text{for all} \quad (a, b) \in \mathbb{R}^{d^2} \times \mathbb{R}^d \tag{4.5}$$

4.1 PDE characterization of the dynamic value function

The first step is as usual to establish the dynamic programming principle (DPP). We observe that a weak dynamic programming principle as introduced in Bouchard and Touzi [13] suffices to prove that the dynamic value function λ is a viscosity solution of the corresponding dynamic programming equation. However, our context is slightly different from that of [13], and we will prove the standard dynamic programming principle.

For bounded controls set U and bounded cost functions, the DPP is shown (implicitly) in Haussmann [21]. El Karoui, Nguyen and JeanBlanc [18] considered a relaxed optimal control problem, and provided a scheme of proof without all details. Our approach is to show that the value function (4.3) coincides with the corresponding relaxation in the sense of [18], and to provide all details for their scheme of proof.

Proposition 4.1. *Let Assumptions 3.1, 3.2, 3.3 hold true, and assume further that λ is locally bounded. Then, for all $\overline{\mathbb{F}}-$stopping time τ with values in $[t,T]$, and all $(a,b) \in \mathbb{R}^{d^2+d}$:*

$$\lambda(t,x) = \inf_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{t,x,a,b}} \mathbb{E}^{\mathbb{P}}\Big[\int_t^\tau \ell(s,X_s,\nu_s)ds + \lambda(\tau, X_\tau)\Big].$$

The proof is reported in section 4.2. The dynamic programming equation is the infinitesimal version of the above dynamic programming principle. Let

$$H(t,x,p,\Gamma) \quad := \quad \inf_{(a,b)\in U}\Big[b\cdot p + \frac{1}{2}a\cdot\Gamma + \ell(t,x,a,b)\Big]. \tag{4.6}$$

We observe that H is continuous under Assumptions 3.1 and 3.3. Indeed, under these two assumptions, for every constant $r > 0$, there is a closed bounded domain $D_r \subset S_d \times \mathbb{R}^d$ such that every subgradient $\nabla_u \ell(t,x,u)$ satisfies $|\nabla_u \ell(t,x,u)| \geq r$, for all $(t,x,u) \in [0,T]\times\mathbb{R}^d\times D_r^c$. Therefore, for every (p,Γ) such that $|(p,\Gamma)| \leq r$, the infimum in (4.6) can be taken in the compact set $U \cap D_r$. This implies that H is a continuous function.

Theorem 4.1. *Let Assumptions 3.1, 3.2, 3.3 hold true, and assume further that λ is locally bounded. Then, λ is a viscosity solution of the dynamic programming equation*

$$-\partial_t\lambda(t,x) - H(t,x,D\lambda,D^2\lambda) \quad = \quad 0,$$

with terminal condition $\lambda(1,x) = \lambda_1(x)$.

The proof is very similar to that of Corollary 5.1 in [13], we report it in Appendix for completeness.

4.2 Proof of the dynamic programming principle

We first prove that the dynamic value function λ is measurable and we can choose " in a measurable way" a family of probabilities $(\mathbb{Q}_{t,x,a,b})_{(t,x,a,b)\in[0,1]\times\mathbb{R}^{2d+d^2}}$ which achieves (or achieves with ε error) the infimum in (4.5).

There are many versions of the measurable selection theorem in the literature, see e.g. Section 12.1 of Stroock and Varadhan [34], Chapter 7 of Bertsekas and Shreve [6], and Chapiter 3 of Dellacherie and Meyer [15]. In our context, we find it convenient to use a result from El Karoui and TAN [19].

Let λ^* be the upper semicontinuous envelope of the function λ, and

$$\tilde{\mathcal{P}}_{t,x,a,b} := \Big\{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{t,x,a,b} \;:\; \mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_t^1 \ell(s,X_s,\nu_s)ds + \lambda_1(X_1)\Big] \leq \lambda^*(t,x)\Big\},$$

$$\tilde{\mathcal{P}} := \big\{(t,x,a,b,\overline{\mathbb{P}}) \;:\; \overline{\mathbb{P}} \in \tilde{\mathcal{P}}_{t,x,a,b}\big\}.$$

In the following statement, for the Borel $\sigma-$field $\mathcal{B}([0,1]\times\mathbb{R}^{2d+d^2})$ of $[0,1]\times\mathbb{R}^{2d+d^2}$ with an arbitrary probability measure μ on it, we denote by $\mathcal{B}^\mu([0,1]\times\mathbb{R}^{2d+d^2})$ its $\sigma-$field completed by μ.

Lemma 4.1. *Let Assumptions 3.1, 3.2, 3.3 hold true, and assume that λ is locally bounded. Then, for any probability measure μ on $\Big([0,1]\times\mathbb{R}^{2d+d^2},\mathcal{B}([0,1]\times\mathbb{R}^{2d+d^2})\Big)$:*
(i) *the function $(t,x,a,b)\mapsto\lambda(t,x)$ is $\mathcal{B}^\mu([0,1]\times\mathbb{R}^{2d+d^2}))-$measurable,*
(ii) *for any $\varepsilon>0$, there is a family of probability $(\bar{\mathbb{Q}}^\varepsilon_{t,x,a,b})_{(t,x,a,b)\in[0,1]\times\mathbb{R}^{2d+d^2}}$ in $\tilde{\mathcal{P}}$ such that $(t,x,a,b)\mapsto\bar{\mathbb{Q}}^\varepsilon_{t,x,a,b}$ is a measurable map from $[0,1]\times\mathbb{R}^{2d+d^2}$ to $\mathbf{M}(\overline{\Omega})$ and*

$$\mathbb{E}^{\bar{\mathbb{Q}}^\varepsilon_{t,x,a,b}}\Big[\int_t^1\ell(s,X_s,\nu_s)ds+\lambda_1(X_1)\Big]\;\;\le\;\;\lambda(t,x)+\varepsilon,\quad\mu-a.s.$$

Proof. By Lemma 3.2, the map $\overline{\mathbb{P}}\mapsto\mathbb{E}^{\overline{\mathbb{P}}}\big[\int_t^1\ell(s,X_s,\nu_s)ds+\lambda_1(X_1)\big]$ is lower semicontinuous, and therefore measurable. Moreover $\tilde{\mathcal{P}}_{t,x,a,b}$ is non empty for every $(t,x,a,b)\in[0,1]\times\mathbb{R}^{2d+d^2}$. Finally, by using the same arguments as in the proof of Lemma 3.3, we see that $\tilde{\mathcal{P}}$ is a closed subset of $[0,1]\times\mathbb{R}^{2d+d^2}\times\mathbf{M}(\overline{\Omega})$. Then, both items of the lemma follow from Corollary 1.20 in El Karoui and TAN [19]. □

We next prove the stability properties of probability measures under conditioning and concatenations at stopping times, which will be the key-ingredients for the proof of the dynamic programming principle.

We first recall some results from Stroock and Varadhan [34] and define some notations.

- For $0\le t\le 1$, let $\overline{\mathcal{F}}_{t,1}:=\sigma((X_s,A_s,B_s):t\le s\le 1)$, and let $\overline{\mathbb{P}}$ be a probability measure on $(\overline{\Omega},\overline{\mathcal{F}}_{t,1})$ with $\overline{\mathbb{P}}((X_t,A_t,B_t)=\eta_t)=1$ for some $\eta\in C([0,t],\mathbb{R}^{2d+d^2})$. Then, there is a unique probability measure $\delta_\eta\otimes_t\overline{\mathbb{P}}$ on $(\overline{\Omega},\overline{\mathcal{F}}_1)$ such that $\delta_\eta\otimes_t\overline{\mathbb{P}}[(X_s,A_s,B_s)=\eta_s,0\le s\le t]=1$ and $\delta_\eta\otimes_t\overline{\mathbb{P}}[A]=\overline{\mathbb{P}}[A]$ for all $A\in\overline{\mathcal{F}}_{t,1}$. In addition, if $\overline{\mathbb{P}}$ is also a probability measure on $(\overline{\Omega},\overline{\mathcal{F}}_1)$, under which a process M defined on $\overline{\Omega}$ is a $\overline{\mathbb{F}}-$martingale after time t, then M is still a $\overline{\mathbb{F}}-$martingale after time t in probability space $(\overline{\Omega},\overline{\mathcal{F}}_1,\eta\otimes_t\overline{\mathbb{P}})$. In particular, for $t\in[0,1]$, a constant $c_0\in\mathbb{R}^{2d+d^2}$, and $\overline{\mathbb{P}}$ satisfying $\overline{\mathbb{P}}((X_t,A_t,B_t)=c_0)=1$, we denote $\delta_{c_0}\otimes_t\overline{\mathbb{P}}:=\delta_{\eta^{c_0}}\otimes_t\overline{\mathbb{P}}$, where $\eta^{c_0}_s=c_0,s\in[0,t]$.
- Let $\bar{\mathbb{Q}}$ be a probability measure on $(\overline{\Omega},\overline{\mathcal{F}}_1)$ and τ a $\overline{\mathbb{F}}-$stopping time. Then, there is a family of measures $(\bar{\mathbb{Q}}_\omega)_{\omega\in\overline{\Omega}}$ such that $\bar{\mathbb{Q}}_\omega((X_t,A_t,B_t)=\omega_t:t\le\tau(\omega))=1$. This is Theorem 1.3.4 of [34], and $(\bar{\mathbb{Q}}_\omega)_{\omega\in\overline{\Omega}}$ is called the regular conditional probability distribution (r.c.p.d.)

