Modeling continuous-time financial markets with capital gains taxes *

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September 2005

Abstract

We formulate a model of continuous-time financial market consisting of a bank account with constant interest rate and one risky asset subject to transaction costs and capital gains taxes. We consider the problem of maximizing expected utility from future consumption in infinite horizon. This is the continuous-time version of the model introduced by Dammon, Spatt and Zhang [11]. The taxation rule is linear so that it allows for tax credits when capital gains losses are experienced. In this context, wash sales are optimal. Our main

*We are grateful to Stathis Tompaidis for numerous discussions on the modeling issue of this paper. We have also benefited from interesting comments and discussions with Victor DeMiguel, Chris Rogers, Jose Scheinkman, and Raman Uppal.

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contribution is to derive lower and upper bounds on the value function in terms of the corresponding value in a tax-free and frictionless model. While the upper bound corresponds to the value function in a tax-free model, the lower bound is a consequence of wash sales. As an important implication of these bounds, we derive an explicit first order expansion of our value function for small interest rate and tax rate coefficients. In order to examine the accuracy of this approximation, we provide a characterization of the value function in terms of the associated dynamic programming equation, and we suggest a numerical approximation scheme based on finite differences and the Howard algorithm. The numerical results show that the first order Taylor expansion is very accurate for reasonable market data.

Key Words and phrases: Optimal consumption and investment in continuous-time, transaction costs, capital gains taxes, finite differences.

JEL classification: G11, E21, C61,C63

1 Introduction

Since the seminal papers of Merton [24],[25], see also Duffie [13], there has been an extensive literature on the problem of optimal consumption and investment decision in financial markets subject to imperfections. We refer to Cox and Huang [7] and Karatzas, Lehoczky and Shreve [20] for the case of incomplete markets, Cvitanić and Karatzas [9] for the case portfolio constraints, Constantinides and Magill [6], Davis and Norman [10], Shreve and Soner [27], Duffie and Sun [14] for the case of transaction costs.

However, the problem of taxes on capital gains received a very limited attention, although taxes represent a much higher percentage than transaction costs in real securities markets. Compared to ordinary income, capital gains are taxed only when the investor sells the security, allowing for a deferral option. One may think that the taxes on capital gains have an appreciable impact on individuals consumption and investment decisions. Indeed, under taxation of capital gains, the portfolio rebalancement implies additional charges, therefore altering the available wealth for future consumption. This possibly induces a depreciation of consumption opportunities compared to a tax-free market. On the other hand, since taxes are paid only when embedded capital gains are actually realized, the investor may choose to defer the realization of capital gains and liquidate his position in case of a capital loss, particularly when the tax code allows for tax credits.

Previous works attempted to characterize consumption and investment decisions of investors who have permanently to choose between two conflicting issues : realize the transfers needed for an optimally diversified portfolio, or use the ability to defer capital gains taxes. The first relevant work is due to Constantinides [5] who shows that the investment and consumption decisions are separable, and that the optimal strategy consists in realizing losses and deferring gains. These results rely heavily on the possibility of short-selling the risky asset. Since capital gain realization are observed in real securities markets, the subsequent literature considers the problem under the no short-sales constraint.

In a multi-period context many challenging difficulties appear because of the path dependency of the problem. The taxation code specifies the basis to which the price of a security has to be compared in order to evaluate the capital gains (or losses). The tax basis is either defined as (i) the specific purchase price of the asset to be sold, (ii) the purchase price of any asset held in the portfolio , or (iii) the weighted average of past purchase prices. In some countries, investors can chose either one of the above definitions of the tax basis.

A deterministic model with the above definition (i) of the tax basis, together with the *first in first out* priority rule for the stock to be sold, has been introduced by Jouini, Koehl and Touzi [18], [19]. An existence result is proved, and the first order conditions of optimality are derived under some conditions. However, the numerical complexity due to the path dependency of the problem was not solved in the context of this model.

A financial model with the above definition (ii) of the taxation rule was considered by Dybvig and Koo [15] in the context of a four-periods binomial model. Some numerical progress was achieved later by DeMiguel and Uppal [12] who were able to consider more periods in the binomial model and/or more stocks. This numerical progress is very limited as these authors were not able to go beyond ten periods in the single-asset framework.

The taxation rule (iii), where the tax basis is the weighted average of past purchase prices, was first considered by Dammon, Spatt and Zhang [11] in the context of the context of a binomial model with short sales constraints and linear taxation rule. The average tax basis is actually used in Canada. Dammon, Spatt and Zhang [11] considered the problem of maximizing the expected discounted utility from future consumption, and provided a numerical analysis of this model based on the dynamic programming principle. The important technical feature of this model is that that the path dependency of the problem is seriously reduced, as the dynamics of the tax basis is Markov. This implies a significant advantage of this model in comparison to [15]. This advantage was further justified by [12] who provided a numerical evidence that the certainty equivalent loss from using the average tax basis (iii) instead of the exact tax basis (ii) is typically less that 1% for a large choice of parameter values.

The analysis of Dammon, Spatt and Zhang was further extended to the multi-asset framework by Gallmeyer, Kaniel and Tompaidis [16]. We also refer to Leland [21] who formulated a similar model to ours, but considered the problem of minimizing the tracking error to a Benchmark index.

In this paper, we formulate a continuous time version of the Dammon, Spatt and Zhang utility maximization problem under capital gains taxes, see Section 2. The financial market consists of a tax-free riskless asset and a risky one. Transfers between the two assets are possibly subject to proportional transaction costs. The holdings in risky asset are subject to the no-short sales constraint, and the total wealth is restricted by the no-bankruptcy condition. The risky asset is subject to taxes on capital gains. The tax basis is defined as the weighted average of past purchase prices, and the taxation rule is linear, thus allowing for tax credits.

