# No Arbitrage Conditions and Liquidity

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**Abstract**

We extend the fundamental theorem of asset pricing to the case of markets with liquidity risk. Our results generalize, when the probability space is finite, those obtained by Kabanov & Stricker (2001), Kabanov, Rásonyi and Stricker (2002, 2003) and by Schachermayer (2004) for markets with proportional transaction costs. More precisely, we restate the notions of consistent and strictly consistent price systems and prove their equivalence to corresponding no arbitrage conditions. We express these results in an analytical form in terms of the subdifferential of the so-called liquidation function. We conclude the paper with a hedging theorem.

## Introduction

In a frictionless discrete-time market, the fundamental theorem of asset pricing states that there is no arbitrage opportunity if and only if there exists an equivalent martingale measure, that is, a probability measure equivalent to the historical one, under which the discounted price process is a martingale. In real financial markets, trading strategies have an impact on price processes, which is a source of liquidity risk. A simple example of liquidity risk is the presence of transaction costs, where the price depends on whether the position is short or long. Markets with transaction costs have been studied by several authors, see for instance Bensaid, Lesne, Pagès and Scheinkman (1992), Jouini and Kallal (1995), Kabanov and Stricker (2001), Kabanov, Rásonyi and Stricker (2002, 2003), Schachermayer (2004). On the other hand, the impact of trading strategies on prices has been considered in many papers, see e.g. Cvitanić and Ma (1996), Frey and Stremme (1997), Schönbucher and Wilmott (2000), Bank and Baum (2004). These authors consider price processes with dynamics affected by the agent's position at a each time. Note that, in practice, only orders, i.e. "instantaneous variations" of the position, can be observed, so that the impact of the portfolio position on the price process has a poor economic justification.

In two recent papers, Çetin, Jarrow and Protter (2004) and Çetin, Jarrow, Protter and Warachka (2002), the authors consider the more relevant case where the exchange sizes have an impact on prices, by introducing a supply curve $S_t(x, \omega)$, where $x$ is the exchange size. In a continuous-time setting, they provide a characterization of the no-arbitrage condition, and discuss the pricing of derivatives. A crucial condition in their work, is that the supply curve is smooth in the $x-$variable. In particular, this excludes classical models with proportional transaction costs, and explains why the no-arbitrage condition in their setting coincides essentially with the no-arbitrage condition on frictionless markets.

The chief goal of this paper is to develop a financial model with liquidity effect in the spirit of Çetin, Jarrow and Protter (2004), avoiding strong smoothness condition on the supply curve at $x = 0$. Since liquidity effects do not allow, in general, to transfer funds from an account to another without any cost, we work in a vector framework, as in the context of markets with transaction costs. Our framework is a generalization, in a finite probability space, of Kabanov and Stricker (2001), Kabanov, Rásonyi, and Stricker (2003), and Schachermayer (2004).

Throughout this paper, we consider a finite probability space $(\Omega, \mathscr{F}, \mathbb{P})$. By $\mathbb{L}^0(E; \mathscr{F})$ we denote the (finite dimensional) space of $\mathscr{F}$-measurable functions with values in the subset $E$ of $\mathbb{R}^d$. Given a subset $G$ of $\mathbb{L}^0(E; \mathscr{F})$, we shall denote by

$$\delta_G(Z) := \sup_{X \in G} \mathbb{E}[X \cdot Z] \quad \text{for all} \quad Z \in \mathbb{L}^0(E; \mathscr{F}),$$

its support function. Here $\cdot$ is the Euclidean inner product.

Given a set $S \subset \mathbb{R}^d$, $S^c$, $\text{int}(S)$, $\text{ri}(S)$, $\overline{S}$, $\text{conv}(S)$, $\text{cone}(S)$ stand respectively for its complement, interior, relative interior, closure, convex hull, conical hull and $\overline{\text{cone}}(S) = \overline{\text{cone}(S)}$. The positive dual cone of $S$ is denoted by $S^*$:

$$S^* := \left\{ y \in \mathbb{R}^d \ : \ x \cdot y \geq 0 \text{ for all } x \in S \right\}.$$

## 1 A discrete-time model with liquidity risk

We consider a discrete-time market with $d$ assets. The stochastic structure is given by a finite filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_j)_{0 \leq j \leq N}, \mathbb{P})$. For $x = (x_1, \ldots, x_d)$ in $\mathbb{R}^d$, we denote by $P_j(\omega, x)$ the *liquidation value* of $x_1$ units of asset 1, ..., and $x_d$ units of asset $d$ at time $j \in \{0, \ldots, N\}$ in the state of the world $\omega \in \Omega$.

We define the solvency region by:

$$K_j(\omega) \quad := \quad \left\{ x \in \mathbb{R}^d \ : \ P_j(\omega, x) \geq 0 \right\} ,$$

so that $-K_j(\omega)$ is the set of portfolios with non-positive purchase cost at time $j \in \{0, \ldots, N\}$, in the state of the world $\omega \in \Omega$.

Throughout this paper, we shall assume that

$$\mathbb{R}^d_+ \quad \subset \quad K_j \, , \tag{1.1}$$

and

$$K_j(\omega) \quad \text{is a closed convex subset of} \quad \mathbb{R}^d \text{ for all } 1 \leq j \leq d \text{ and } \omega \in \Omega \, . \tag{1.2}$$

The latter condition can of course be ensured by convenient assumptions on the liquidation function $P_j$. For instance, (1.2) holds whenever $P_j(\omega, \cdot)$ is quasi-concave and upper semi-continuous, see e.g. Hiriart-Urruty and Lemaréchal (1993).

**Example 1.1 (Frictionless Markets)** Let $\{S_j, \ 0 \leq j \leq N\}$ be an adapted process with values in $\mathbb{R}^d_+$, and consider the liquidation function $P_j(\omega, x) := x \cdot S_j(\omega)$. Clearly, Conditions (1.1) and (1.2) are satisfied. This is the classical context where the sets $K_j(\omega)$ are half-spaces with normal vector $S_j(\omega)$.

**Example 1.2 (discrete-time version of Çetin, Jarrow and Protter (2002))** The financial market consists in one risky asset and a bank account, i.e. $d = 2$. For each $x \in \mathbb{R}$, let $S_j(\omega, x) \in \mathbb{R}_+$ be the price to pay in order to purchase $x$ units of the risky asset. In this framework, the liquidation function is defined by $P_j(\omega, (x, y)) = y + x \, S_j(\omega, -x)$. The authors make the crucial assumption that $S_j(\omega, x)$ is a $C^2$ function of $x$. Therefore, their framework does not include the classical financial market model with proportional transaction costs. Also, there is no convexity assumption in their paper and their main example is the specification $S_j(\omega, x) = S_j(\omega, 0)e^{-\alpha x}$. However, our analysis only requires the convexity of the solvency sets near the origin, so that this specification can easily be incorporated in our context.

**Example 1.3 (Proportional Transaction Costs, Kabanov (1999))** , Define the solvency regions by the following polyhedral cones

$$K_j = \left\{ x \in \mathbb{R}^d \ : \ x^i S^i_j + \sum_{k=0}^{d} \left( a^{ki} S^k_j - (1 + \lambda^{ik}) a^{ik} S^i_j \right) \geq 0, \ 1 \leq i \leq d, \text{ for some } a \in \mathscr{M}^+_d \right\}$$

where $\mathscr{M}^+_d$ is the set of $d \times d$ matrices with nonnegative entries and $\lambda^{ik}$ is the proportional transaction cost corresponding to transfers from $i$ to $k$. Following Bouchard, Kabanov and

Touzi (2001), the corresponding liquidation function is given by $P_j(x) := \sup\{w \in \mathbb{R} : x - we_1 \in K_j\}$, where $e_1 = (1, 0, \ldots, 0)$. Conditions (1.1) and (1.2) are also satisfied in this context.