Lemma 4.2. *Let $\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{t,x,a,b}$, τ an $\overline{\mathbb{F}}-$stopping time taking value in $[t,1]$, and $(\bar{\mathbb{Q}}_\omega)_{\omega\in\overline{\Omega}}$ be a r.c.p.d. of $\overline{\mathbb{P}}|\overline{\mathcal{F}}_\tau$. Then there is a $\overline{\mathbb{P}}-$null set $N\in\overline{\mathcal{F}}_\tau$ such that $\delta_{\omega_{\tau(\omega)}}\otimes_{\tau(\omega)}\bar{\mathbb{Q}}_\omega\in\overline{\mathcal{P}}_{\tau(\omega),\omega_{\tau(\omega)}}$ for all $\omega\notin N$.*

Proof. Since $\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{t,x,a,b}$, it follows from Theorem II.2.21 of Jacod and Shiryaev [24] that

$$(X_s-B_s)_{t\le s\le 1},\quad\big((X_s-B_s)^2-A_s\big)_{t\le s\le 1}$$

are all local martingales after time t. Then it follows from Theorem 1.2.10 of Stroock and Varadhan [34] together with a localization technique that there is a $\overline{\mathbb{P}}-$null set $N_1\in\overline{\mathcal{F}}_\tau$ such that they are still local martingales after time $\tau(\omega)$ both under $\bar{\mathbb{Q}}_\omega$ and $\delta_{\omega_{\tau(\omega)}}\otimes_{\tau(\omega)}\bar{\mathbb{Q}}_\omega$, for all $\omega\notin N_1$. It is clear, moreover, that $\nu\in U$, $d\bar{\mathbb{Q}}_\omega\times dt-$a.e.

on $\overline{\Omega} \times [\tau(\omega), 1]$ for $\overline{\mathbb{P}}-$a.e. $\omega \in \overline{\Omega}$. Then there is $\overline{\mathbb{P}}-$null set $N \in \overline{\mathcal{F}}_\tau$ such that $\delta_{\omega_{\tau(\omega)}} \otimes_{\tau(\omega)} \bar{\mathbb{Q}}_\omega \in \overline{\mathcal{P}}_{\tau(\omega),\omega_{\tau(\omega)}}$ for every $\omega \notin N$. □

Lemma 4.3. *Let Assumptions 3.1, 3.2, 3.3 hold true, and assume that λ is locally bounded. Let $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{t,x,a,b}$, $\tau \geq t$ a $\overline{\mathbb{F}}-$stopping time, and $(\bar{\mathbb{Q}}_\omega)_{\omega\in\overline{\Omega}}$ a family of probability measures such that $\bar{\mathbb{Q}}_\omega \in \overline{\mathcal{P}}_{\tau(\omega),\omega_{\tau(\omega)}}$ and $\omega \mapsto \bar{\mathbb{Q}}_\omega$ is $\overline{\mathcal{F}}_\tau-$measurable. Then there is a unique probability measure, denoted by $\overline{\mathbb{P}} \otimes_{\tau(\cdot)} \bar{\mathbb{Q}}_\cdot$, in $\overline{\mathcal{P}}_{t,x,a,b}$, such that*

$$\overline{\mathbb{P}} \otimes_{\tau(\cdot)} \bar{\mathbb{Q}}_\cdot = \overline{\mathbb{P}} \quad \text{on} \quad \overline{\mathcal{F}}_\tau, \quad \text{and } (\delta_\omega \otimes_{\tau(\omega)} \bar{\mathbb{Q}}_\omega)_{\omega\in\overline{\Omega}} \text{ is a r.c.p.d. of } \overline{\mathbb{P}} \otimes_{\tau(\cdot)} \bar{\mathbb{Q}}_\cdot|\overline{\mathcal{F}}_\tau \tag{4.7}$$

Proof. The existence and uniqueness of the probability measure $\overline{\mathbb{P}}\otimes_{\tau(\cdot)}\bar{\mathbb{Q}}_\cdot$ on $(\overline{\Omega}, \overline{\mathcal{F}}_1)$, satisfying (4.7), follows from Theorem 6.1.2 of [34]. It remains to prove that $\overline{\mathbb{P}}\otimes_{\tau(\cdot)}\bar{\mathbb{Q}}_\cdot \in \overline{\mathcal{P}}_{t,x,a,b}$.

Since $\bar{\mathbb{Q}}_\omega \in \overline{\mathcal{P}}_{\tau(\omega),\omega_{\tau(\omega)}}$, X is a $\delta_\omega \otimes_{\tau(\omega)} \bar{\mathbb{Q}}_\omega-$semimartingale after time $\tau(\omega)$, characterized by (A, B). Then, the processes $X - B$ and $(X - B)^2 - A$ are local martingales under $\delta_\omega \otimes_{\tau(\omega)} \bar{\mathbb{Q}}_\omega$ after time $\tau(\omega)$. By Theorem 1.2.10 of [34] together with a localization argument, they are still local martingales under $\overline{\mathbb{P}} \otimes_{\tau(\cdot)} \bar{\mathbb{Q}}_\cdot$. Hence, the required result follows from Theorem II.2.21 of [24]. □

We have now collected all the ingredients for the proof of the dynamic programming principle.

Proof of Proposition 4.1 Let τ be an $\overline{\mathbb{F}}-$stopping time taking value in $[t, 1]$. We proceed in two steps.

1. For $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{t,x,a,b}$, we denote by $(\bar{\mathbb{Q}}_\omega)_{\omega\in\overline{\Omega}}$ a r.c.p.d. of $\overline{\mathbb{P}}|\mathcal{F}_\tau$, and $\overline{\mathbb{P}}^\omega_\tau := \delta_{\omega_{\tau(\omega)}} \otimes_{\tau(\omega)} \bar{\mathbb{Q}}_\omega$. By the representation (4.5) of λ, together with the tower property of conditional expectations, we see that

$$\begin{aligned}
\lambda(t,x) &= \inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{t,x,a,b}} \mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_t^\tau \ell(s,X_s,\nu_s)ds + \int_\tau^1 \ell(s,X_s,\nu_s)ds + \lambda_1(X_1)\Big] \\
&= \inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{t,x,a,b}} \mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_t^\tau \ell(s,X_s,\nu_s)ds + \mathbb{E}^{\overline{\mathbb{P}}^\omega_\tau}\Big\{\int_\tau^1 \ell(s,X_s,\nu_s)ds + \lambda_1(X_1)\Big\}\Big] \\
&\geq \inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{t,x,a,b}} \mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_t^\tau \ell(s,X_s,\nu_s)ds + \lambda(\tau,X_\tau)\Big],
\end{aligned} \tag{4.8}$$

where the last inequality follows from the fact that $\overline{\mathbb{P}}^\omega_\tau \in \overline{\mathcal{P}}_{\tau(\omega),\omega_{\tau(\omega)}}$ by Lemma 4.2.