In the context of this financial market, we consider the problem of maximizing expected utility from future consumption in infinite horizon. The investor preferences are described by the power utility which exhibits a constant relative risk aversion coefficient. This simplification is only needed in order to reduce the numerical complexity by taking advantage of the homogeneity property of the power utility function.

As in the tax-free case of Constantinides and Magill [6], and Davis and Norman [10], there is no explicit description of the value function and the optimal consumption-investment policy. We therefore concentrate on the approximation aspect and we obtain the two following main results.

• In Section 5, we derive a general upper bounds on the value function, and a lower bound in the transaction costs free model.

A first important implication of these bounds, is an *explicit* first order Taylor expansion of the value function when there are no transaction costs. This explicit approximation of the value function is valid for models with small interest rate and tax parameters. However, our numerical experiments indicate that this approximation is satisfactory with realistic values of interest rate and tax parameters.

The lower bound is derived as the limit of the value implied by a sequence of strategies which mimics the Merton optimal strategy in a Merton-type fictitious frictionless financial market with tax-deflated drift and volatility coefficients. The risk premium of this fictitious financial market is smaller than that of the original market. So, even if the optimal strategy in our problem is not available in explicit form, our first order expansion is accompanied by an explicit strategy which achieves "the first order maximal utility value". Therefore, this sequence of strategies can be viewed as a first-order maximizing sequence for the problem of optimal investment under capital gains taxes.

The investment component of this approximation sequence exhibits a smaller exposition to the risky asset, which is consistent with the numerical results of Dammon, Spatt and Zhang [11] in the context of the tax forgiveness at death hypothesis. Then, the presence of taxes appears as a possible explanation of the risk premium puzzle highlighted by Mehra and Prescott [23].

• In order to evaluate the accuracy of our first order Taylor expansion, we report in Section 6 a characterization of the value function in terms of the associated dynamic programming equation. The rigorous derivation of these results involves heavy technicalities and is therefore reported in the accompanying paper [4]. In order to obtain a satisfactory uniqueness result, which is crucial for the justification of our numerical results of this paper, we introduce in [4] a convenient approximation of our value function.

As a technical by-product of our analysis, we obtain the continuity (and even Lipschitz-continuity, up to a change of variable) of the value function. We recall that, in the tax-free models of [24, 6, 10], the value function is immediately seen to be concave, and the continuity is therefore trivial. Under capital gains taxes, this argument fails, and the numerical results of Section 7 suggest that the value function is indeed not concave.

This characterization of the value function, in terms of the associated dynamic programming equation, is exploited in order to define a numerical approximation based on the finite differences and the Howard algorithm. The convergence of our numerical procedure is guaranteed by the general result of Barles and Souganidis [3]. The precise description of our algorithm together with some numerical results are displayed in Section 7. In particular, for reasonable market data, our explicit first order Taylor expansion of the value function is remarkably close to the numerical approximation obtained by the finite differences algorithm.

The numerical approximation of the optimal strategy displays a bang-bang behavior as expected in our singular control problem. As in the transaction costs context of [10], the state space is partitioned in three regions : the *no-transaction region* NT, the *buy region* B, and the *sell region* S. In NT, the optimal investor holds his position on the financial market, and does not perform any trading. In S, the optimal trader sells immediately part of his holdings in risky assets so that his position is instantaneously removed to the NT region. In particular, this region contains all capital loss positions, since wash sales are shown to be optimal in the absence of transaction costs. Finally, in the B region, the optimal investor buys immediately some amount of risky asset, thus removing instantaneously the position to the NT region. In contrast with the transaction costs framework of [10], these regions are not cones.

NOTATIONS : For a domain $\mathbf{D}$ in $\mathbb{R}^n$, we denote by $\mathrm{USC}(\mathbf{D})$ (resp. $\mathrm{LSC}(\mathbf{D})$) the collection of all upper semi-continuous (resp. lower semi-continuous) functions from $\mathbf{D}$ to $\mathbb{R}$. The set of continuous functions from $\mathbf{D}$ to $\mathbb{R}$ is denoted by $\mathrm{C}^0(\mathbf{D}) := \mathrm{USC}(\mathbf{D}) \cap \mathrm{LSC}(\mathbf{D})$. For a parameter $\delta > 0$, we say that a function $f : \mathbf{D} \longrightarrow \mathbb{R}$ has $\delta-$polynomial growth if

$$\sup_{x\in\mathbf{D}} \frac{|f(x)|}{1+|x|^\delta} \;<\; \infty\,.$$

We finally denote by $\mathrm{USC}_\delta(\mathbf{D}) := \{f \in \mathrm{USC}(\mathbf{D}) \;:\; f \text{ has } \delta\text{−polynomial growth}\}$. The sets $\mathrm{LSC}_\delta(\mathbf{D})$ and $\mathrm{C}^0_\delta(\mathbf{D})$ are defined similarly.

2 Consumption-investment models with capital gains taxes

2.1 The financial assets

Throughout this paper, we consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with a standard scalar Brownian motion $W = \{W_t, 0 \le t\}$, and we denote by $\mathbb{F}$ the $\mathbb{P}$-completion of the natural filtration of the Brownian motion.

We consider a financial market consisting of one bank account with constant interest rate $r > 0$, and one risky asset with price process evolving according to the Black and Scholes model :

$$dP_t \;=\; P_t\left[(r+\theta\sigma)dt \;+\; \sigma\, dW_t\right]\,, \tag{2.1}$$

where $\theta > 0$ is a constant risk premium, and $\sigma > 0$ is a constant volatility parameter. The positivity restriction on the risk premium coefficient ensures that positive investment in the risky asset is interesting. The shares of stock are assumed to be infinitely divisible.

The transfers of wealth between the two assets are subject to constant proportional transaction costs represented by the coefficients $\lambda, \mu \in [0,1)$, so that the bid and ask prices at time t of the risky asset are given by $(1-\mu)P_t$ and $(1+\lambda)P_t$.