**Example 1.4 (Concave liquidation function)** Let $P_j(\omega, .)$ be a concave function on $\mathbb{R}^d$, with $P_j(\omega, 0) = 0$. Then Conditions (1.1) and (1.2) hold. Observe that the concavity of $P_j$ implies that $P_j(\omega, \mu x) \leq \mu P_j(\omega, x)$ for all $x \in \mathbb{R}^d$ and $\mu \geq 1$. Hence the unit price of any portfolio $x$ is non-increasing in the exchanged volume.

**Definition 1.1** *A portfolio is an $\mathbb{R}^d$-valued adapted process $Y = \{Y_j,\ 0 \leq j \leq N\}$. We say that $Y$ is self-financing if:*

$$\forall j \in \{1, \ldots, N\}, \quad Y_j - Y_{j-1} \in \mathbb{L}^0(-K_j, \mathscr{F}_j).$$

Let

$$A_N \quad := \quad \sum_{j=0}^{N} \mathbb{L}^0\left(-K_j, \mathscr{F}_j\right).$$

In other words, $A_N$ is the set of the values at time $N$ of self-financing portfolios attainable from a non-positive initial capital.

Note that the set $A_N$ is only convex. It is not a cone in general.

As usual, we say that there is no arbitrage opportunity if the following condition is satisfied:

**NA** $\quad A_N \cap \mathbb{L}^0(\mathbb{R}^d_+, \mathscr{F}_N) = \{0\}$.

This means that it is not possible to produce a nonzero gain with nonnegative components in every state of the world at time $N$.

In Section 2, we introduce two stronger notions of no-arbitrage which extend the concepts of weak and robust no-arbitrage of Kabanov and Stricker (2001), and Schachermayer (2004) for markets with proportional transaction costs. In Sections 3 and 4, we show that these two concepts can be characterized by the existence of so called *(strictly) consistent price systems*, see Definition 2.2 below. We restate these characterizations in Section 5 in terms of the liquidation functions $P_j$. The final section of the paper deals with the super-hedging problem, i.e. given a contingent claim $G \in \mathbb{L}^0(\mathbb{R}^d, \mathscr{F}_N)$, we characterize the initial positions $y \in \mathbb{R}^d$ from which $G$ can be hedged without risk.

**Remark 1.1** When the solvency sets $K_j(\omega)$ are closed convex cones the no-arbitrage principle is a necessary condition for an equilibrium on the financial market. Indeed, when there is an arbitrage opportunity, the optimal demand function of each agent is infinite in the direction of such an arbitrage opportunity, so that the market clearing condition can not be satisfied. When $K_j(\omega)$ is not a cone for some $j \in \{0, \ldots, d\}$, this motivation of the no-arbitrage principle is not valid anymore, as the agent cannot increase unboundedly her expected utility.

# 2 Price systems and no-arbitrage concepts

## 2.1 Consistent price systems

In the context of a financial market with proportional transaction costs, Kabanov and Stricker (2001) isolated two kinds of *pricing systems*, called consistent and strictly consistent price processes in Schachermayer (2004). These concepts extend the classical Radon-Nykodim density of risk neutral measures in frictionless markets.

**Definition 2.1** *A $(0,\infty)^d$−valued martingale $Z = \{Z_j,\, 0 \le j \le N\}$ is called :*

(i) *a consistent price system if $Z_j$ takes values in $K_j^*$, for any $j \in \{0, \ldots, N\}$,*

(ii) *a strictly consistent price system if $Z_j$ takes values in* $\mathrm{ri}(K_j^*)$*, for any $j \in \{0, \ldots, N\}$.*

The existence of such price systems is as usual obtained by applying the Hahn-Banach Theorem to separate the sets $\mathbb{L}^0(\mathbb{R}^d_+, \mathscr{F}_N)$ and $A_N$.

1. In financial markets with proportional transaction costs, the cone property of the set $A_N$ is crucial in order to prove that the separating hyperplane contains zero.

2. Observe that any price system defines a separating hyperplane of $\mathbb{L}^0(\mathbb{R}^d_+, \mathscr{F}_N)$ and $T_N := \overline{\mathrm{cone}}(A_N)$, the closed cone generated by $A_N$, as illustrated in Figure 1. This suggests to extend the classical characterization results of the no-arbitrage concepts by passing to the cone generated by $A_N$.

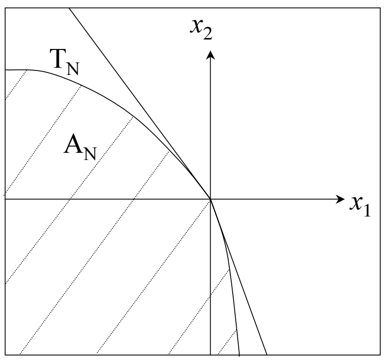


Fig. 1: Both $A_N$ and $T_N$ can be separated from $\mathbb{L}^0(\mathbb{R}^d_+, \mathscr{F}_N)$ by the same hyperplane

3. In Figure 2, the intersection of the sets $T_N$ and $\mathbb{L}^0(\mathbb{R}^d_+, \mathscr{F}_N)$ is not reduced to $\{0\}$, although the no-arbitrage condition holds. In particular, there is no consistent price system in this example. This suggests that Definition 2.1 does not provide the appropriate dual concepts in the context where $A_N$ is not a cone.

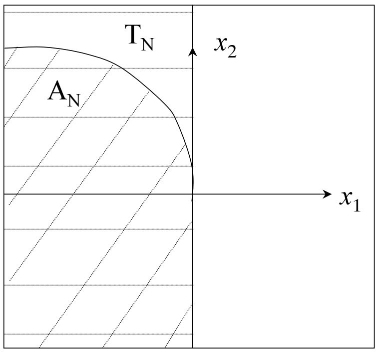


Fig. 2: $T_N$ does not necessarily satisfy **NA** when $A_N$ does

4. The situation illustrated in Figure 2 can occur for instance if $K_j$ is tangent to $\mathbb{L}^0(\mathbb{R}^d_+, \mathscr{F}_j)$ for some $j \in \{0, \ldots, N\}$. Notice that it can occur even in a situation where $\mathbb{L}^0(\overline{\text{cone}}(K_j), \mathscr{F}_j) \cap \mathbb{L}^0(\mathbb{R}^d_+, \mathscr{F}_j) = \{0\}$ for every $j \in \{0, \ldots, N\}$. This is illustrated by the following example which was communicated by Professor W. Schachermayer.

**Example 2.1** Suppose $\Omega = \{\omega_1, \omega_2\}$. Define a two-date market consisting in one risk-free asset with constant price 1 and one risky asset with prices $S_0$ at time 0, $S_1(\omega_1) = 2$ at time 1 in the state of the world $\omega_1$, and $S_1(\omega_2) = 1$ at time 1 in the state of the world $\omega_2$. Assume that the liquidation functions are of the following form:

$$P_0(x_1, x_2) = x_1 + x_2 - cx_1^2 \text{ and } P_1(x_1, x_2) = x_1 + x_2 S_1$$

for a given constant $c > 0$. We easily show that there is no arbitrage opportunity.