2. For $\varepsilon > 0$, let $(\bar{\mathbb{Q}}^\varepsilon_{t,x,a,b})_{[0,1]\times\mathbb{R}^{2d+d^2}}$ be the family defined in Lemma 4.1, and denote $\bar{\mathbb{Q}}^\varepsilon_\omega := \bar{\mathbb{Q}}^\varepsilon_{\tau(\omega),\omega_{\tau(\omega)}}$. Then $\omega \mapsto \bar{\mathbb{Q}}^\varepsilon_\omega$ is $\overline{\mathcal{F}}_\tau-$measurable. Moreover, for all $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{t,x,a,b}$, we may construct by Lemmas 4.1 and 4.3 $\overline{\mathbb{P}} \otimes_{\tau(\cdot)} \bar{\mathbb{Q}}_\cdot \in \overline{\mathcal{P}}_{t,x,a,b}$ such that

$$\mathbb{E}^{\overline{\mathbb{P}}\otimes_{\tau(\cdot)}\bar{\mathbb{Q}}_\cdot}\left[\int_t^T \ell(s,X_s,\nu_s)ds + \lambda_1(X_1)\right] \leq \mathbb{E}^{\overline{\mathbb{P}}}\left[\int_t^\tau \ell(s,X_s,\nu_s)ds + \lambda(\tau,X_\tau)\right] + \varepsilon.$$

By the arbitrariness of $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{t,x,a,b}$ and $\varepsilon > 0$, together with the representation (4.5) of λ, this implies that the reverse inequality to (4.8) holds true, and the proof is complete. □

We conclude this section by the

Proof of Lemma 3.1 By the same arguments as in Lemma 4.1, we can easily deduce that λ_0 is $\mathcal{B}^{\mu_0}(\mathbb{R}^d)-$ measurable, and we just need to prove that

$$\mu_0(\lambda_0) = \inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}(\mu_0)} \mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_0^1 \ell(s,X_s,\nu_s)ds + \lambda_1(X_1)\Big].$$

Given a probability measure $\overline{\mathbb{P}} \in \overline{\mathcal{P}}(\mu_0)$, we can get a family of conditional probabilities $(\bar{\mathbb{Q}}_\omega)_{\omega\in\Omega}$ such that $\bar{\mathbb{Q}}_\omega \in \overline{\mathcal{P}}_{0,\omega_0}$, which implies that

$$\mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_0^1 \ell(s,X_s,\nu_s)ds + \lambda_1(X_1)\Big] \geq \mu_0(\lambda_0), \quad \forall \overline{\mathbb{P}} \in \overline{\mathcal{P}}(\mu_0).$$

On the other hand, for every $\varepsilon > 0$ and $\mu_0 \in \mathbf{M}(\mathbb{R}^d)$, we can select a measurable family of $(\bar{\mathbb{Q}}^\varepsilon_x \in \overline{\mathcal{P}}_{0,x,0,0})_{x\in\mathbb{R}^d}$ such that

$$\mathbb{E}^{\bar{\mathbb{Q}}^\varepsilon_x}\Big[\int_0^1 \ell(s,X_s,\nu_s)ds + \lambda_1(X_1)\Big] \leq \lambda_0(x) + \varepsilon,\ \ \mu_0 - \text{a.s.}$$

and then construct a probability measure $\mu_0 \otimes_0 \bar{\mathbb{Q}}^\varepsilon_\cdot \in \overline{\mathcal{P}}(\mu_0)$ by concatenation such that

$$\mathbb{E}^{\mu_0\otimes_0\bar{\mathbb{Q}}^\varepsilon_\cdot}\Big[\int_0^1 \ell(s,X_s,\nu_s)ds + \lambda_1(X_1)\Big] \leq \mu_0(\lambda_0) + \varepsilon,\ \ \forall\varepsilon > 0,$$

which completes the proof. □

5 Numerical approximation

In this section, we provide an implementable numerical scheme for the approximation of the value function $V(\mu_0,\mu_1)$ in the Markovian context where $L(t,\mathbf{x},u) = \ell(t,\mathbf{x}(t),u)$, under Assumptions 3.1, 3.2, 3.3, and 3.4. By our duality result of Theorem 3.1 together with Theorem 3.2, we have that

$$V = \mathcal{V} := \sup_{\lambda_1\in C_b(\mathbb{R}^d)} v(\lambda_1) = \overline{\mathcal{V}} := \sup_{\lambda_1\in C_b^\infty(\mathbb{R}^d)} v(\lambda_1) \quad \text{where} \quad v(\lambda_1) := \mu_0(\lambda_0) - \mu_1(\lambda_1)$$

and the function λ_0 is defined in (3.2). We shall require the following additional conditions to hold.

Assumption 5.1. $\int_{\mathbb{R}^d} |x|(\mu_0+\mu_1)(dx) < \infty$.

Assumption 5.2. *U is compact, and ℓ is Lipschitz in x uniformly in (t,u).*

Throughout this section, we denote:

$$M \ \ := \sup_{(t,x,u)\in[0,1]\times\mathbb{R}^d\times U} |u| + |\ell(t,0,u)| + |\nabla_x \ell(t,x,u)|.$$

where $\nabla_x\ell(t,x,u)$ is the gradient of ℓ with respect to x which exists a.e. under Assumption 5.2.

5.1 First approximations

Let Lip^0_K denote the collection of all bounded $K-$Lipschitz-continuous functions $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}$ with $\phi(0) = 0$, and denote $\mathrm{Lip}^0 := \cup_{K>0}\mathrm{Lip}^0_K$. Since $v(\lambda_1 + c) = v(\lambda_1)$ for any $\lambda_1 \in C_b(\mathbb{R}^d)$ and $c \in \mathbb{R}$, we deduce that:

$$V \;=\; \sup_{\lambda_1 \in \mathrm{Lip}^0} v(\lambda_1).$$

As a first approximation, we introduce the function:

$$V^K \;:=\; \sup_{\lambda_1 \in \mathrm{Lip}^0_K} v(\lambda_1). \tag{5.1}$$

Under Assumptions 5.1 and 5.2, we easily verify that $V < \infty$, see Lemma 5.1 below. Then, it is immediate that:

$$(V^K)_{K>0} \text{ is increasing and } V^K \longrightarrow V \quad \text{as} \quad K \to \infty. \tag{5.2}$$

Our next approximation restricts the space variable x to the bounded subsets $O_R := (-R, R)^d$ of $\mathbb{R}^d$, $R > 0$. Let τ_R be the first exit time of the canonical process X from O_R:

$$\tau_R \;:=\; \inf\{t : X_t \notin O_R\},$$

and define for all bounded functions $\lambda_1 \in C_b(\mathbb{R}^d)$:

$$\lambda^R(t,x) \;:=\; \inf_{\mathbb{P}\in\mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}}\Big[\int_t^{\tau_R\wedge 1} \ell(s, X_s, \nu^{\mathbb{P}}_s)ds + \lambda_1(X_{\tau_R\wedge 1})\Big].$$

By similar arguments as in Theorem 4.1, λ^R is a viscosity solution of equation

$$-\partial_t\lambda^R(t,x) - H(t,x,D\lambda^R, D^2\lambda^R) = 0, \quad (t,x) \in [0,1) \times O_R, \tag{5.3}$$

with boundary conditions

$$\lambda^R(t,x) = \lambda_1(x) \quad \text{for all} \quad (t,x) \in ([0,1) \times \partial O_R) \cup (\{1\} \times O_R). \tag{5.4}$$

Here ∂O_R denotes the boundary of O_R. Moreover, from discussions in Example 3.6 of Crandall et al. [14], it satisfies a comparison result. Then λ^R is the unique bounded viscosity solution of (5.3) with boundary condition (5.4).

Lemma 5.1. *Under Assumption 5.2, let $\lambda_1 \in Lip^0_K$ be arbitrary. Then λ and λ^R are Lipschitz-continuous, and there is a constant C depending on M such that:*

$$|\lambda(t,0)| + |\lambda^R(t,0)| + |\nabla_x\lambda(t,x)| + |\nabla_x\lambda^R(t,x)| \le C(1+K), \quad (t,x) \in [0,1] \times \mathbb{R}^d.$$

Proof. We only provide the estimates for λ, those for λ_R follows from the same arguments. First, by Assumption 5.2 together with the fact that λ_1 is $K-$Lipschitz and $\lambda_1(0) = 0$, for every $\mathbb{P} \in \mathcal{P}_{t,0}$,

$$\mathbb{E}^{\mathbb{P}}\Big[\int_t^1 \ell(s, X_s, \nu^{\mathbb{P}}_s)ds + \lambda_1(X_1)\Big] \;\le\; M + (M+K)\sup_{t\le s\le 1}\mathbb{E}^{\mathbb{P}}\big|X_s\big|.$$

Recall that X is continuous semimartingale under $\mathbb{P}$ whose finite variation part and quadratic variation of the martingale part are both bounded by constant M. Separating the two parts and using Cauchy-Schwarz's inequality, it follows that $\mathbb{E}^{\mathbb{P}}\big|X_s\big| \le M + \sqrt{M},\ \forall t \le s \le 1$, and then $|\lambda(t,0)| \le M + (M+K)(M+\sqrt{M})$.