Remark 2.1 The only place where positive transaction costs parameters $\lambda, \mu > 0$ are needed is the uniqueness results of Proposition 6.2 and Theorem 6.1. These uniqueness results are crucial for the convergence of the numerical scheme of Section 7 based on the finite differences and the Howard algorithm.

2.2 Taxation rule on capital gains

The sales of the stock are subject to *taxes on capital gains*. The amount of tax to be paid for each sale of risky asset, at time t, is computed by comparison of the current price P_t to an index B_t defined as the weighted average price of the shares purchased by the investor up to time t. When $P_t \leq B_t$, i.e. the current price of the risky asset is greater than the weighted average price, the investor would *realize a capital gain* by selling the risky asset. Similarly, when $P_t \geq B_t$, the sale of the risky asset corresponds to the *realization of a capital loss*.

In order to better explain the definition of the tax basis B, we provide the following example taken from the official Canadian tax code, see the document *Capital Gains 2004* p21 on www.cra.gc.ca.

The following table reports transactions performed by an individual on shares of STU Ltd, and how the tax basis of the individual changes over time

Transaction	Price P (Dollars)	number of shares (unitless)	Portfolio (unitless)	Tax basis B (Dollars)
Purchase at t_1	15.00	100	100 : 15.00/share	**15.00**
Purchase at t_2	20.00	150	100 : \$15.00/share	
			150 : \$20.00/share	**18.00**
Sale at t_3	-	200	20 : \$15.00/share	
		$= \frac{4}{5}(100 + 150)$	30 : \$20.00/share	**18.00**
Purchase at t_4	21.00	350	20 : \$15.00/share	
			30 : \$20.00/share	
			350 : \$21.00/share	**20.625**

Just after a sale transaction, the tax basis is not changed. However sales do alter the tax basis starting from the date of the next purchase. Notice however that the tax basis is only affected by the number of shares which has been sold, and not by the sale price.

The sale of a unit share of stock at some time t is subject to the payment of an amount of tax computed according to the tax basis of the portfolio at time t. In this paper we consider a linear taxation rule, i.e. this amount of tax is given by

$$\ell(P_t - B_t) \quad := \quad \alpha(1-\mu)\,(P_t - B_t) \ , \tag{2.2}$$

where $\alpha \in (0,1)$ is a constant tax rate coefficient. When the tax basis is smaller than the spot price, the investor realizes a capital gain. Then, by selling one unit of risky asset at the bid price $(1-\mu)P_t$, the amount of tax to be paid is $(1-\mu)\alpha(P_t - B_t)$. When the tax basis is larger than the spot price, the investor receives the tax credit $(1-\mu)\alpha(B_t - P_t)$ for each unit of asset sold at time t.

Remark 2.2 Our definition of the tax basis B is slightly different from that of

Dammon, Spatt and Zhang [11] who set the tax basis to be equal to the spot price whenever the average purchase price exceeds the current price. In the absence of transaction costs, this does not affect the results, see Proposition 5.4 below.

Remark 2.3 In the context of the linear taxation rule, it is not obvious that the investor can not take advantage of these tax credits and do better than in a tax-free market. Of course, this would not be acceptable from the economic viewpoint. The upper bound which will be stated in Proposition 5.5 shows that the presence of tax credits does not produce such a non-desirable effect.

2.3 Consumption-investment strategies

We denote by X_t the position on the bank, Y_t the position on the risky assets account, and

$$K_t \;:=\; B_t \frac{Y_t}{P_t}, \qquad t \geq 0, \tag{2.3}$$

the position on the risky asset account evaluated at the basis price. The trading in risky asset is subject the no-short sales constraint

$$Y_t \;\geq\; 0 \qquad \mathbb{P}-\text{a.s.} \quad \text{for all} \quad t \geq 0, \tag{2.4}$$

and the position of the investor is required to satisfy the solvency condition

$$\begin{aligned} Z_t &:= X_t + (1-\mu)Y_t - \ell\,(P_t - B_t)\;\frac{Y_t}{P_t} \\ &= X_t + (1-\mu)\left[(1-\alpha)Y_t + \alpha K_t\right] \;\geq\; 0 \qquad \mathbb{P}-\text{a.s.} \end{aligned} \tag{2.5}$$

i.e. the total wealth of the investor, after liquidation of the risky asset position, is non-negative at any point in time.

Trading on the financial market is described by means of the transfers between the two investment opportunities defined by two $\mathbb{F}-$adapted, right-continuous and non-decreasing processes $L = \{L_t,\, t \geq 0\}$ and $M = \{M_t,\, t \geq 0\}$ with $L_{0^-} = M_{0^-} = 0$.

The amount transferred from the bank to the non-risky asset account at time t is given by dL_t and corresponds to a purchase of risky asset. The amount transferred from the risky asset account to the bank at time t is given by $Y_{t-}dM_t$ and corresponds to a sale of risky asset. The example of calculation of the tax basis of a portfolio, displayed in the above table, shows the importance of expressing the sales in terms of proportions of the total holdings in risky asset.

In order to ensure that the no short-sales constraint (2.4) holds, we restrict the jumps of M by

$$\Delta M_t \;\leq\; 1 \quad \text{for} \quad t \geq 0 \;\; \mathbb{P}-\text{a.s.} \tag{2.6}$$

With these notations, the evolution of the wealth on the risky asset account is given by

$$dY_t \;=\; Y_t \frac{dP_t}{P_t} + dL_t \;-\; Y_{t-}dM_t \,. \tag{2.7}$$

and, by definition of the tax basis B and (2.3), we have :

$$dK_t \;=\; dL_t \;-\; K_{t-}dM_t \,. \tag{2.8}$$

Observe that the contribution of the sales in the dynamics of K_t is evaluated at the basis price. For any given initial condition (Y_{0-}, K_{0-}) equations (2.7)-(2.8) define a unique $\mathbb{F}-$adapted process (Y, K) with values in $\mathbb{R}^2_+$, the non-negative orthant of $\mathbb{R}^2$.