Consider now the "tangent market", i.e. take the smallest cones containing the previous solvency regions. Then, the liquidation functions are given by:

$$\bar{P}_0(x_1, x_2) = x_1 + x_2 \text{ and } \bar{P}_1(x_1, x_2) = x_1 + x_2 S_1.$$

Obviously, the "strategy" $(-1, 1)$, whose initial value is 0, is an arbitrage since it leads to the payoff 1 in state $\omega_1$ and 0 in state $\omega_2$.

## 2.2 Nearly consistent price systems

In the previous subsection, we illustrated that the notion of consistent price system is not appropriate in the context where $A_N$ is not a cone. In particular, Figure 2 shows that the main difficulty is related to the small portfolios near the origin which may converge towards a tangent portfolio. In order to circumvent this difficulty, our main idea is to approximate the set $A_N$ by a family of subsets which behave *nicely* as in Figure 1. We thus define the sets

$$K_j^\varepsilon \;:=\; K_j \cap \operatorname{cone}\left(\Theta_j^\varepsilon\right) \quad \text{for every} \quad j \in \{0, \ldots, n\}$$

with

$$\Theta_j^\varepsilon \quad := \quad \left\{ x \in K_j \;:\; \sum_{i=1}^{d} |x_i| = \varepsilon \right\}$$

and

$$A_N^\varepsilon \;:=\; \sum_{j=0}^{N} \mathbb{L}^0\left(-K_j^\varepsilon, \mathscr{F}_j\right).$$

Note that the family $\{K_j^\varepsilon \,;\, \varepsilon > 0\}$ is decreasing.

Figure 3 illustrates our construction.

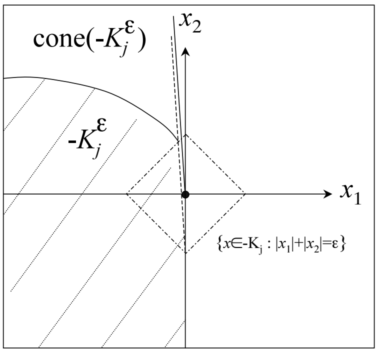


Fig. 3: Both $-K_j^\varepsilon$ and cone$(-K_j^\varepsilon)$ satisfy **NA**

The following easy result shows that we can proceed to the analysis of the no-arbitrage condition by working separately on each of the $A_N^\varepsilon$.

**Lemma 2.1** *The following properties are equivalent:*

(i) **NA**

(ii) $A_N^\varepsilon \cap \mathbb{L}^0(\mathbb{R}_+^d, \mathscr{F}_N) = \{0\}$ *for all* $\varepsilon > 0$

(iii) cone$\,(A_N^\varepsilon) \cap \mathbb{L}^0(\mathbb{R}_+^d, \mathscr{F}_N) \;=\; \{0\}$ *for all* $\varepsilon > 0$.

**Proof.** We only show that (ii) $\Longrightarrow$ (i), as all other implications are trivial. Suppose that (ii) holds and that there exists a non zero $X = \sum_{j=0}^{N} \xi_j$ in $A_N \cap \mathbb{L}^0(\mathbb{R}_+^d, \mathscr{F}_N)$. If $X \neq 0$, then

$$\varepsilon \;:=\; \min\left\{\sum_{i=1}^{d}|(\xi_j(\omega))_i|\;;\;0\le j\le N,\,\omega\in\Omega,\,\xi_j(\omega)\neq 0\right\}\;>\;0\,,$$

and we obtain a contradiction by observing that $X\in A_N^\varepsilon\cap\mathbb{L}^0(\mathbb{R}^d_+,\mathscr{F}_N)=\{0\}$. $\square$

We now generalize the concepts of price systems.

**Definition 2.2** *A family $\{Z^\varepsilon,\,\varepsilon>0\}$ of martingales, with values in $(0,\infty)^d$, is called :*

(i) *a nearly consistent price system if $Z_j^\varepsilon$ takes values in $K_j^{\varepsilon *}$, for every $j\in\{0,\ldots,N\}$ and $\varepsilon>0$,*

(ii) *a nearly strictly consistent price system if $Z_j^\varepsilon$ takes values in $\mathrm{ri}\left(K_j^{\varepsilon *}\right)$, for every $j\in\{0,\ldots,N\}$ and $\varepsilon>0$.*

**Remark 2.1** Let us specialize the discussion to the case where the solvency sets $K_j(\omega)$ are closed convex cones of $\mathbb{R}^d$. Then, it is immediately checked that $K_j^\varepsilon=K_j$, and the notion of nearly (strictly) consistent price system reduces to the notion of (strictly) consistent price system.

### 2.3 Dual characterization of no-arbitrage concepts

In this subsection, we state a characterization result of the existence of nearly (strictly) consistent price systems in terms of a suitable strengthening of Condition NA which relies on the approximating family $(A_N^\varepsilon)_{\varepsilon>0}$ of $A_N$ introduced in the previous subsection.

The first concept, called robust no-arbitrage condition, is given by the following condition :

**NA$^{\mathbf{r}}$** For every $\varepsilon>0$, there is a family $\{\tilde K_j^\varepsilon,\,0\le j\le N\}$ of closed convex subsets of $\mathbb{R}^d$ such that :

$$K_j^\varepsilon\setminus F_j^\varepsilon\;\subset\;\mathrm{int}\left(\tilde K_j^\varepsilon\right)\quad\text{and}\quad \tilde A_N^\varepsilon\cap\mathbb{L}^0(\mathbb{R}^d_+,\mathscr{F}_N)\;=\;\{0\}\,,$$

where

$$F_j^\varepsilon\;:=\;\overline{\mathrm{cone}}(K_j^\varepsilon)\cap\overline{\mathrm{cone}}(-K_j^\varepsilon)\quad\text{and}\quad \tilde A_N^\varepsilon\;:=\;\sum_{j=0}^{N}\mathbb{L}^0\left(-\tilde K_j^\varepsilon,\mathscr{F}_j\right)\,.$$

Forgetting the technical aspect of turning $A_N$ into $A_N^\varepsilon$, Condition **NA$^{\mathbf{r}}$** says that the no-arbitrage condition still holds for slightly better market conditions.

1. In the context of markets with proportional transaction costs $F_j^\varepsilon = F_j = K_j \cap (-K_j)$ is independent of $\varepsilon$, and represents the frictionless directions in the financial market. It is then natural to exclude these directions, as there is no possible improvement of the market conditions when exchanges are not subject to transaction costs.

2. In our context, we still need to exclude the linear space $F_j^\varepsilon$ although these directions are not necessarily feasible in our financial market, and can only be approximated by suitable sequences of strategies.

**Remark 2.2** The above condition $\mathbf{NA^r}$ is in agreement with the robust no-arbitrage condition introduced by Schachermayer (2004) in the context of a financial market with proportional transaction costs. This follows immediately from the fact that $K_j^\varepsilon = K_j$ and $F_j^\varepsilon = F_j$ when $K_j$ is a closed convex cone.

We next introduce our second concept of no-arbitrage, that we call weak no-arbitrage condition.

$\mathbf{NA^w}$ For every $\varepsilon > 0$, there is a convex cone $C^\varepsilon \subset L^0(\mathbb{R}^d; \mathscr{F}_N)$ such that

$$\mathbb{L}^0(\mathbb{R}^d_+, \mathscr{F}_N) \setminus \{0\} \subset \text{int}(C^\varepsilon) \quad \text{and} \quad A_N^\varepsilon \cap C^\varepsilon = \{0\}.$$

The interpretation of this condition is very similar to that of $\mathbf{NA^r}$. Condition $\mathbf{NA^w}$ says that there is no arbitrage even if we relax slightly the definition of positivity.