We next prove that λ is Lipschitz and provide the corresponding estimate. Observe that $\mathcal{P}_{t,y} = \big\{\mathbb{P} := \tilde{\mathbb{P}} \circ (X+y-x)^{-1}\ :\ \tilde{\mathbb{P}} \in \mathcal{P}_{t,x}\big\}$. Then

$$\begin{aligned}
& |\lambda(t,x) - \lambda(t,y)| \\
\le\ & \sup_{\mathbb{P}\in\mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}}\Big| \int_t^1 \ell(s,X_s,\nu_s^{\mathbb{P}}) - \ell(s,X_s+y-x,\nu_s^{\mathbb{P}})ds + \lambda_1(X_1) - \lambda_1(X_1+y-x)\Big| \\
\le\ & (M+K)|y-x|
\end{aligned}$$

by the Lipschitz property of ℓ and λ in x. □

Define

$$\lambda_0^R := \lambda^R(0,.),\ v^R(\lambda_1) := \mu_0(\lambda_0^R \mathbf{1}_{O_R}) - \mu_1(\lambda_1 \mathbf{1}_{O_R}),\ \text{and } V^{K,R} := \sup_{\lambda_1 \in \mathrm{Lip}_0^K} v^R(\lambda_1). \quad (5.5)$$

In the special case where U is a singleton, equation (5.3) degenerates to the heat equation, Barles, Daher and Romano [2] proved that the error $\lambda - \lambda^R$ satisfies a large deviation estimate as $R \to \infty$. The next result extends this estimate to our context.

Proposition 5.1. *Let Assumption 5.2 and 5.1 hold true, we denote $|x| := \max_{i=1}^d |x_i|$ for $x \in \mathbb{R}^d$ and choose $R > 2M$. Then, there is a constant C such that:*
(i) *for all $K > 0$, $\lambda_1 \in Lip_0^K$ and $|x| \le R - M$,*

$$|\lambda^R - \lambda|(t,x) \le C(1+K)e^{-(R-M-|x|)^2/2M},$$

(ii) *for all $K > 0$:*

$$\big|V^{K,R} - V^K\big| \ \le\ C(1+K)\Big(e^{-R^2/8M+R/2} + \int_{O_{R/2}^c} (1+|x|)(\mu_0+\mu_1)(dx)\Big). \quad (5.6)$$

Proof. 1. For arbitrary $(t,x) \in [0,1]\times\mathbb{R}^d$ and $\mathbb{P} \in \mathcal{P}_{t,x}$, we denote $Y^i := \sup_{0\le s\le 1}\big|X_s^i\big|$ where X^i is the $i-$th component of the canonical process X. By the Dubins-Schwartz time-change theorem (see e.g. Theorem 4.6, Chapter 3 of Karatzas and Shreve [25]), we may represent the continuous local martingale part of X^i as a time-changed Brownian motion W. Since the characteristics of X are bounded by M, we see that:

$$\begin{aligned}
S^i(R) \ :=\ \mathbb{P}[Y^i \ge R] \ &\le\ \mathbb{P}\Big[\sup_{0\le t\le M} |W_t| \ge R - |x_i| - M\Big] \\
&\le\ 2\mathbb{P}\Big[\sup_{0\le t\le M} W_t \ge R - |x_i| - M\Big] \\
&=\ 4\Big(1 - \mathbf{N}\big(R_{|x_i|}^M\big)\Big), \quad (5.7)
\end{aligned}$$

where $R_{|x_i|}^M := (R - M - |x_i|)/\sqrt{M}$, $\mathbf{N}$ be the cumulative distribution function of the standard normal distribution $N(0,1)$, and the last equality follows from the reflection principle of the Brownian motion. Then by integration by parts as well as (5.7),

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Big[Y^i \mathbf{1}_{Y^i\geq R}\Big] &= RS^i(R) + \int_R^\infty S^i(z)dz. \\
&\leq 4\int_R^\infty \frac{1}{\sqrt{M}}\frac{1}{\sqrt{2\pi}}\exp\Big(\frac{(z-M-|x_i|)^2}{2M}\Big)z\ dz \\
&= 4(|x_i|+M)\Big(1-\mathbf{N}\big(R^M_{|x_i|}\big)\Big) + \frac{4\sqrt{M}}{\sqrt{2\pi}}\exp\Big(-\frac{(R^M_{|x_i|})^2}{2}\Big).
\end{aligned}$$

We further remark that for any $R>0$,

$$(1-\mathbf{N}(R)) = \int_R^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt \leq \frac{1}{R}\int_R^\infty \frac{1}{\sqrt{2\pi}}te^{-\frac{t^2}{2}}dt = \frac{1}{\sqrt{2\pi}}\frac{1}{R}e^{\frac{-R^2}{2}}.$$

2. By definitions of λ, λ^R, it follows that for all (t,x) such that $|x|\leq R-M$,

$$\begin{aligned}
|\lambda-\lambda^R|(t,x) &\leq \sup_{\mathbb{P}\in\mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}}\Big[\int_{\tau_R\wedge 1}^1 |\ell(s,X_s,\nu_s^{\mathbb{P}})|ds + \big|\lambda_1(X_{\tau_R\wedge 1})-\lambda_1(X_1)\big|\Big] \\
&\leq \sup_{\mathbb{P}\in\mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}}\Big[\Big(M+\sqrt{d}KR+(M+K)\sup_{t\leq s\leq 1}|X_s|\Big)\mathbf{1}_{\tau_R<1}\Big] \\
&\leq \sup_{\mathbb{P}\in\mathcal{P}_{t,x}} \mathbb{E}^{\mathbb{P}}\Big[\sum_{i=1}^d \Big(M+\sqrt{d}KR+\sqrt{d}(M+K)Y_i\Big)\mathbf{1}_{Y_i\geq R}\Big] \\
&\leq C(1+K)e^{-(R^M_{|x|})^2/2}, \qquad (5.8)
\end{aligned}$$

for some constant C depending on M and d. This completes the proof of (i).
3. To prove item (ii) of the proposition, we start with:

$$\begin{aligned}
\big|V^{K,R}-V^K\big| &= \Big|\sup_{\lambda_1\in\mathrm{Lip}_K^0}\big\{\mu_0(\lambda_0^R\mathbf{1}_{O_R})-\mu_1(\lambda_1\mathbf{1}_{O_R})\big\} - \sup_{\lambda_1\in\mathrm{Lip}_K^0}\big\{\mu_0(\lambda_0)-\mu_1(\lambda_1)\big\}\Big| \\
&\leq \sup_{\lambda_1\in\mathrm{Lip}_K^0}\big|\mu_0(\lambda_0^R\mathbf{1}_{O_R})-\mu_0(\lambda_0)\big| + K\int_{O_R^c}|x|\mu_1(dx).
\end{aligned}$$

Now for all $\lambda_1\in\mathrm{Lip}_K^0$, we estimate that:

$$\begin{aligned}
\big|\mu_0(\lambda_0^R\mathbf{1}_{O_R})-\mu_0(\lambda_0)\big| &\leq \mu_0\Big(\big|\lambda_0^R-\lambda_0\big|\mathbf{1}_{O_{\frac{R}{2}}}\Big) + \mu_0\Big(\big(\big|\lambda_0^R\big|+\big|\lambda_0\big|\big)\mathbf{1}_{(O_{\frac{R}{2}})^c}\Big) \\
&\leq C(1+K)\Big(\int_{O_{\frac{R}{2}}} e^{-(R^M_{|x|})^2/2}\mu_0(dx) + \int_{(O_{\frac{R}{2}})^c}(1+|x|)\mu_0(dx)\Big),
\end{aligned}$$

where we used Lemma 5.1 together with item (i) of the present proposition. Observing that $(R^M_{|x|})^2 \geq R^2/4M - R + M$ on $O_{\frac{R}{2}}$, this implies that

$$\big|\mu_0(\lambda_0^R\mathbf{1}_{O_R})-\mu_0(\lambda_0)\big| \leq C(1+K)\Big(e^{-R^2/8M+R/2} + \int_{(O_{\frac{R}{2}})^c}(1+|x|)\mu_0(dx)\Big)\Big),$$

and the required estimate follows. □

5.2 A finite differences approximation

In the remaining part of this paper, we restrict the discussion to the one-dimensional case

$$d = 1 \quad \text{so that} \quad O_R = (-R, R).$$

Let $(l, r) \in \mathbb{N}^2$ and $h = (\Delta t, \Delta x) \in (\mathbb{R}^+)^2$ be such that $l\Delta t = 1$ and $r\Delta x = R$. Denote $x_i := i\Delta x$, $t_k := k\Delta t$ and define the discrete grids:

$$\mathcal{N} := \{x_i \; : \; i \in \mathbb{Z}\}, \qquad \mathcal{N}_R := \mathcal{N} \cap (-R, R),$$

$$\mathcal{M}_{T,R} := \big\{(t_k, x_i) \; : \; (k, i) \in \mathbb{Z}^+ \times \mathbb{Z}\big\} \cap ([0, 1] \times (-R, R)).$$

The terminal set, boundary set as well as interior set of $\mathcal{M}_{T,R}$ are denoted by

$$\partial_T \mathcal{M}_{T,R} := \big\{(1, x_i) \; : \; x_i \in \mathcal{N}_R\big\}, \; \partial_R \mathcal{M}_{T,R} := \big\{(t_k, \pm R) \; : \; k = 0, \cdots, l\big\},$$

$$\mathring{\mathcal{M}}_{T,R} := \mathcal{M}_{T,R} \setminus (\partial_T \mathcal{M}_{T,R} \cup \partial_R \mathcal{M}_{T,R}).$$