In addition to the trading activities, the investor consumes in continuous time at the rate $C = \{C_t,\, t \geq 0\}$. Here, C is an $\mathbb{F}-$adapted process with

$$C \;\geq\; 0 \quad \text{and} \quad \int_0^T C_t dt \;<\; \infty \;\; \mathbb{P}-\text{a.s. for all } T > 0\,. \tag{2.9}$$

Then, the bank component of the wealth process satisfies the dynamics

$$\begin{aligned} dX_t &= (rX_t - C_t)\, dt - (1+\lambda)dL_t + (1-\mu)Y_{t-}dM_t - \ell\,(P_t - B_{t-})\,\frac{Y_{t-}dM_t}{P_t} \\ &= (rX_t - C_t)\, dt - (1+\lambda)dL_t + (1-\mu)\,[(1-\alpha)Y_{t-} \;+ \alpha K_{t-}]\, dM_t\,. \end{aligned} \tag{2.10}$$

Since the processes Y and K have been previously defined, the above dynamics uniquely defines an $\mathbb{F}-$adapted process X valued in $\mathbb{R}$, for any given initial condition X_{0-}.

For later use, we report the dynamics of the corresponding liquidation value process defined in (2.5), which follows from (2.7)-(2.8)-(2.10) :

$$dZ_t = (rZ_t - C_t)\,dt + (1-\mu)\left[(1-\alpha)\,Y_t\left(\frac{dP_t}{P_t} - rdt\right) - r\alpha K_t dt\right] - (\lambda+\mu)dL_t\,. \tag{2.11}$$

Definition 2.1 (i) *A consumption investment strategy is a triple of $\mathbb{F}-$adapted processes $\nu = (C, L, M)$ where C satisfies (2.9), L, M are non-decreasing, right-continuous, $L_{0-} = M_{0-} = 0$, and the jumps of M satisfy (2.6).*
(ii) *Given an initial condition $s = (x, y, k) \in \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+$, and a consumption-investment strategy ν, we denote by $S^{s,\nu} = (X^{s,\nu}, Y^{s,\nu}, K^{s,\nu})$ the unique strong solution of (2.10)-(2.7)-(2.8) with initial condition $S^{s,\nu}_{0-} = s$.*
(ii) *Given an initial condition $s = (x, y, k) \in \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+$, a consumption-investment strategy ν is said to be $s-$admissible if the corresponding state process $S^{s,\nu}$ satisfies the no-bankruptcy constraint (2.5). We shall denote by $\mathcal{A}(s)$ the collection of all $s-$admissible consumption-investment strategies.*

The admissibility conditions imply that the process $S^{s,\nu}$ is valued in the closure $\bar{\mathcal{S}}$ of

$$\mathcal{S} \;=\; \left\{(x,y,k)\in\mathbb{R}^3 \;:\; x + (1-\mu)\left[(1-\alpha)y + \alpha k\right] > 0\;,\; y>0\;,\; k>0\right\}\;. \tag{2.12}$$

We partition the boundary of $\mathcal{S}$ into $\partial\mathcal{S} \;=\; \partial^z\mathcal{S}\,\cup\,\partial^y\mathcal{S}\,\cup\,\partial^k\mathcal{S}$ with

$$\partial^y\mathcal{S} \;:=\; \left\{(x,y,k)\in\bar{\mathcal{S}} \;:\; y=0\,\right\}\;,\qquad \partial^k\mathcal{S} \;:=\; \left\{(x,y,k)\in\bar{\mathcal{S}} \;:\; k=0\,\right\}\;,$$

and

$$\partial^z\mathcal{S} \;=\; \left\{(x,y,k)\in\bar{\mathcal{S}} \;:\; z := x + (1-\mu)\left[(1-\alpha)y + \alpha k\right] = 0\,\right\}\;.$$

2.4 The consumption-investment problem

The investor preferences are characterized by a power utility function with constant relative risk aversion coefficient $1-p \in (0,1)$:

$$U(c) \ := \ \frac{c^p}{p} \quad \text{for all} \quad c \geq 0\,.$$

The restriction of the relative risk aversion coefficient to the interval $(0,1)$ does not correspond to observed values on real financial markets. We do impose this condition in order to simplify the analysis of this paper, as the boundary condition on $\partial^z\mathcal{S}$ is easily obtained, see Proposition 5.3.

For every initial data $s \in \bar{\mathcal{S}}$ and any admissible strategy $\nu \in \mathcal{A}(s)$, we introduce the investment-consumption criterion

$$J_T(s,\nu) \ := \ \mathbb{E}\left[\int_0^T e^{-\beta t}U(C_t)dt + e^{-\beta T}U(Z_T^{s,\nu})\mathbf{1}_{\{T<\infty\}}\right]\,, \quad T \in \mathbb{R}_+ \cup \{+\infty\}\,. \tag{2.13}$$

The consumption-investment problem is defined by

$$V(s) \ := \ \sup_{\nu \in \mathcal{A}(s)} \ J_\infty(s,\nu)\,, \qquad s \in \bar{\mathcal{S}}\,. \tag{2.14}$$

We shall assume that the parameters r, θ, σ, p and β satisfy the condition :

$$\frac{\beta}{p} - r - \frac{\theta^2}{2(1-p)} \quad > \quad 0\,, \tag{2.15}$$

which has been pointed out as a sufficient condition for the finiteness of the value function in the context of a financial market without taxes in [24] and [27].

Remark 2.4 In the absence of taxes on capital gains, i.e. $\alpha = 0$, it is easy to deduce from the concavity of U that the value function V is concave. The numerical results exhibited in Section 7 reveal that this property is no longer valid when $\alpha > 0$.

3 Review of the tax-free models

In this section, we briefly review the solution of the consumption-investment problem when the financial market is free from taxes on capital gains. The properties of the corresponding value function are going to be useful to state relevant bounds for the maximal utility achieved in a financial market with taxes.