To the best of our knowledge, Condition $\mathbf{NA^w}$ has not been introduced before in the financial context. It is however related to a notion of efficiency in vector optimization theory, see e.g. Luc (1989).

**Remark 2.3** Clearly, $\mathbf{NA^w}$ implies $\mathbf{NA}$. When $A_N$ is known to be a closed convex cone, as in the case of financial markets with proportional transaction costs, we shall see later that equivalence holds between $\mathbf{NA^w}$ and $\mathbf{NA}$. See Remark (3.1).

We now state the two first main results of this paper, which will be proved in the subsequent sections.

**Theorem 2.1** *Condition* $\mathbf{NA^w}$ *holds if and only if there exists a nearly consistent price system.*

**Theorem 2.2** *Condition* $\mathbf{NA^r}$ *holds if and only if there exists a nearly strictly consistent price system.*

## 3 Characterization of the weak no-arbitrage condition

In this section, we prove the characterization of Condition $\mathbf{NA^w}$ stated in Theorem 2.1.

Fix $\varepsilon > 0$ and suppose $\mathbf{NA^w}$ holds. Then there is a closed convex cone $C^\varepsilon$ such that

$$\mathbb{L}^0(\mathbb{R}^d_+, \mathscr{F}_N) \setminus \{0\} \;\subset\; \mathrm{int}(C^\varepsilon) \quad \text{and} \quad A^\varepsilon_N \cap C^\varepsilon \;=\; \{0\} \;.$$

By the (large) separation Theorem, there exists a non-zero $Z^\varepsilon \in L^0(\mathbb{R}^d, \mathscr{F}_N)$ such that

$$\mathbb{E}[Z^\varepsilon \cdot Y] \;\le\; 0 \;\le\; \mathbb{E}[Z^\varepsilon \cdot X] \quad \text{for all} \quad Y \in A^\varepsilon_N \text{ and } X \in C^\varepsilon \,. \tag{3.1}$$

The second inequality in (3.1) shows that $Z^\varepsilon \in (C^\varepsilon)^*$. Since $\mathbb{L}^0(\mathbb{R}^d_+, \mathscr{F}_N) \setminus \{0\} \;\subset\; \mathrm{int}(C^\varepsilon)$, this implies that $Z^\varepsilon$ takes values in $(0,\infty)^d$. Set $Z^\varepsilon_j := \mathbb{E}\left[Z^\varepsilon | \mathscr{F}_j\right]$. Since $\mathbb{L}^0(-K^\varepsilon_j; \mathscr{F}_j) \subset A^\varepsilon_N$, the process $\{Z^\varepsilon_j, \, 0 \le j \le N\}$ is a martingale with values in $(0,\infty)^d$, and

$$\delta_{\mathbb{L}^0(-K^\varepsilon_j, \mathscr{F}_j)}(Z^\varepsilon_j) \;=\; 0 \quad \text{for every} \quad j \in \{0, \ldots, N\} \,. \tag{3.2}$$

Let $X$ be an arbitrary $\mathscr{F}_j-$measurable random variable with values in $-K^\varepsilon_j$, and set $X' := X \mathbf{1}_{\{Z^\varepsilon_j \cdot X > 0\}}$. Then $X' \in \mathbb{L}^0(-K^\varepsilon_j, \mathscr{F}_j)$, and it follows from (3.2) that $\mathbb{E}[X' \cdot Z^\varepsilon_j] \le 0$. Hence $\mathbb{P}(X \cdot Z^\varepsilon_j \le 0) = 1$. From the arbitrariness of $X \in \mathbb{L}^0(-K^\varepsilon_j, \mathscr{F}_j)$, we conclude that $Z^\varepsilon_j$ takes values in $K^{\varepsilon *}_j$.

Conversely, suppose there exists a consistent price system $\{Z^\varepsilon; \varepsilon > 0\}$. For any $\varepsilon > 0$, set

$$M^\varepsilon \;:=\; \left\{ X \in \mathbb{L}^0(\mathbb{R}^d_+, \mathscr{F}_N) \;:\; \mathbb{E}[Z^\varepsilon_N \cdot X] = 1 \right\} \,,$$

$$N^\varepsilon \;:=\; \left\{ X \in \mathbb{L}^0(\mathbb{R}^d; \mathscr{F}_N) \;:\; |\mathbb{E}[Z^\varepsilon_N \cdot X]| < \frac{1}{2} \right\} \,,$$

and

$$C^\varepsilon \;:=\; \overline{\mathrm{cone}}\,(M^\varepsilon + N^\varepsilon) \;.$$

Observe that

$$\mathbb{E}[Z^\varepsilon_N \cdot X] \;>\; 0 \quad \text{for every} \quad X \in C^\varepsilon \setminus \{0\} \;. \tag{3.3}$$

Clearly, $\mathbb{L}^0(\mathbb{R}^d_+, \mathscr{F}_N) \subset C^\varepsilon$. Since $\mathrm{int}\,(C^\varepsilon) \supset M^\varepsilon$, it follows that $\mathrm{int}\,(C^\varepsilon) \supset \mathbb{L}^0(\mathbb{R}^d_+, \mathscr{F}_N) \setminus \{0\}$. In order to conclude the proof, it remains to show that $A^\varepsilon_N \cap C^\varepsilon = \{0\}$. To see this, let $X := \sum_{j=0}^N \xi_j$, for some $\xi_j \in \mathbb{L}^0(-K^\varepsilon_j; \mathscr{F}_j)$, $0 \le j \le N$. Then $\mathbb{E}(X \cdot Z^\varepsilon_N) \le 0$ by definition of $Z^\varepsilon$, which contradicts (3.3) if $X \neq 0$. □

**Remark 3.1** Assume that $A_N$ is a closed convex cone, and let us show that the notion **NA** is equivalent to (the apparently stronger notion) $\mathbf{NA^w}$ . Under **NA** , it follows from a strict separation theorem that there exists a consistent price system $Z$ in the sense of Definition 2.1, see Kabanov and Stricker (2001). Next, define the set $C := \overline{\text{cone}}\,(M+N)$, where

$$M \quad := \quad \left\{ X \in \mathbb{L}^0\,(\mathbb{R}^d_+, \mathscr{F}_N) \ : \ \mathbb{E}[Z_N \cdot X] = 1 \right\} ,$$

and

$$N \quad := \quad \left\{ X \in \mathbb{L}^0\,(\mathbb{R}^d; \mathscr{F}_N) \ : \ |\mathbb{E}[Z_N \cdot X]| < \frac{1}{2} \right\} .$$

As in the previous proof, it is easily checked that $A_N \cap C = \{0\}$, and that $\mathbb{L}^0\,(\mathbb{R}^d_+, \mathscr{F}_N) \setminus \{0\} \subset \text{int}(C)$. Hence $\mathbf{NA^w}$ holds.

# 4 Characterization of the robust no-arbitrage condition

In this section, we prove the characterization of Condition $\mathbf{NA^r}$ stated in Theorem 2.2.

In the sequel, we denote by

$$T^\varepsilon_N \quad := \quad \overline{\text{cone}}\,(A^\varepsilon_N)$$

the closed cone generated by $A^\varepsilon_N$. In order to prove the required result, we shall first show that $\mathbf{NA^r}$ implies that the set $\text{cone}(A^\varepsilon_N)$ is closed, for every $\varepsilon > 0$. Then the robust no-arbitrage condition implies that $T^\varepsilon_N \cap \mathbb{L}^0\,(\mathbb{R}^d_+, \mathscr{F}_N) = \{0\}$, for all $\varepsilon > 0$, thus reducing our problem to the classical cone context. The rest of the proof essentially follows the arguments of Schachermayer (2004).