We shall use the finite differences method to solve the dynamic programming equation (5.3), (5.4) on the grid $\mathcal{M}_{T,R}$. For a function w defined on $\mathcal{M}_{T,R}$, we introduce the discrete derivatives of w:

$$D^{\pm} w(t_k, x_i) := \frac{w(t_k, x_{i\pm1}) - w(t_k, x_i)}{\Delta x} \quad \text{and} \quad (bD)w := b^+ D^+ w + b^- D^- w \text{ for } b \in \mathbb{R},$$

where $b^+ := \max(0, b)$, $b^- := \max(0, -b)$; and

$$D^2 w(t_k, x_i) := \frac{w(t_k, x_{i+1}) - 2w(t_k, x_i) + w(t_k, x_{i-1})}{\Delta x^2}.$$

We now define the function $\hat{\lambda}^{h,R}$(or $\hat{\lambda}^{h,R,\hat{\lambda}_1}$ to precise its dependence on the boundary condition $\hat{\lambda}_1$) on the grid $\mathcal{M}_{T,R}$ by the following explicit finite differences approximation of the dynamic programming equation (5.3):

$$\begin{aligned} \hat{\lambda}^{h,R}(t_k, x_i) &= \Big(\hat{\lambda}^{h,R} + \Delta t \inf_{u=(a,b)\in U} \Big\{\ell(\cdot, u) + (bD)\hat{\lambda}^{h,R} + \frac{1}{2} a D^2 \hat{\lambda}^{h,R}\Big\}\Big)(t_{k+1}, x_i) \;\; \text{on } \mathring{\mathcal{M}}_{T,R} \\ \hat{\lambda}^{h,R}(t_k, x_i) &= \hat{\lambda}_1(x_i) \;\; \text{on } \partial_T \mathcal{M}_{T,R} \cup \partial_R \mathcal{M}_{T,R}, \end{aligned} \tag{5.9}$$

and we introduce the following natural approximation of v^R:

$$\hat{v}_h^R(\hat{\lambda}_1) := \mu_0\Big(\text{lin}^R[\hat{\lambda}_0^{h,R}]\Big) - \mu_1\Big(\text{lin}^R[\hat{\lambda}_1]\Big) \quad \text{where} \quad \hat{\lambda}_0^{h,R} := \hat{\lambda}^{h,R}(0, \cdot), \tag{5.10}$$

and for all function ϕ defined on the grid $\mathcal{N}_R$ we denote by $\text{lin}^R[\phi]$ the linear interpolation of ϕ extended by zero outside $[-R, R]$.

We shall also assume that the discretization parameters $h = (\Delta t, \Delta x)$ satisfy the CFL condition

$$\Delta t \left(\frac{|b|}{\Delta x} + \frac{|a|}{\Delta x^2}\right) \le 1 \quad \text{for all} \quad (a, b) \in U. \tag{5.11}$$

Then the scheme (5.9) is $L^\infty-$monotone, so that the convergence of the scheme is guaranteed by the monotonic scheme method of Barles and Souganidis [4]. For our next result, we assume that the following error estimate holds.

Assumption 5.3. *There are positive constants $L_{K,R}$, ρ_1, ρ_2 which are independent of $h = (\Delta t, \Delta x)$, such that*

$$\mu_0\Big(\big|\mathrm{lin}^R[\hat{\lambda}_0^{h,R}] - \lambda_0 \mathbf{1}_{[-R,R]}\big|\Big) \le L_{K,R}(\Delta t^{\rho_1} + \Delta x^{\rho_2}), \forall \lambda_1 \in Lip_0^K \text{ and } \hat{\lambda}_1 = \lambda_1|_{\mathcal{N}_R}.$$

Let $\mathrm{Lip}_0^{K,R}$ be the collection of all functions on the grid $\mathcal{N}_R$ defined as restrictions of functions in Lip_0^K:

$$\mathrm{Lip}_0^{K,R} \quad := \quad \big\{\hat{\lambda}_1 := \lambda_1|_{\mathcal{N}_R} \text{ for some } \lambda_1 \in \mathrm{Lip}_0^K\big\}. \tag{5.12}$$

The above approximation of the dynamic value function λ suggests the following natural approximation of the minimal transportation cost value:

$$V_h^{K,R} \quad := \quad \sup_{\hat{\lambda}_1 \in \mathrm{Lip}_0^{K,R}} \hat{v}_h^R(\hat{\lambda}_1) = \sup_{\hat{\lambda}_1 \in \mathrm{Lip}_0^{K,R}} \mu_0\Big(\mathrm{lin}^R[\hat{\lambda}_0^{h,R}]\Big) - \mu_1\Big(\mathrm{lin}^R[\hat{\lambda}_1]\Big). \tag{5.13}$$

Remark 5.1. *Under the additional condition that ℓ is uniformly $\frac{1}{2}-$Hölder in t with constant M, then in spirit of the analysis in Barles and Jakobsen [3], Assumption 5.3 holds true with $\rho_1 = \frac{1}{4}$, $\rho_2 = \frac{1}{10}$ and $L_{K,R} = C(1 + K + KR)$ with some constant C depending on M.*

Theorem 5.1. *Let Assumption 5.3 be true, then with the constants $L_{K,R}$, ρ_1, ρ_2 introduced in Assumption 5.3, we have*

$$\big|V_h^{K,R} - V^{K,R}\big| \quad \le \quad L_{K,R}(\Delta t^{\rho_1} + \Delta x^{\rho_2}) + K\Delta x.$$

Proof. First, given $\lambda_1 \in \mathrm{Lip}_0^K$, we take $\hat{\lambda}_1 := \lambda_1|_{\mathcal{N}_R} \in \mathrm{Lip}_0^{K,R}$, then clearly $|\mathrm{lin}^R[\hat{\lambda}_1] - \lambda_1|_{L^\infty([-R,R])} \le K\Delta x$, and it follows from Assumption 5.3 and (5.5) as well as (5.10) that $v^R(\lambda_1) \le \hat{v}_h^R(\hat{\lambda}_1) + L_{K,R}(\Delta t^{\rho_1} + \Delta x^{\rho_2}) + K\Delta x$. Hence,

$$V^{K,R} \le V_h^{K,R} + L_{K,R}(\Delta t^{\rho_1} + \Delta x^{\rho_2}) + K\Delta x.$$

Next, given $\hat{\lambda} \in \mathrm{Lip}_0^{K,R}$, let $\lambda_1 := \mathrm{lin}[\hat{\lambda}_1] \in \mathrm{Lip}_0^K$ be the linear interpolation of $\hat{\lambda}_1$, it follows from Assumption 5.3 that $\hat{v}_h^R(\hat{\lambda}_1) \le v^R(\lambda_1) + L_{K,R}(\Delta t^{\rho_1} + \Delta x^{\rho_2})$, and therefore,

$$V_h^{K,R} \le V^{K,R} + L_{K,R}(\Delta t^{\rho_1} + \Delta x^{\rho_2}).$$

□

Remark 5.2. *In the $d-$dimensional case, we can use the generalized finite differences method to approximate $V^{K,R}$. To construct the generalized finite difference scheme, we refer to section 5 of Kushner [27] when every $a \in S_d$ for $(a, b) \in U$ are diagonal dominated, and to Bonnans and Zidani [12] as well as Bonnans, Ottenwaelter and Zidani [9] for general cases.*

5.3 Gradient projection algorithm

In this section, we suggest a numerical scheme to approximate $V_h^{K,R} = \sup_{\hat\lambda_1 \in \mathrm{Lip}_0^{K,R}} \hat v_h^R(\hat\lambda_1)$ in (5.13). The crucial observation for our methodology is the following. By $B(\mathcal{N}_R)$, we denote the set of all bounded function on $\mathcal{N}_R$.