3.1 The Merton frictionless model

In the classical formulation of the tax-free consumption-investment problem [24], the investment control variable is described by means of a unique process π which represents the proportion of wealth invested in risky assets at each time, and the consumption process C is expressed as a proportion c of the total wealth :

$$d\bar{Z}_t \;=\; \bar{Z}_t \left[(r - c_t)dt + \pi_t\sigma(\theta dt + dW_t)\right] \;. \tag{3.1}$$

In this context, a consumption-investment admissible strategy is a pair of adapted processes (c, π) such that c is nonnegative and

$$\int_0^T c_t dt \;+\; \int_0^T |\pi_t|^2 dt \;<\; \infty \quad \mathbb{P}-\text{a.s. for all} \quad T \;>\; 0 \;.$$

We shall denote by $\bar{\mathcal{A}}$ the collection of all such consumption-investment strategies. For every initial condition $z \geq 0$ and strategy $(c, \pi) \in \bar{\mathcal{A}}$, there is a unique strong solution to (3.1) that we denote by $\bar{Z}^{z,c,\pi}$. The frictionless consumption-investment problem is

$$\bar{V}(z) \;:=\; \sup_{(c,\pi)\in\bar{\mathcal{A}}} \mathbb{E}\left[\int_0^\infty e^{-\beta t} U\left(c_t \bar{Z}_t^{z,c,\pi}\right) du\right] \;. \tag{3.2}$$

Theorem 3.1 ([24]) *Let Condition* (2.15) *hold. Then, for all* $z \geq 0$ *:*

$$\bar{V}(z) \;=\; \gamma(r,\theta)\,\frac{z^p}{p} \;, \quad \textit{where} \quad \gamma(r,\theta) \;:=\; \left(\frac{\beta - p\,r}{1-p} - \frac{p\,\theta^2}{2\,(1-p)^2}\right)^{p-1} \;,$$

and the constant consumption-investment strategy

$$\bar{\pi} := \frac{\theta}{(1-p)\sigma}, \qquad \bar{c} := \gamma(r, \theta),$$

is optimal.

Remark 3.1 The reduction of the model of Section 2 to the frictionless case, i.e. $\alpha = \lambda = \mu = 0$, does not alter the value function. Indeed, the tax-free version of our model ($\alpha = 0$) is known to converge to the Merton value function as the transaction costs coefficients shrink to zero, see e.g. [17]. However, the investment strategies in our formulation are constrained to have bounded variation. Since the Merton optimal strategy is well-known to be unique and has unbounded variation, it follows that existence fails to hold in our formulation.

3.2 The consumption-investment problem under transaction costs

We now consider the reduction of the model of Section 2 to the case $\alpha = 0$. Then, the state variable K (or equivalently B) does not affect the problem. Given an initial position $s^0 = (x, y)$ in the bank and the stock account, we define the set of s^0-admissible consumption-investment strategies $\mathcal{A}^0(s^0) := \mathcal{A}(x, y, 0)$, and we re-write the tax-free problem into

$$V^0\left(s^0\right) = \sup_{(C,L,M)\in\mathcal{A}^0(s^0)} \mathbb{E}\left[\int_0^\infty e^{-\beta t} U\left(C_t\right) dt\right] \tag{3.3}$$

which is now defined on the domain

$$\bar{\mathcal{S}}^0 := \left\{ s^0 = (x, y) \in \mathbb{R}^2 \ : \ x + (1-\mu)y \geq 0 \ \text{ and } \ y \geq 0 \right\}. \tag{3.4}$$

This problem was first formulated by Constantinides and Magill [6]. The authors found that the investor trades only when the proportion of wealth allocated to the risky asset is outside a certain region, and they conjectured that this region is a cone containing Merton's optimal line. Davis and Norman [10] provided a rigorous formulation and analysis for this problem and proved that the optimal buying and selling policies are the local times of the two-dimensional process of bank and stock holdings at the boundaries of a wedge shaped region. Their work has been revisited and completed by Shreve and Soner [27]. Observe that our model differs slightly from [10] and [27] by (i) the presence of the no-short sales constraint (2.4), and (ii) the transfers from the stock account to the bank in our model are expressed as a proportion of the wealth on the stock account. This does not alter the solution of the problem, since the optimal strategy outlined in the previous literature induces a positive optimal holding in stock, see Theorem 9.2 in [27].

We next partition the boundary of the set $\bar{\mathcal{S}}^0$ into

$$\partial^y\mathcal{S}^0 := \left\{(x,y) \in \bar{\mathcal{S}}^0 \ : \ y = 0\right\} \quad \text{and} \quad \partial^z\mathcal{S}^0 := \left\{(x,y) \in \bar{\mathcal{S}}^0 \ : \ x+(1-\mu)y = 0\right\},$$

and introduce two vectors in $\mathbb{R}^2$:

$$\mathbf{G^b} := \begin{pmatrix} 1+\lambda \\ -1 \end{pmatrix}, \ \mathbf{G^s} := \begin{pmatrix} -(1-\mu) \\ 1 \end{pmatrix}.$$

For a smooth function $\varphi : \bar{\mathcal{S}}^0 \longrightarrow \mathbb{R}$ and $s^0 = (x,y) \in \bar{\mathcal{S}}^0$, we denote

$$\mathcal{L}\varphi\left(s^0\right) := -\beta\varphi\left(s^0\right) + rx\varphi_x\left(s^0\right) + (r+\theta\sigma)y\varphi_y\left(s^0\right) + \frac{1}{2}\sigma^2y^2\varphi_{yy}\left(s^0\right) + \tilde{U}\left(\varphi_x\left(s^0\right)\right), \tag{3.5}$$

where

$$\tilde{U}(\xi) := \sup_{c>0}\left[U(c) - c\xi\right], \quad \text{for all } \xi > 0 .$$