## 4.1 Closedness of attainable sets

**Lemma 4.1** *Let $C$ be a closed convex subset of $\mathbb{R}^d$, let $\varepsilon > 0$, and set $C^\varepsilon := C \cap \text{cone}(\Theta^\varepsilon)$, with $\Theta^\varepsilon := \left\{ x \in C \ : \ \sum_{i=1}^d |x_i| = \varepsilon \right\}$. Then the set $\text{cone}(C^\varepsilon)$ is closed in $\mathbb{R}^d$.*

**Proof.** Fix $\varepsilon > 0$. Since $\Theta^\varepsilon$ is a compact set that does not contain 0, $\text{cone}(C^\varepsilon)$ is a closed convex cone (see Rockafellar (1970), Corollary 9.6.1). Obviously, $\text{cone}(C^\varepsilon)$ $=\text{cone}(\Theta^\varepsilon)$. Hence, $\text{cone}(C^\varepsilon)$ is closed. □

To prove the closedness of cone $(A^\varepsilon_N)$, we verify the following criterion for the closedness of sums of closed convex cones.

**Lemma 4.2** *[Rockafellar (1970), Corollary 9.1.3] Let $C_1, \ldots, C_m$ be non-empty closed convex cones in $\mathbb{R}^n$, and set $D_j := C_j \cap (-C_j)$. Assume that*

$$\{(z_1, \ldots, z_m) \in C_1 \times \cdots \times C_m \ : \ z_1 + \cdots + z_m = 0\} \quad \subset \quad D_1 \times \cdots \times D_m\,.$$

*Then $C_1 + \cdots + C_m$ is a closed convex cone of $\mathbb{R}^n$.*

**Proposition 4.1** *Let Condition* $\mathbf{NA^r}$ *hold. Then* $\mathrm{cone}\,(A_N^\varepsilon)$ *is closed.*

**Proof.** 1. We first observe that

$$\mathrm{cone}\,(A_N^\varepsilon) \;=\; \sum_{j=0}^{N} \mathrm{cone}\left(\mathbb{L}^0\left(-K_j^\varepsilon, \mathscr{F}_j\right)\right) .$$

The inclusion $\mathrm{cone}\,(A_N^\varepsilon) \subset \sum_{j=0}^{N} \mathrm{cone}\left(\mathbb{L}^0\left(-K_j^\varepsilon, \mathscr{F}_j\right)\right)$ is trivial. Conversely, let $\lambda_j \geq 0$, and $\xi_j \in \mathbb{L}^0\left(-K_j^\varepsilon, \mathscr{F}_j\right)$ for $j \in \{0, \ldots, m\}$. Set $\bar{\lambda} = \max\{\lambda_0, \ldots, \lambda_m\}$, and consider two cases.

- If $\bar{\lambda} = 0$, then $\xi := \sum_{i=0}^{m} \lambda_i \xi = 0 \in \mathrm{cone}\,(A_N^\varepsilon)$.

-If $\bar{\lambda} > 0$, then $\xi := \bar{\lambda} \sum_{i=0}^{m} \hat{\lambda}_i \xi$, where $\hat{\lambda}_i := \bar{\lambda}^{-1} \lambda_i \in [0,1]$. By convexity of $K_j^\varepsilon$, we see that $\lambda_i \xi \in \mathbb{L}^0\left(-K_j^\varepsilon, \mathscr{F}_j\right)$, and therefore $\xi \in \mathrm{cone}\,(A_N^\varepsilon)$.

2. We next observe that

$$\mathrm{cone}\left(\mathbb{L}^0\left(-K_j^\varepsilon, \mathscr{F}_j\right)\right) \;=\; \mathbb{L}^0\left(-\mathrm{cone}\left(K_j^\varepsilon\right), \mathscr{F}_j\right)$$

for every $j \in \{0, \ldots, m\}$. We only need to prove that the right hand-side set is contained in the left hand-side one. Let $\lambda$ and $\xi$ be non-zero random variables with values in $\mathbb{R}_+$ and $-K_j^\varepsilon$, respectively. Set $\bar{\lambda} := \max\{\lambda(\omega) \;: \omega \in \Omega\}$. Then $\lambda\xi = \bar{\lambda}\hat{\xi}$, with $\hat{\xi} \in \mathbb{L}^0\left(-K_j^\varepsilon, \mathscr{F}_j\right)$ by convexity of $K_j^\varepsilon$.

3. In view of the previous steps, we have to show that

$$\sum_{j=0}^{N} \mathbb{L}^0\left(-\mathrm{cone}\left(K_j^\varepsilon\right), \mathscr{F}_j\right) \qquad \text{is closed.}$$

To do this, we shall verify the sufficient condition of Lemma 4.2 by adapting an argument from Schachermayer (2004). Let $\xi_j \in \mathbb{L}^0\left(-\mathrm{cone}\left(K_j^\varepsilon\right), \mathscr{F}_j\right)$, for $j \in \{0, \ldots, N\}$, be such that $\xi_0 + \cdots + \xi_N = 0$. Without loss of generality, we can assume that $\xi_j \in \mathbb{L}^0\left(-K_j^\varepsilon, \mathscr{F}_j\right)$. Suppose that $\xi_i \notin \mathbb{L}^0\left(-\mathrm{cone}\left(F_i^\varepsilon\right), \mathscr{F}_i\right)$ for some $i \in \{0, \ldots, N\}$, then it follows from $\mathbf{NA^r}$ that $\xi_i$ takes values in $\mathrm{int}\left(-\tilde{K}_i^\varepsilon\right)$. We then can find a random variable $\hat{\xi}_i \in \mathbb{L}^0\left(-\tilde{K}_i^\varepsilon, \mathscr{F}_i\right)$ such that $\hat{\xi}_i - \xi_i$ is a non-zero random variable with values in $\mathbb{R}_+^d$. set $\hat{\xi}_j = \xi_j$ for $j \neq i$, and observe that $\hat{\xi}_0 + \cdots + \hat{\xi}_N \in \tilde{A}_N^\varepsilon \cap \mathbb{L}^0(\mathbb{R}_+^d, \mathscr{F}_N)$. This is in contradiction with Condition $\mathbf{NA^r}$. $\square$

## 4.2 Some properties of closed convex sets

**Lemma 4.3** *Let $C$ be a convex subset of $\mathbb{R}^d$, and set $D := \overline{\text{cone}}(C) \cap \overline{\text{cone}}(-C)$. Then $D = \{0\}$ if and only if* $\text{int}\,(C^*) \neq \emptyset$.

**Proof.** The space $D$ is reduced to $\{0\}$ if and only if $\text{int}(\overline{\text{cone}}(C)^*) \neq \emptyset$. But $\overline{\text{cone}}(C)^* = C^*$, which concludes the proof □

Lemma 4.3 is the key-result in order to prove the following characterization of the relative interior of the polar of a closed convex set.