Proposition 5.2. *Under the CFL condition* (5.11)*, the function* $\hat\lambda_1 \mapsto \hat v_h^R(\hat\lambda_1)$ *is concave on* $B(\mathcal{N}_R)$*.*

Proof. Let $\bar u = (\bar u_{k,i})_{0\le k<l,-r<i<r}$, with $\bar u_{k,i} = (\bar a_{k,i}, \bar b_{k,i}) \in U$, we introduce $\overline{\lambda}^{h,\bar u,\hat\lambda_1}$ (or just $\overline{\lambda}^{h,\bar u}$ if there is no risk of ambiguity) as the unique solution of the discrete linear system on $\mathcal{M}_{T,R}$ with a given $\hat\lambda_1$:

$$\begin{cases} \overline{\lambda}^{h,\bar u}(t_k,x_i) = \Big(\overline{\lambda}^{h,\bar u} + \Delta t\Big(\ell(\cdot,\bar u_{k,i}) + (\bar b_{k,i}D)\overline{\lambda}^{h,\bar u} + \bar a_{k,i}D^2\overline{\lambda}^{h,\bar u}\Big)\Big)(t_{k+1},x_i) \text{ on } \mathring{\mathcal{M}}_{T,R}, \\ \overline{\lambda}^{h,\bar u}(t_k,x_i) = \hat\lambda_1(x_i), \quad \text{for } (t_k,x_i) \in \partial_T\mathcal{M}_{T,R} \cup \partial_R\mathcal{M}_{T,R}. \end{cases} \tag{5.14}$$

Let $\overline{\lambda}_0^{h,\bar u} := \overline{\lambda}^{h,\bar u}(0,\cdot)$, and define:

$$\overline{v}_h^{R,\bar u}(\hat\lambda_1) \;:=\; \mu_0\big(\mathrm{lin}^R[\overline{\lambda}_0^{h,\bar u}]\big) - \mu_1\big(\mathrm{lin}^R[\hat\lambda_1]\big).$$

We claim that

$$\hat v_h^R(\hat\lambda_1) \;=\; \inf_{\bar u \in U^{l(2r-1)}} \overline{v}_h^{R,\bar u}(\hat\lambda_1). \tag{5.15}$$

Indeed, under the CFL condition (5.11), the finite difference scheme (5.9) as well as (5.14) are both $L^\infty-$monotone in sense of Barles and Souganidis [4]. Moreover, the linear interpolation $\hat\lambda_0 \mapsto \mathrm{lin}^R[\hat\lambda_0]$ is also monotone. Then taking infimum step by step in (5.9) and (5.13) is equivalent to taking infimum globally in (5.15).

Finally, the concavity of $\hat\lambda_1 \mapsto \hat v_h^R(\hat\lambda_1)$ follows from its representation as the infimum of linear maps in (5.15). □

By the previous proposition, $V_h^{K,R}$ consists in the maximization of a concave function, and a natural scheme to approximate it is the gradient projection algorithm.

Remark 5.3. *Since* U *is compact by Assumption 5.2, then for every function* $\hat\lambda_1$*, we have the optimal control* $\hat u(\hat\lambda_1) = \big(\hat u_{k,i}(\hat\lambda_1)\big)_{0\le k<l,-r<i<r}$ *such that*

$$\hat\lambda_0^{h,R} = \overline{\lambda}_0^{h,\hat u(\hat\lambda_1)} \quad \textit{and} \quad \hat v_h^R(\hat\lambda_1) = \overline{v}_h^{R,\hat u(\hat\lambda_1)}(\hat\lambda_1). \tag{5.16}$$

Now we are ready to give the gradient projection algorithm for $V_h^{K,R}$ in (5.13). Given a function $\varphi \in B(\mathcal{N}_R)$, we denote by $P_{\mathrm{Lip}_0^{K,R}}(\varphi)$ the projection of φ on $\mathrm{Lip}_0^{K,R}$, where $\mathrm{Lip}_0^{K,R} \subset B(\mathcal{N}_R)$ is defined in (5.12). Of course, the projection depends on the choice of the norm equipping $B(\mathcal{N}_R)$ which in turn has serious consequences on the numerics. We shall discuss this important issue later.

Let $\gamma := (\gamma_n)_{n\ge 0}$ be a sequence of positive constants, we propose the following algorithm:

Algorithm 5.1. *To solve problem* (5.13)*:*

- *1, Let* $\hat{\lambda}_1^0 := 0$.
- *2, Given* $\hat{\lambda}_1^n$*, compute the super-gradient* $\nabla \hat{v}_h^R(\hat{\lambda}_1^n)$ *of* $\hat{\lambda}_1 \mapsto \hat{v}_h^R(\hat{\lambda}_1)$ *at* $\hat{\lambda}_1^n$.
- *3, Let* $\hat{\lambda}_1^{n+1} = P_{Lip_0^{K,R}}\big(\hat{\lambda}_1^n + \gamma_n \nabla \hat{v}_h^R(\hat{\lambda}_1^n)\big)$.
- *4, Go back to step 2.*

In the following, we shall discuss the computation of super-gradient $\nabla \hat{v}_h^R(\hat{\lambda}_1)$, the projection $P_{\mathrm{Lip}_0^{K,R}}$ as well as the convergence of the above gradient projection algorithm.

5.3.1 Super-gradient

Let $\hat{\lambda}_1 \in B(\mathcal{N}_R)$ be fixed. Then, by Remark 5.3, we may find an optimal control $\hat{u}(\hat{\lambda}_1) = \big(\hat{u}_{k,i}(\hat{\lambda}_1)\big)_{0 \le k < l, -r \le i \le r}$, where $\hat{u}_{k,i}(\hat{\lambda}_1) = (\hat{a}_{k,i}(\hat{\lambda}_1), \hat{b}_{k,i}(\hat{\lambda}_1)) \in U$, for system (5.15). We then denote by g^j the unique solution of the following linear system on $\mathcal{M}_{T,R}$, for every $-r \le j \le r$:

$$\begin{cases} g^j(t_k, x_i) = \Big(g^j + \Delta t\Big(\big(\hat{b}_{k,i}(\hat{\lambda}_1) D\big) g^j + \hat{a}_{k,i}(\hat{\lambda}_1) D^2 g^j\Big)\Big)(t_{k+1}, x_i) \text{ on } \mathring{\mathcal{M}}_{T,R}, \\ g^j(t_k, x_i) = \delta_{i,j}, \quad \text{for } (t_k, x_i) \in \partial_T \mathcal{M}_{T,R} \cup \partial_R \mathcal{M}_{T,R}. \end{cases} \tag{5.17}$$

Denote $g_0^j := g^j(0, \cdot)$ and δ_j be a function on $\mathcal{N}_R$ defined by $\delta_j(x_i) := \delta_{i,j}$.

Proposition 5.3. *Let CFL condition* (5.11) *hold true, then the vector*

$$\nabla \hat{v}_h^R(\hat{\lambda}_1) \quad := \quad \Big(\mu_0\big(\mathrm{lin}^R[g_0^j]\big) - \mu_1\big(\mathrm{lin}^R[\delta_j]\big)\Big)_{-r \le j \le r} \tag{5.18}$$

is a super-gradient of $\varphi \in B(\mathcal{N}_R) \mapsto \hat{v}_h^R(\varphi) \in \mathbb{R}$ *at* $\hat{\lambda}_1$.

Proof. Consider the system (5.14) introduced in the proof of Proposition 5.2. Under the CFL condition (5.11), by (5.15), we have for every perturbation $\Delta\hat{\lambda}_1 \in B(\mathcal{N}_R)$,

$$\hat{v}_h^R(\hat{\lambda}_1 + \Delta\hat{\lambda}_1) = \overline{v}_h^{R, \hat{u}(\hat{\lambda}_1 + \Delta\hat{\lambda}_1)}(\hat{\lambda}_1 + \Delta\hat{\lambda}_1) \le \overline{v}_h^{R, \hat{u}(\hat{\lambda}_1)}(\hat{\lambda}_1 + \Delta\hat{\lambda}_1),$$

which implies that

$$\hat{v}_h^R(\hat{\lambda}_1 + \Delta\hat{\lambda}_1) - \hat{v}_h^R(\hat{\lambda}_1) \quad \le \quad \overline{v}_h^{R, \hat{u}(\hat{\lambda}_1)}(\hat{\lambda}_1 + \Delta\hat{\lambda}_1) - \overline{v}_h^{R, \hat{u}(\hat{\lambda}_1)}(\hat{\lambda}_1).$$

We next observe that for fixed $\hat{\lambda}_1$, the function $\varphi \longmapsto \overline{v}_h^{R, \hat{u}(\hat{\lambda}_1)}(\varphi)$ is linear, it follows that

$$\Big(\overline{v}_h^{R, \hat{u}(\hat{\lambda}_1)}(\hat{\lambda}_1 + \delta_j) - \overline{v}_h^{R, \hat{u}(\hat{\lambda}_1)}(\hat{\lambda}_1)\Big)_{-r \le j \le r} \tag{5.19}$$

is a super-gradient of $\varphi \mapsto \hat{v}_h^R(\varphi)$ at $\hat{\lambda}_1$. Finally, by (5.14) and (5.17), $g^j(t_k, x_i) = \overline{\lambda}^{\hat{u}(\hat{\lambda}_1), \hat{\lambda}_1 + \delta_j}(t_k, x_i) - \overline{\lambda}^{\hat{u}(\hat{\lambda}_1), \hat{\lambda}_1}(t_k, x_i)$, where $\overline{\lambda}^{\hat{u}(\hat{\lambda}_1), \hat{\lambda}_1 + \delta_j}$ is the solution of (5.14) with boundary condition $\hat{\lambda}_1 + \delta_j$. By the definition of $\overline{v}_h^{R, \bar{u}}(\hat{\lambda}_1)$ in (5.15), it follows that the super-gradient (5.19) is equivalent to $\nabla \hat{v}_h^R(\hat{\lambda}_1)$ defined in (5.18). □

5.3.2 Projection

To compute the projection $P_{\mathrm{Lip}_0^{K,R}}(\varphi)$, $\forall\varphi \in B(\mathcal{N}_R)$, we need to equip $B(\mathcal{N}_R)$ with a specific norm. In order to obtain a simple projection algorithm, we shall introduce an invertible linear map between $B(\mathcal{N}_R)$ and $\mathbb{R}^{2r+1}$, then equip on $B(\mathcal{N}_R)$ the norm induced by the classical L^2-norm on $\mathbb{R}^{2r+1}$.