Theorem 3.2 *Let $\lambda+\mu > 0$, and assume that Condition (2.15) holds. Then, the function V^0 is the unique constrained viscosity solution in the class $\mathrm{C}^0_p\left(\bar{\mathcal{S}}^0\right)$ of*

$$\begin{cases} \min\left\{-\mathcal{L}V^0 , \ \mathbf{G^b}\cdot DV^0 , \ \mathbf{G^s}\cdot DV^0\right\} = 0 \ \ \textit{in } \bar{\mathcal{S}}^0 \setminus \partial^z\mathcal{S}^0 \\ V^0(x,y) = 0 \ \ \textit{for } (x,y) \in \partial^z\mathcal{S}^0 , \end{cases} \tag{3.6}$$

i.e. V^0 *is a viscosity supersolution (resp. subsolution) of* (3.6) *on* $\bar{\mathcal{S}}^0 \setminus \partial^y \mathcal{S}^0$ *(resp. on* $\bar{\mathcal{S}}^0$*).*

We do not report the precise definition of the notion of constrained viscosity solutions, as it is similar to Definition 6.1. For the general definition, we refer to [8].

The above viscosity property corresponds to Theorem 3.2 of [4] in the particular case $\alpha = 0$, and differs from the results reported in [27] by the state constraint $y \geq 0$.

Further characterization of the solution of this problem is available in [10] and [27]. But no explicit solution is available. An asymptotic expansion of the value function for small transaction costs has been obtained by Whalley and Wilmott [29], and Janeček and Shreve [17]. We also refer to Barles and Soner [2] for similar approximation result in the context of utility indifference valuation of contingent claims.

4 The zero interest rates case

In this section, we show that, when the interest rate is zero, the value function V, of the optimal investment problem under capital gains taxes, reduces to a tax-free problem with modified transaction costs parameters.

Let $r = 0$, and set $\tilde{X}^\nu := X^\nu + (1-\mu)\alpha K^\nu$, $\tilde{Y}^\nu := (1-\alpha)Y^\nu$ for any $\nu = (C, L, M) \in \mathcal{A}(s)$. Then, it follows from (2.7)-(2.8)-(2.10) that

$$\begin{aligned} d\tilde{X}^\nu_t &= -C_t dt - (1+\tilde{\lambda})d\tilde{L}_t - (1-\mu)\tilde{Y}^\nu_t dM_t \\ d\tilde{Y}^\nu_t &= d\tilde{L}_t - \tilde{Y}^\nu_t dM_t \end{aligned}$$

where

$$\tilde{L} := (1-\alpha)L \quad \text{and} \quad \tilde{\lambda} := \frac{\lambda + \alpha\mu}{1-\alpha}.$$

In terms, of these new state variables, the no-bankruptcy constraint (2.5) translates into

$$Z_t^\nu \;=\; \tilde{X}_t^\nu + (1-\mu)\tilde{Y}_t^\nu \;\geq\; 0\,.$$

Observe that the dynamics of the pair $(\tilde{X}^\nu, \tilde{Y}^\nu)$ do not involve the state variable K^ν. Then, the optimal consumption-investment problem under capital gains taxes and proportional transaction costs defined by (λ, μ) reduce to a tax-free optimal consumption-investment problem with modified transaction costs parameters $(\tilde{\lambda}, \mu)$. We summarize these finding in the following statement.

Proposition 4.1 *Let $r = 0$. Then, for any $s = (x, y, k) \in \bar{\mathcal{S}}$, we have*

$$V(s) \;=\; \tilde{V}^0\left(x + (1-\mu)\alpha k, (1-\alpha)y\right)\;,$$

where $\tilde{V}^0$ is the value function defined in (3.3) with transaction costs parameters $(\tilde{\lambda}, \mu)$. Moreover, if $\left(\tilde{C}^*, \tilde{L}^*, \tilde{M}^*\right)$ is a solution of $\tilde{V}^0(s)$, then the strategy (C^*, L^*, M^*) defined by $(C^*, L^*, M^*) := \left(\tilde{C}^*, (1-\alpha)^{-1}\tilde{L}^*, \tilde{M}^*\right)$ is a solution of $V(s)$.

Observe that $\tilde{\lambda} \geq \lambda$. Then, we deduce from the above statement that for any $s = (x, y, k) \in \bar{\mathcal{S}}$:

$$V(s) \;=\; \tilde{V}^0\left(\tilde{s}^0\right) \;\leq\; V^0\left(\tilde{s}^0\right)\;, \quad \text{with} \quad \tilde{s}^0 \;:=\; (x + (1-\mu)\alpha k, (1-\alpha)y)$$

where V^0 is the value function defined in (3.3). This means that the maximal utility under capital gains taxes can not be higher than the maximal utility in the corresponding tax-free model, despite the compensation for the capital losses.

In the rest of this paper, we will concentrate on the case where

$$r \;>\; 0\,,$$

and we will prove that the upper bound $V(s) \leq V^0\left(\tilde{s}^0\right)$ is still valid.