**Lemma 4.4** *Let $C \neq \emptyset$ be a closed convex subset of $\mathbb{R}^d$, and set $D_C := \overline{\text{cone}}(C) \cap \overline{\text{cone}}(-C)$. For $y \in \mathbb{R}^d$, the following properties are equivalent:*

(i) $y \in \text{ri}(C^*)$,

(ii) $y \in \tilde{C}^* \setminus \{0\}$ *for some non-empty closed convex set $\tilde{C} \subset \mathbb{R}^d$ such that $C \setminus D_C \subset \text{int}(\tilde{C})$.*

**Proof.** The proof relies on a reduction to the case where the space $D_C$ is $\{0\}$, as in Schachermayer (2004).

1. Since $D_C$ is a (closed) subspace, we consider the quotient space $\mathbb{R}^d/D_C$ and the canonical projection $\pi : \mathbb{R}^d \longrightarrow \mathbb{R}^d/D_C$ defined by $\pi(x) = \pi(y)$ if and only if $x - y \in D_C$. Set $\widehat{C} := C/D_C$, and observe that $D_{\widehat{C}} = D_C/D_C = \{0\}$. It then follows from Lemma 4.3 that $\text{int}\left(\widehat{C}^*\right) \neq \emptyset$.

Now, since $\pi$ is continuous, a vector $z \in \mathbb{R}^d$ is in $\text{ri}(C^*)$ if and only if $\pi(z)$ belongs to $\text{ri}(\widehat{C}^*) = \text{int}(\widehat{C}^*)$. Hence, in the following, we can assume without loss of generality that $D_C = \{0\}$.

2. Let $z$ be an arbitrary element of $\text{int}(C^*)$. Then

$$x \cdot z > 0 \quad \text{for all} \quad x \in C \setminus \{0\}.$$

Set $\tilde{C} := \{x \in \mathbb{R}^d : x \cdot z \geq 0\}$: we have $C \setminus \{0\} \subset \text{int}(\tilde{C})$.

Conversely, let $z \in \mathbb{R}^d \setminus \{0\}$ be satisfying (ii), i.e. there is a closed convex set $\tilde{C}$ such that $C \setminus \{0\} \subset \text{int}(\tilde{C})$, and

$$z \neq 0 \quad \text{and} \quad x \cdot z \geq 0 \quad \text{for all} \quad x \in \tilde{C}.$$

Then, if $x \in \text{int}(\tilde{C})$, $x \cdot z > 0$. In particular, for all $x \in C \setminus \{0\}$, $x \cdot z > 0$, i.e. $z \in \text{int}(C^*)$, which proves that (i) implies (ii). □

### 4.3 Proof of Theorem 2.2

Suppose $\mathbf{NA^r}$ holds and take an arbitrary $\varepsilon > 0$. Then, by Lemma 2.1 and Proposition 4.1, $\tilde{T}_N^\varepsilon \cap \mathbb{L}^0(\mathbb{R}_+^d, \mathscr{F}_N) = \{0\}$, where $\tilde{T}_N^\varepsilon$ is the tangent cone to $\tilde{A}_N^\varepsilon$ at 0. Hence, applying the separation theorem, for all $Y \in \mathbb{L}^0(\mathbb{R}_+^d, \mathscr{F}_N)$, there exists a random variable $Z^\varepsilon$ such that:

$$\forall X \in \tilde{T}_N^\varepsilon, \quad \mathbb{E}(Z^\varepsilon \cdot X) \leq 0 < \mathbb{E}(Z^\varepsilon \cdot Y).$$

Thus $Z^\varepsilon$ is in $\mathbb{L}^0(\mathbb{R}_+^d, \mathscr{F}_N)$ and satisfies

$$\delta_{\tilde{T}_N^\varepsilon}(Z^\varepsilon) = 0$$

which implies

$$\delta_{\tilde{A}_N^\varepsilon}(Z^\varepsilon) = 0$$

since $\tilde{A}_N^\varepsilon$ is a convex subset of $\tilde{T}_N^\varepsilon$ containing 0. Set $Z_j^\varepsilon = \mathbb{E}[Z_N^\varepsilon | \mathscr{F}_j]$ for all $j \in \{0, \ldots, N\}$. The adapted process $\{Z_j^\varepsilon,\, 0 \leq j \leq N\}$ is an $\mathbb{R}_+^d \backslash \{0\}$-valued martingale such that:

$$\delta_{\mathbb{L}^0(-\tilde{K}_j^\varepsilon, \mathscr{F}_j)}(Z_j^\varepsilon) = 0 \text{ for } 0 \leq j \leq N$$

since $\mathbb{L}^0(-\tilde{K}_j^\varepsilon; \mathscr{F}_j) \subset \tilde{A}_N^\varepsilon$ and $\mathbb{E}(Z_N^\varepsilon \cdot X) = \mathbb{E}[\mathbb{E}(Z_N^\varepsilon \cdot X)|\mathscr{F}_j] = \mathbb{E}(Z_j^\varepsilon \cdot X)$ for all $\mathscr{F}_j$-measurable random variable $X$.

Finally $Z_j^\varepsilon(\omega) \in \tilde{K}_j^{\varepsilon *}(\omega) \backslash \{0\}$ for all $\omega \in \Omega$. Indeed, as in the proof of Theorem 3.1, take $X \in \mathbb{L}^0(-\tilde{K}_j^\varepsilon; \mathscr{F}_j)$. The random variable $X' = X \mathbb{1}_{\{Z_j^\varepsilon \cdot X > 0\}}$ belongs to $\mathbb{L}^0(-\tilde{K}_j^\varepsilon; \mathscr{F}_j)$, so that $\mathbb{E}[X' \cdot Z_j^\varepsilon] \leq 0$, i.e. $\mathbb{P}(X \cdot Z_j^\varepsilon \leq 0) = 1$.

We deduce from Lemma 4.4 that $Z_j^\varepsilon \in \operatorname{ri} K_j^{\varepsilon *}$.

Conversely, suppose there exists a nearly strictly consistent price system $\{Z^\varepsilon\,;\, \varepsilon > 0\}$. We set, for all $\varepsilon > 0$, $j \in \{0, \ldots, N\}$ and $\omega \in \Omega$ :

$$-\tilde{K}_j^\varepsilon(\omega) = \{x \in \mathbb{R}^d : Z_j^\varepsilon(\omega) \cdot x \leq 0\}.$$

Then $-K_j^\varepsilon \setminus F_j^\varepsilon \subset \operatorname{int}(-\tilde{K}_j^\varepsilon)$. If $X = \sum_{j=0}^N \xi_j \in \tilde{A}_N \cap \mathbb{L}^0(\mathbb{R}_+^d, \mathscr{F}_N)$, let

$$\varepsilon := \min \left\{ \sum_{i=1}^d |(\xi_j(\omega))_i| \,;\, 0 \leq j \leq N,\, \omega \in \Omega,\, \xi_j(\omega) \neq 0 \right\}$$

Then:

$$\mathbb{E}(Z_N^\varepsilon \cdot X) \leq 0.$$

But since $X \in \mathbb{L}^0(\mathbb{R}_+^d, \mathscr{F}_N)$ and since the components of $Z_N^\varepsilon$ are positive, we deduce that $X = 0$. Thus $\tilde{A}_N \cap \mathbb{L}^0(\mathbb{R}_+^d, \mathscr{F}_N) = \{0\}$. □

# 5 Interpretation in Terms of the Liquidation Function

In this section, we assume that there is a *numéraire*, and that the liquidation value $P_j(x,y)$ of $(x,y)$ at time $j$ can be written as

$$P_j(x,y) = Q_j(x) + y$$

where $y \in \mathbb{R}$ is the "*numéraire* part" of the portfolio $(x,y) \in \mathbb{R}^d$. In addition, we assume that, for all $(j,\omega) \in \{0,\ldots,N\} \times \Omega$,

$$Q_j(\omega,\cdot) \quad \text{is concave and} \quad Q_j(\omega,0) = 0\,. \tag{5.1}$$

Notice that the concavity assumption on $Q_j(\omega,\cdot)$ can be weakened by assuming that $Q_j(\omega,\cdot)$ is locally concave at the origin.