Let us define the invertible linear map $\mathcal{T}_R$ from $B(\mathcal{N}_R)$ to $\mathbb{R}^{2r+1}$ as

$$\psi_i = \mathcal{T}_R(\varphi)_i := \begin{cases} \varphi(x_{i+1}) - \varphi(x_i), & i = 1, \cdots, r, \\ \varphi(0), & i = 0, \\ \varphi(x_{i-1}) - \varphi(x_i), & i = -1, \cdots, -r, \end{cases}$$

and define the norm $|\cdot|_R$ on $B(\mathcal{N}_R)$ (easily be verified) by

$$|\varphi|_R := |\mathcal{T}_R(\varphi)|_{L^2(\mathbb{R}^{2r+1})}, \quad \forall\varphi \in B(\mathcal{N}_R).$$

Notice that

$$\begin{aligned} \mathcal{T}_R\mathrm{Lip}_0^{K,R} &:= \left\{\psi = \mathcal{T}_R\varphi : \ \varphi \in \mathrm{Lip}_0^{K,R}\right\} \\ &= \left\{\psi = (\psi_i)_{-r\le i\le r} \in [-K\Delta x, K\Delta x]^{2r+1} : \ \psi_0 = 0\right\}. \end{aligned}$$

Then the projection $P_{\mathrm{Lip}_0^{K,R}}$ from $B(\mathcal{N}_R)$ to $\mathrm{Lip}_0^{K,R}$ under norm $|\cdot|_R$ is equivalent to the projection $P_{\mathcal{T}_R\mathrm{Lip}_0^{K,R}}$ from $\mathbb{R}^{2r+1}$ to $\mathcal{T}_R\mathrm{Lip}_0^{K,R}$ under the L^2-norm, which is simply written as

$$\left(P_{\mathcal{T}_R\mathrm{Lip}_0^{K,R}}(\psi)\right)_i = \begin{cases} 0, & \text{if } \ i = 0, \\ (K\Delta x) \wedge \psi_i \vee (-K\Delta x), & \text{otherwise.} \end{cases}$$

5.3.3 Convergence rate

Now, let us give a convergence rate for the above gradient projection algorithm. In preparation, we first provide an estimate for the norm of super-gradients.

Proposition 5.4. *Suppose that CFL condition* (5.11) *hold, then* $|\hat{v}_h^R(\varphi_1) - \hat{v}_h^R(\varphi_2)| \le 2|\varphi_1 - \varphi_2|_\infty$ *for every* $\varphi_1,\ \varphi_2 \in B(\mathcal{N}_R)$. *And therefore, the super-gradient* $\nabla\hat{v}_h^R$ *satisfies*

$$\left|\nabla\hat{v}_h^R(\hat{\lambda}_1)\right|_R \le 2\sqrt{\frac{R}{\Delta x} + 1}, \ \text{for all } \hat{\lambda}_1 \in B(\mathcal{N}). \tag{5.20}$$

Proof. Under CFL condition, the scheme (5.9) is $L^\infty-$monotone, then $\left|\hat{\lambda}_0^{h,R,\varphi_1} - \hat{\lambda}_0^{h,R,\varphi_2}\right|_\infty \le |\varphi_1 - \varphi_2|_\infty$, and it follows from the definition of $\hat{v}_h^R$ in (5.10) that

$$|\hat{v}_h^R(\varphi_1) - \hat{v}_h^R(\varphi_2)| \le 2|\varphi_1 - \varphi_2|_\infty. \tag{5.21}$$

Next, by Cauchy-Schwarz inequality,

$$|\varphi_1 - \varphi_2|_\infty \le \max\left(\sum_{i=0}^{r}\left|\mathcal{T}_R(\varphi_1 - \varphi_2)_i\right|, \sum_{i=0}^{-r}\left|\mathcal{T}_R(\varphi_1 - \varphi_2)_i\right|\right) \le \sqrt{r+1}|\varphi_1 - \varphi_2|_R.$$

Together with (5.21), this implies that (5.20) holds for every super-gradient $\nabla \hat{v}_h^R(\hat{\lambda}_1)$. $\square$

Let us finish this section by providing a convergence rate for our gradient projection algorithm. Denote

$$\Pi := \max_{\varphi_1,\varphi_2 \in \mathrm{Lip}_0^{K,R}} |\varphi_1 - \varphi_2|_R^2 \le 2r(K\Delta x)^2 \le 2K^2 R\Delta x,$$

it follows from section 5.3.1 of Ben-Tal and Nemirovski [5] that

$$\begin{aligned} 0 \le V_h^{K,R} - \max_{n\le N} \hat{v}_h^R(\hat{\lambda}_1^n) &\le \frac{\Pi + \sum_{n=1}^N \gamma_n^2 |\nabla \hat{v}_h^R(\hat{\lambda}_1^n)|_R^2}{\sum_{n=1}^N \gamma_n} \\ &\le \frac{2K^2 R\Delta x + 4\left(\frac{R}{\Delta x}+1\right)\sum_{n=1}^N \gamma_n^2}{\sum_{n=1}^N \gamma_n}. \end{aligned} \tag{5.22}$$

We have several choices for the series $\gamma = (\gamma_n)_{n\ge 1}$:

- Divergent series : $\gamma_n \ge 0$, $\sum_{n=1}^\infty \gamma_n = +\infty$ and $\sum_{n=1}^\infty \gamma_n^2 < +\infty$, then the right hand side of (5.22) converges to 0 as $N \to \infty$.
- Optimal stepsizes : $\gamma_n = \frac{\sqrt{2\Pi}}{|\nabla \hat{v}_h^R(\hat{\lambda}_1^n)|_R \sqrt{n}}$, [5] shows that

  $$V_h^{K,R} - \max_{n\le N} \hat{v}_h^R(\hat{\lambda}_1^n) \le C_1 \frac{(\max_{1\le n\le N} |\nabla \hat{v}_h^R(\hat{\lambda}_1^n)|_R)\cdot\sqrt{2\Pi}}{\sqrt{N}} \le C\frac{K(R+\sqrt{R\Delta x})}{\sqrt{N}},$$

  for some constant C independent of K, R, Δt, Δx and N.

5.4 Numerical Example

We finally give a numerical example for the above algorithm. Let us consider the one-dimensional case $d = 1$, and $U = U_1 \times \{0\}$ where U_1 is a compact interval in $\mathbb{R}^+$, i.e. X is a one-dimensional martingale under $\mathbb{P}$ for all $\mathbb{P} \in \mathcal{P}(\mu_0, \mu_1)$. Let the transportation cost be given by:

$$\ell(t,x,a,b) = a \quad \text{so that} \quad J(\mathbb{P}) = \mathbb{E}^{\mathbb{P}}\langle X\rangle_1.$$

This example is motivated by an application in financial mathematics, where $\langle X\rangle_1$ is the payoff of a financial derivative called *variance swap*. Then, the minimum cost of transportation is the minimum no-arbitrage price of the variance swap given the possibility of dynamic trading the underlying asset, with price X, together with the static trading of the European options maturing at time 1 with all possible strikes.