5 First properties of the value function

5.1 Monotonicity and Homogeneity

Proposition 5.1 *The value function V is nondecreasing with respect to each of the variables x, y, and k.*

Proof. Let $s := (x, y, k)$ be in $\bar{\mathcal{S}}$, and $s' := (x', y', k')$ such that $s' - s \in \mathbb{R}^3_+$. Clearly s' is in $\bar{\mathcal{S}}$. In order to prove the required result, it is sufficient to show that $\mathcal{A}(s) \subset \mathcal{A}(s')$. Let $\nu = (C, L, M)$ be an arbitrary strategy in $\mathcal{A}(s)$, and we claim that the liquidation value process $Z^{s'\nu}$ is non-negative, so that $\nu \in \mathcal{A}(s')$. Indeed, for $t \geq 0$, we directly compute that

$$\hat{Y}_t \; := \; Y_t^{s',\nu} - Y_t^{s,\nu} \;\; = \;\; (y' - y) e^{\left(r + \sigma\theta - \sigma^2/2\right)t + \sigma W_t - M_t^c} \prod_{0 \leq s \leq t} (1 - \Delta M_s) \; \geq \; 0 \quad \mathbb{P} - \text{a.s.} \, ,$$

$$\hat{K}_t \; := \; K_t^{s',\nu} - K_t^{s,\nu} \;\; = \;\; (k' - k) e^{-M_t^c} \prod_{0 \leq s \leq t} (1 - \Delta M_s) \; \geq \; 0 \, , \quad \mathbb{P} - \text{a.s.}$$

where we denoted by M^c the continuous part of M, and

$$e^{-rt} \left(X_t^{s',\nu} - X_t^{s,\nu} \right) \;\; = \;\; x' - x + (1 - \mu) \int_0^t e^{-ru} \left[(1 - \alpha) \hat{Y}_{u-} + \alpha \hat{K}_{u-} \right] dM_u \; \geq \; 0 \, .$$

Then, $Z_t^{s',\nu} \geq Z_t^{s,\nu} \geq 0$ since $\nu \in \mathcal{A}(s)$, and therefore ν is in $\mathcal{A}(s')$. □

We next state a homogeneity property of V which is implied by the choice of the power utility function. This feature will be used, in the numerical approximation of this paper, to reduce the dimensionality of the state space.

Proposition 5.2 *The value function V satisfies the following homogeneity property*

$$V(\delta s) \; = \; \delta^p \, V(s) \quad \textit{for all} \quad s \in \bar{\mathcal{S}} \; \textit{and} \; \delta > 0 \, .$$

Proof. 1. Let $\nu = (C, L, M)$ be an arbitrary strategy in $\mathcal{A}(s)$, and define the strategy $\nu' := (\delta C, \delta L, M)$. We easily verify that $S^{\delta s, \nu'} = \delta S^{s,\nu} \in \bar{\mathcal{S}}$, which implies that ν' is in $\mathcal{A}(\delta s)$, and therefore

$$V(\delta s) \;\geq\; \mathbb{E}\left[\int_0^\infty e^{-\beta u} U(\delta C_u) du\right] \;=\; \delta^p J_\infty(\delta, \nu)\,,$$

where the last equality follows from the homogeneity property of the utility function U. By the arbitrariness of ν in $\mathcal{A}(s)$, this shows that $V(\delta s) \geq \delta^p V(s)$.
2. The reverse inequality follows immediately from the first step of this proof by writing $V(s) = V\left(\delta^{-1}\delta s\right) \geq \delta^{-p} V\left(\delta s\right)$. □

We finally discuss the value function on the boundary of the state space $\mathcal{S}$. Observe that there is no *a priori* information on the boundary components $\partial^y \mathcal{S}$ and $\partial^k \mathcal{S}$. This is one source of difficulty in the numerical part of this paper, as this state constraint problem needs a special treatment, see [4]. On the remaining boundary $\partial^z \mathcal{S}$, the following result states that the value function is zero.

Proposition 5.3 *For every $s \in \partial^z \mathcal{S}$, we have $V(s) = 0$.*

Proof. Let s be in $\partial^z \mathcal{S}$, and ν be in $\mathcal{A}(s)$. By the definition of the set admissible controls, the process $Z^{s,\nu}$ is non-negative. By Itô's Lemma together with the non-negativity of C, K, and the non-decrease of L, this provides

$$0 \;\leq\; e^{-rt} Z_t^{s,\nu} \;\;\leq\;\; (1-\mu)(1-\alpha) \int_0^t e^{-ru} Y_u^{s,\nu} \sigma \left[\theta du + dW_u\right]\;.$$

Let $\mathbb{Q}$ be the probability measure equivalent to $\mathbb{P}$ under which the process $\tilde{W} := \{\theta u + W_u,\; u \geq 0\}$ is a Brownian motion. The process appearing on the right-hand side of the last inequality is a $\mathbb{Q}-$supermartingale as a non-negative $\mathbb{Q}-$local martingale. By taking expected values under $\mathbb{Q}$, it then follows from the last inequalities that $Z^{s,\nu} = Y^{s,\nu} = K^{s,\nu} = C = L \equiv 0$. We have then proved that for $s \in \partial^z \mathcal{S}$, any admissible strategy $\nu = (C, L, M) \in \mathcal{A}(s)$ is such that $C = L \equiv 0$, implying that $V(s) = 0$. $\square$

5.2 Optimality of wash sales in the absence of transaction costs

Throughout this subsection, we consider a transaction costs free financial market, and we prove that it is always worth realizing capital losses whenever the tax basis exceeds the spot price of the risky asset. In other words, given $s = (x, y, k) \in \bar{\mathcal{S}}$, every admissible strategy $\nu \in \mathcal{A}(s)$, with $K_\tau^{s,\nu} > Y_\tau^{s,\nu}$ (i.e. $B_\tau^{s,\nu} > P_\tau$) for some stopping time τ, can be improved strictly by realizing the capital loss on the entire portfolio at time τ. This property is observed in practice, and is known as a *wash sale*. It was stated in [11], and embedded directly in the definition of the tax basis.

This result can be understood easily. Observe that any wash sale implies an immediate decrease of the holdings in risky assets evaluated at the basis price K, while the total holdings in risky assets remain unchanged. Since the dynamics of K in (2.8) is autonomous and $K \geq 0$, it follows that wash sales imply a permanent decrease of the K variable. We now observe from the dynamics of the Z variable in (2.11) that this in turn implies an increase of the after-tax liquidation value of the portfolio. Since such an increase may be used to increase the consumption rate, this shows that wash sales induce an increase of the total consumption.