Given a function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, its epigraph is the set defined by

$$\mathcal{E}\mathrm{pi}(f) = \{(x,r) \in \mathbb{R}^d \times \mathbb{R} \,:\, f(x) \le r\}$$

and its super-differential at $x_0 \in \mathbb{R}^d$ is the set defined by

$$\partial f(x_0) = \{y \in \mathbb{R}^d \,:\, \forall x \in \mathbb{R}^d,\ f(x) \le f(x_0) + y\cdot(x - x_0)\}.$$

**Lemma 5.1** *Let* $J = \{(x,y) \in \mathbb{R}^{d-1} \times \mathbb{R} : q(x) + y \ge 0\}$*, where* $q : \mathbb{R}^{d-1} \longrightarrow \mathbb{R}$ *is concave and let* $z = (z_1,z_2) \in \mathbb{R}^{d-1} \times (0,+\infty)$*,* $z \neq 0$*. Then:*

$$z \in J^* \Longleftrightarrow \frac{z_1}{z_2} \in \partial q(0)$$

*and*

$$z \in \mathrm{ri}(J^*) \Longleftrightarrow \frac{z_1}{z_2} \in \mathrm{ri}(\partial q(0)).$$

**Proof.**

$$\begin{aligned}
z \in J^* &\Longleftrightarrow \forall (x_1,x_2) \in J,\ x_1\cdot z_1 + x_2 z_2 \ge 0\\
&\Longleftrightarrow \left[\forall (x_1,x_2) \in \mathbb{R}^d,\ -q(x_1) \le x_2 \Longrightarrow x_2 \ge -\tfrac{x_1\cdot z_1}{z_2}\right]\\
&\Longleftrightarrow \forall x_1 \in \mathbb{R}^{d-1},\ q(x_1) \le \tfrac{z_1}{z_2}\cdot x_1\\
&\Longleftrightarrow \tfrac{z_1}{z_2} \in \partial q(0)
\end{aligned}$$

which proves the first assertion. To prove the second assertion, write

$$J^* = \left\{(z_1,z_2) : \frac{z_1}{z_2} \in \partial q(0)\right\} = \left\{z_2\left(\frac{z_1}{z_2},1\right)\ ;\ \frac{z_1}{z_2} \in \partial q(0)\right\}.$$

Then it follows from Corollary 6.8.1 in Rockafellar (1970) that $z \in \mathrm{ri}(J^*)$ if and only if $\frac{z_1}{z_2} \in \mathrm{ri}(\partial q(0))$. □

We next use the previous lemma in order to restate the characterizations of $\mathbf{NA^w}$ and $\mathbf{NA^r}$ in terms of the $Q_j$'s. For every $\varepsilon > 0$, we define

$$\check{Q}_j^\varepsilon(\omega,x) \quad := \quad \sup\left\{y \ : \ Q(\omega,x) \geq y \text{ and } (x,y) \in \mathrm{cone}(\Theta_j^\varepsilon)`\right\}$$

where $\Theta_j^\varepsilon := \left\{(x,y) \in K_j \ : \ |y| + \sum_{i=1}^{d-1} |x_i| = \varepsilon\right\}$.

Let

$$Q_j^\varepsilon(\omega,\cdot) \quad \text{be the u.s.c. concave hull of} \quad \check{Q}_j^\varepsilon(\omega,\cdot)\,.$$

**Proposition 5.1** *(i)* $\mathbf{NA^w}$ *holds if and only if there exists a family* $\{Z^\varepsilon\,;\,\varepsilon > 0\}$ *of* $(0,\infty)^{d-1} \times (0,\infty)$*-valued martingales* $Z^\varepsilon = (\bar{Z}^\varepsilon, \zeta^\varepsilon)$ *such that*

$$\frac{\bar{Z}_j^\varepsilon}{\zeta_j^\varepsilon} \ \in \ \mathbb{L}^0\left(\partial Q_j^\varepsilon(0), \mathscr{F}_j\right) \quad \textit{for all} \quad j \in \{0,\ldots,N\} \ \textit{and} \ \varepsilon > 0\,.$$

*(ii)* $\mathbf{NA^r}$ *holds if and only if there exists a family* $\{Z^\varepsilon\,;\,\varepsilon > 0\}$ *of* $(0,\infty)^{d-1} \times (0,\infty)$*-valued martingales* $Z^\varepsilon = (\bar{Z}^\varepsilon, \zeta^\varepsilon)$ *such that*

$$\frac{\bar{Z}_j^\varepsilon}{\zeta_j^\varepsilon} \ \in \ \mathbb{L}^0\left(\mathrm{ri}(\partial Q_j^\varepsilon(0)), \mathscr{F}_j\right) \quad \textit{for all} \quad j \in \{0,\ldots,N\} \ \textit{and} \ \varepsilon > 0\,.$$

**Proof.** We first remark that $K_j^\varepsilon = \mathcal{E}\mathrm{pi}(-Q) \cap \mathrm{cone}(\Theta_j^\varepsilon)$, so that

$$(K_j^\varepsilon)^* \quad = \quad (\mathcal{E}\mathrm{pi}(-Q) \cap \mathrm{cone}(\Theta_j^\varepsilon))^*\,.$$

Thus,

$$(K_j^\varepsilon)^* = (\mathcal{E}\mathrm{pi}(-\check{Q}_j^\varepsilon))^* = (\overline{\mathrm{co}}(\mathcal{E}\mathrm{pi}(-\check{Q}_j^\varepsilon)))^*.$$

Since $\overline{\mathrm{co}}(\mathcal{E}\mathrm{pi}(-\check{Q}_j^\varepsilon))$ is the epigraph of the l.s.c. convex hull $-Q_j^\varepsilon$ of $-\check{Q}_j^\varepsilon$, we conclude by applying Lemma 5.1. □

**Corollary 5.1** *Assume* $A_N$ *is a cone.*
*(i)* $\mathbf{NA^w}$ *holds if and only if there exists a* $(0,+\infty)^{d-1} \times (0,+\infty)$*-valued martingale* $Z = (\bar{Z},\zeta)$ *such that* $\frac{\bar{Z}_j}{\zeta_j} \in \mathbb{L}^0\left(\partial Q_j(0), \mathscr{F}_j\right)$*, for all* $j \in \{0,\ldots,N\}$.

*(ii)* $\mathbf{NA^r}$ *holds if and only if there exists a* $(0,+\infty)^{d-1} \times (0,+\infty)$*-valued martingale* $Z = (\bar{Z},\zeta)$ *such that* $\frac{\bar{Z}_j}{\zeta_j} \in \mathbb{L}^0\left(\mathrm{ri}\left(\partial Q_j(0)\right), \mathscr{F}_j\right)$*, for all* $j \in \{0,\ldots,N\}$.