Suppose that $\mathcal{P}(\mu_0, \mu_1)$ is nonempty, it follows from the duality Theorems 3.1 and 3.2 that

$$\begin{aligned} V(\mu_0,\mu_1) &= \inf_{\mathbb{P}\in\mathcal{P}(\mu_0,\mu_1)} \mathbb{E}^{\mathbb{P}} \int_0^1 \alpha_t^{\mathbb{P}} dt = \inf_{\mathbb{P}\in\mathcal{P}(\mu_0,\mu_1)} \mathbb{E}^{\mathbb{P}}\langle X\rangle_1 \\ &= \inf_{\mathbb{P}\in\mathcal{P}(\mu_0,\mu_1)} \mathbb{E}^{\mathbb{P}}\left[(X_1 - X_0)^2\right] = \int_{\mathbb{R}} x^2 \mu_1(dx) - \int_{\mathbb{R}} x^2 \mu_0(dx). \end{aligned} \tag{5.23}$$

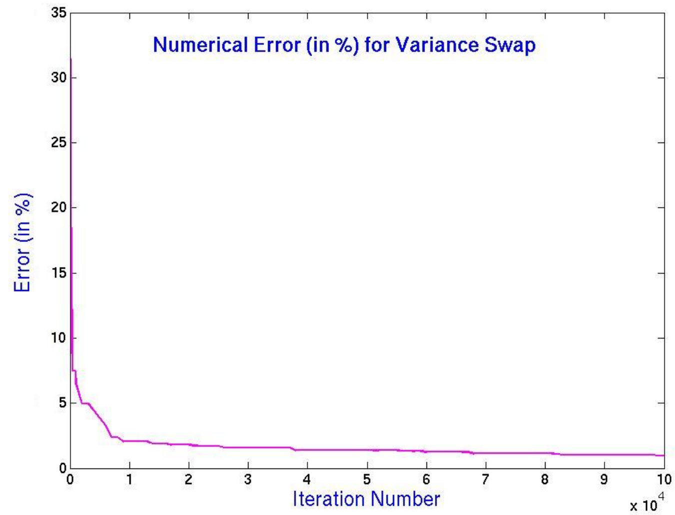

Figure 1: Numerical example: $\mu_i = N(0, \sigma_i^2)$ with $\sigma_0 = 0.1$, $\sigma_1 = 0.2$, $U = [0, 0.1] \times \{0\}$, $K = 1.5$, $R = 1$, $\Delta x = 0.1$, $\Delta t = 0.025$.

We choose μ_i as normal distribution $N(0, \sigma_i^2)$ with $\sigma_0 = 0.1$, $\sigma_1 = 0.2$ and $U = [0, 0.1] \times \{0\}$, we implement the scheme suggested in the previous subsection, and we compare to the explicit solution (5.23). The numerical result shows that with 10^5 iterations (which takes no more than 1 minute of calculation on a standard computer), the relative error is less than 1%, see Figure 1.

A Appendix

We first report a theorem which provides the unique canonical decomposition of a continuous semimartingale under different filtrations. In particular, it follows that an Itô process has a diffusion representation, by taking the filtration generated by itself. This is in fact Theorem 7.17 of Liptser and Shiryayev [28] in 1-dimensional case, or Theorem 4.3 of Wong [35] in multi-dimensional case.

Theorem A.1. *In a filtrated space $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le 1}, \mathbb{P})$ (here Ω is not necessary the canonical space), a process X is a continuous semimartingale with canonical decomposition:*

$$X_t = X_0 + B_t + M_t,$$

where $B_0 = M_0 = 0$, and $B = (B_t)_{0 \le t \le 1}$ is of finite variation and $M = (M_t)_{0 \le t \le 1}$ a local martingale. In addition, suppose that there are measurable and $\mathbb{F}-$adapted processes (α, β) such that

$$B_t = \int_0^t \beta_s ds, \quad \int_0^1 \mathbb{E}|\beta_s| ds < \infty, \quad \text{and } A_t := \langle M \rangle_t = \int_0^t \alpha_s ds.$$

Let $\mathbb{F}^X = (\mathcal{F}_t^X)_{0 \le t \le 1}$ be the filtration generated by process X and $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{0 \le t \le 1}$ be a filtration such that $\mathcal{F}_t^X \subseteq \bar{\mathcal{F}}_t \subseteq \mathcal{F}_t$. Then X is still a continuous semimartingale under

$\bar{\mathbb{F}}$, *whose canonical decomposition is given by*

$$X_t = X_0 + \int_0^t \bar{\beta}_s ds + \bar{M}_t \quad \textit{with } \bar{A}_t := \langle \bar{M} \rangle_t = \int_0^t \bar{\alpha}_s ds,$$

where

$$\bar{\beta}_t = \mathbb{E}(\beta_t | \bar{\mathcal{F}}_t) \quad \textit{and} \quad \bar{\alpha}_t = \alpha_t, \quad d\mathbb{P} \times dt - a.e.$$

Proof of Theorem 4.1. The characterization of the value function as viscosity solution is a natural result of the dynamic programming principle. Here, we give a proof, similar to that of Corollary 5.1 in [13], in our context.

1, We first prove the subsolution property. Suppose that $(t_0, x_0) \in [0,1) \times \mathbb{R}^d$ and $\phi \in C_c^\infty([0,1) \times \mathbb{R}^d)$ is a smooth function such that

$$0 = (\lambda - \phi)(t_0, x_0) > (\lambda - \phi)(t, x), \quad \forall (t,x) \neq (t_0, x_0).$$

By adding $\varepsilon \left(|t - t_0|^2 + |x - x_0|^4 \right)$ to $\phi(t,x)$, we can suppose that

$$\phi(t,x) \geq \lambda(t,x) + \varepsilon \left(|t - t_0|^2 + |x - x_0|^4 \right) \tag{A.1}$$

without loosing generality. Assume to the contrary that

$$-\partial_t \phi(t_0, x_0) - H\left(t_0, x_0, D_x \phi(t_0, x_0), D^2_{xx} \phi(t_0, x_0)\right) > 0,$$

we shall derive a contradiction. Indeed, by definition of H, there is $c > 0$ and $(a,b) \in U$ such that

$$-\partial_t \phi(t,x) - b \cdot D_x \phi(t,x) - \frac{1}{2} a \cdot D^2_{xx} \phi(t,x) - \ell(t,x,a,b) > 0, \quad \forall (t,x) \in B_c(t_0, x_0),$$

where $B_c(t_0,x_0) := \{(t,x) \in [0,1) \times \mathbb{R}^d \ : \ |(t,x) - (t_0,x_0)| \leq c\}$. Let $\tau := \inf\{t \geq t_0 \ : (t, X_t) \notin B_c(t_0, x_0)\}$, then

$$\begin{aligned} \lambda(t_0, x_0) = \phi(t_0, x_0) &\geq \inf_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{t_0, x_0, 0, 0}} \mathbb{E}^{\overline{\mathbb{P}}} \left[\int_t^\tau \ell(s, X_s, \nu_s) ds + \phi(\tau, X_\tau) \right] \\ &\geq \inf_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{t_0, x_0, 0, 0}} \mathbb{E}^{\overline{\mathbb{P}}} \left[\int_t^\tau \ell(s, X_s, \nu_s) ds + \lambda(\tau, X_\tau) \right] + \eta, \end{aligned}$$

where η is some positive constant from (A.1) and the definition of τ. This is a contradiction to Proposition 4.1.

2, For the supersolution property, we assume to the contrary that there is $(t_0, x_0) \in [0,1) \times \mathbb{R}^d$ and smooth function ϕ satisfying

$$0 = (\lambda - \phi)(t_0, x_0) < (\lambda - \phi)(t, x), \quad \forall (t,x) \neq (t_0, x_0).$$

and

$$-\partial_t \phi(t_0, x_0) - H\left(t_0, x_0, D_x \phi(t_0, x_0), D^2_{xx} \phi(t_0, x_0)\right) < 0,$$

We also suppose without loosing generality that

$$\phi(t,x) \leq \lambda(t,x) - \varepsilon \left(|t - t_0|^2 + |x - x_0|^4 \right). \tag{A.2}$$

By continuity of H, there is $c > 0$ such that for all $(t,x) \in B_c(t_0,x_0)$ and every $(a,b) \in U$,

$$-\partial_t\phi(t,x) - b\cdot D_x\phi(t,x) - \frac{1}{2}a\cdot D^2_{xx}\phi(t,x) - \ell(t,x,a,b) < 0.$$

Let $\tau := \inf\{t \geq t_0 \;:\; (t,X_t) \notin B_c(t_0,x_0)\}$, then

$$\begin{aligned}
\lambda(t_0,x_0) = \phi(t_0,x_0) &\leq \inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{t_0,x_0,0,0}} \mathbb{E}^{\overline{\mathbb{P}}}\left[\phi(\tau,X_\tau) + \int_{t_0}^{\tau} \ell(s,X_s,\nu_s)ds\right] \\
&\leq \inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{t_0,x_0,0,0}} \mathbb{E}^{\overline{\mathbb{P}}}\left[\lambda(\tau,X_\tau) + \int_{t_0}^{\tau} \ell(s,X_s,\nu_s)ds\right] - \eta
\end{aligned}$$

for some $\eta > 0$ by (A.2), which is a contradiction to Proposition 4.1. □

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