**Proof.** This is a direct application of Theorems 2.1 and 2.2, and Lemma 5.1. □

## 6 Hedging

In this section, we focus on the problem of super-replication of contingent claims in our financial market with liquidity risk. Our main result is an extension of the classical dual formulation of this problem. We introduce the following notation: for every $\varepsilon > 0$, set

$$D_j^\varepsilon(\omega) \;:=\; \left\{ z \in \mathbb{R}^d \;:\; \sup_{x \in K_j^\varepsilon(\omega)} -x \cdot z < \infty \right\},$$

for $\omega \in \Omega$ and $j \in \{0, \ldots, N\}$. We then denote by $\mathscr{Z}^\varepsilon$ the collection of all martingales $Z = \{Z_j,\, 0 \le j \le N\}$ with

$$Z_j \;\in\; D_j^\varepsilon \cap (0,\infty)^d \quad \text{for all} \quad j \in \{0, \ldots, N\}.$$

Note that, from the results of the previous sections, $\mathscr{Z}^\varepsilon \neq \emptyset$ whenever $\mathbf{NA^w}$ or $\mathbf{NA^r}$ are satisfied. We also observe that

$$\delta_{A_N^\varepsilon}(Z_N) \;<\; \infty \quad \text{for every} \quad Z \;\in\; \mathscr{Z}^\varepsilon.$$

Indeed, for all $(\zeta_t, 0 \le t \le N) \in \prod_{j=0}^N \mathbb{L}^0(-K_j, \mathscr{F}_j)$, we have

$$\mathbb{E}\left[ Z_N \cdot \sum_{j=0}^N \zeta_j \right] \;=\; \sum_{j=0}^N \mathbb{E}[Z_j \cdot \zeta_j] \;\le\; \sum_{j=0}^N \delta_{\mathbb{L}^0(-K_j^\varepsilon, \mathscr{F}_j)}(Z_j),$$

and therefore

$$\delta_{A_N^\varepsilon}(Z_N) \le \sum_{j=0}^N \delta_{\mathbb{L}^0(-K_j^\varepsilon, \mathscr{F}_j)}(Z_j) \;<\; \infty.$$

We need the following sets:

$$\Gamma \;:=\; \left\{ y \in \mathbb{R}^d \;:\; y + Y_N = G \text{ for some } Y \in \mathscr{A} \right\}$$

and

$$D \;:=\; \left\{ y \in \mathbb{R}^d \;:\; \text{there exists } \varepsilon > 0 \text{ s.t. } \mathbb{E}[(G - y) \cdot Z_N] - \delta_{A_N^\varepsilon}(Z_N) \le 0 \text{ for all } Z \in \mathscr{Z}^\varepsilon \right\}.$$

**Theorem 6.1** *Let $G \in \mathbb{L}^0(\mathbb{R}^d, \mathscr{F}_N)$. Assume $\mathscr{Z}^\varepsilon \neq \emptyset$ for all $\varepsilon > 0$. Then*

$$\operatorname{int}(D) \subset \Gamma \subset D.$$

**Proof.** Let $y \in \Gamma$, i.e. $y = G - Y_N$ where $Y_N = \sum_{j=0}^N \xi_j$ for some $(\xi_0, \ldots, \xi_N) \in \prod_{j=0}^N \mathbb{L}^0(-K_j, \mathscr{F}_j)$. Set

$$\varepsilon \;:=\; \min\left\{ \sum_{i=1}^d |(\xi_j(\omega))_i| \;;\; 0 \le j \le N,\, \omega \in \Omega,\, \xi_j(\omega) \neq 0 \right\}.$$

Then

$$\mathbb{E}[(G-y)\cdot Z_N] - \delta_{A_N^\varepsilon}(Z_N) \quad \leq \quad 0$$

for all $Z \in \mathscr{Z}^\varepsilon$. Hence $y \in D$ and we have proved that

$$\Gamma \quad \subset \quad D\,.$$

Now, suppose that $y \notin \Gamma$. Then:

$$A_N^\varepsilon \cap \{G-y\} = \emptyset$$

for all $\varepsilon > 0$. Fix an arbitrary $\varepsilon > 0$. By the (large) separation Theorem, there exists a non-zero $Z$ such that

$$0 \;\leq\; \delta_{A_N^\varepsilon}(Z) \quad \leq \quad \mathbb{E}[(G-y)\cdot Z]\,. \tag{6.1}$$

Then $Z_j = \mathbb{E}(Z \mid \mathscr{F}_j)$ defines a martingale such that $Z_j \in D_j^\varepsilon$ $(0 \leq j \leq N)$.

Let $Z^\lambda := (1-\lambda)Z + \lambda\tilde{Z}$ for some $\tilde{Z} \in \mathscr{Z}^\varepsilon$ and $\lambda \in [0,1]$. Then $Z^\lambda \in \mathscr{Z}^\varepsilon$.

We deduce the following inequalities from (6.1) and the convexity of $\delta_{A_N^\varepsilon}$:

$$\begin{aligned}
0 \;&\leq\; \mathbb{E}\left[(G-y)\cdot Z^\lambda\right] - \delta_{A_N^\varepsilon}(Z^\lambda) + \mathbb{E}\left[(G-y)\cdot(Z-Z^\lambda)\right] - (\delta_{A_N^\varepsilon}(Z) - \delta_{A_N^\varepsilon}(Z^\lambda))\\
&\leq\; \mathbb{E}\left[(G-y)\cdot Z^\lambda\right] - \delta_{A_N^\varepsilon}(Z^\lambda) + \lambda\mathbb{E}\left[(G-y)\cdot(Z-\tilde{Z})\right] - \lambda(\delta_{A_N^\varepsilon}(Z) - \delta_{A_N^\varepsilon}(\tilde{Z}))\\
&\leq\; \sup_{Z\in\mathscr{Z}^\varepsilon}\left\{\mathbb{E}\left[(G-y)\cdot Z\right] - \delta_{A_N^\varepsilon}(Z)\right\}\\
&\quad\; +\lambda\left\{\mathbb{E}\left[(G-y)\cdot(Z-\tilde{Z})\right] - (\delta_{A_N^\varepsilon}(Z) - \delta_{A_N^\varepsilon}(\tilde{Z}))\right\}
\end{aligned}$$

and letting $\lambda \longrightarrow 0$, we have:

$$0 \quad \leq \quad \sup_{Z\in\mathscr{Z}^\varepsilon}\left\{\mathbb{E}\left[(G-y)\cdot Z\right] - \delta_{A_N^\varepsilon}(Z)\right\}. \tag{6.2}$$

Take any $Z^\varepsilon \in \mathscr{Z}^\varepsilon$ and set $y_n := y - \frac{1}{n}e$ where $e = (1,\ldots,1)$. By (6.2), we find:

$$\mathbb{E}[(G-y_n)\cdot Z_N^\varepsilon] - \delta_{A_N^\varepsilon}(Z_N^\varepsilon) \;=\; \mathbb{E}\left[(G-y)\cdot Z_N^\varepsilon\right] - \delta_{A_N^\varepsilon}(Z_N^\varepsilon) + \frac{1}{n}e\cdot Z_0^\varepsilon > 0$$

which shows that $y \in \overline{D^c}$. Since $y_n \to y$, we have $y \in \overline{D^c}$. Hence $\Gamma^c \subset \overline{D^c}$ and therefore $(\overline{D^c})^c \subset \Gamma$.

The required inclusion follows from the observation that $(\overline{D^c})^c = \mathrm{int}(D)$. □

**Remark 6.1** - When $D$ is closed (which is the case when $A_N$ is a cone), Theorem 6.1 says that

$$\overline{\Gamma} \quad = \quad D.$$

If in addition $A_N$ is closed, then

$$\Gamma = D.$$

This is the case in financial markets with proportional transaction costs.

- The support function of $A_N^\varepsilon$ plays the role of a penalty function. When $A_N$ is a cone, this penalty function reduces to the indicator function of some closed convex subset, and we recover the usual result for markets with transaction costs.

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