Proposition 5.4 *Let $\lambda = \mu = 0$, and consider some $s \in \bar{\mathcal{S}}$ and $\nu = (C, L, M) \in \mathcal{A}(s)$. Assume that $K_\tau^{s,\nu} > Y_\tau^{s,\nu}$ a.s. for some finite stopping time τ. Then there exists an admissible strategy $\tilde{\nu} = \left(\tilde{C}, \tilde{L}, \tilde{M}\right) \in \mathcal{A}(s)$ such that*

$$Y^{\tilde{\nu}} = Y^{\nu}, \quad \Delta\tilde{M} - \Delta M = \mathbf{1}_{\{t\}}(\tau) \quad \text{and} \quad J_\infty(s, \tilde{\nu}) > J_\infty(s, \nu),$$

i.e. wash sale is optimal.

To prove this result, we start by the following lemma.

Lemma 5.1 *In the setting of Proposition 5.4, set* $(L', M') := (L, M) + (1, 1)(1 - \Delta M_\tau)1_{t\geq\tau}$*. Then* $\nu' = (C, L', M') \in \mathcal{A}(s)$ *and the resulting state process satisfies*

$$Y^{s,\nu'} = Y^{s,\nu},\ Z^{s,\nu'} \geq Z^{s,\nu},\ K^{s,\nu'} \leq K^{s,\nu} \quad a.s.\ and \quad Z_t^{\nu'} > Z_t^{\nu}\ a.s.\ on\ \{t > \tau\}\ .$$

Proof. 1. Since ν and ν' differ only by the jump at the stopping time τ, and $\Delta L'_\tau = \Delta M'_\tau$, we have

$$Y^{s,\nu'} \;=\; Y^{s,\nu}\ , \quad \text{and} \quad \left(Z_t^{s,\nu'}, K_t^{s,\nu'}\right) \;=\; \left(Z_t^{s,\nu}, K_t^{s,\nu}\right) \quad \text{for all } t < \tau\ .$$

Since $\lambda = \mu = 0$, it follows from (2.11) that the processes $Z^{s,\nu}$ and $Z^{s,\nu'}$ have continuous paths. Hence $Z_\tau^{s,\nu'} = Z_\tau^{s,\nu}$.

2. For $t > \tau$, we compute directly from (2.8) that

$$K_t^{s,\nu'} - K_t^{s,\nu} \;=\; \left(K_\tau^{s,\nu'} - K_\tau^{s,\nu}\right)\ e^{-M_t^c + M_\tau^c} \prod_{\tau<u\leq t} (1 - \Delta M_u)$$

Observe that the newly defined strategy ν' consists in selling out the whole portfolio at time τ, as $\Delta M'_\tau = 1$. Hence $K_\tau^{s,\nu'} = Y_\tau^{s,\nu'} = Y_\tau^{s,\nu}$, and

$$K_t^{s,\nu'} - K_t^{s,\nu} \;=\; \left(Y_\tau^{s,\nu} - K_\tau^{s,\nu}\right)\ e^{-M_t^c + M_\tau^c} \prod_{\tau<u\leq t} (1 - \Delta M_u) \;>\; 0 \quad \text{for } t \geq \tau\,,$$

since $Y_\tau^{s,\nu} - K_\tau^{s,\nu} < 0$, by definition of τ.

3. We finally compute directly from (2.11) that

$$e^{-rt}\left(Z_t^{s,\nu'} - Z_t^{s,\nu}\right) \;=\; -r\alpha \int_\tau^t e^{-ru}\left(K_u^{s,\nu'} - K_u^{s,\nu}\right) du \;>\; 0\,, \quad \text{for } t > \tau\,,$$

by Step 2 of this proof. Hence $Z^{s,\nu'} \geq 0$ and $\nu' \in \mathcal{A}(s)$. □

Proof of Proposition 5.4. Let $\nu' = (C, L', M')$ be the transformation of of the consumption-investment strategy ν introduced in the previous Lemma 5.1, and define the strategy $\tilde{\nu} = (\tilde{C}, \tilde{L}, \tilde{M})$ by :

$$\tilde{C}_t \; := \; C_t \; + \; \xi \left(Z_t^{s,\tilde{\nu}} - Z_t^{s,\nu} \right) 1_{t \geq \tau} \quad \text{and} \quad \left(\tilde{L}, \tilde{M} \right) \; := \; (L', M') \; , \qquad (5.1)$$

where ξ is an arbitrary positive constant. Observe that $(Y^{s,\tilde{\nu}}, K^{s,\tilde{\nu}}) = (Y^{s,\nu'}, K^{s,\nu'})$, and $Z_t^{s,\tilde{\nu}} = Z_t^{s,\nu'} = Z_t^{s,\nu}$ for $t \leq \tau$. In particular, $K^{s,\tilde{\nu}} - K^{s,\nu} = K^{s,\nu'} - K^{s,\nu} \leq 0$ by Lemma 5.1. In order to check the admissibility of the strategy $\tilde{\nu}$, we directly compute that :

$$\begin{aligned} e^{-r(t-\tau)} \left(Z_t^{s,\tilde{\nu}} - Z_t^{s,\nu} \right) \;&=\; Z_\tau^{s,\tilde{\nu}} - Z_\tau^{s,\nu} - r\alpha \int_\tau^t e^{-r(u-\tau)} \left(K_u^{s,\tilde{\nu}} - K_u^{s,\nu} \right) du \\ &\quad + \xi \int_\tau^t e^{-r(u-\tau)} \left(Z_u^{s,\tilde{\nu}} - Z_u^{s,\nu} \right) du \\ &\geq\; -\xi \int_\tau^t e^{-r(u-\tau)} \left(Z_u^{s,\tilde{\nu}} - Z_u^{s,\nu} \right) du \; . \end{aligned}$$

By the Gronwall inequality, this implies that $Z_t^{\tilde{\nu}} > Z_t^{\nu}$ on $\{t > \tau\}$, and therefore $\tilde{C} > C$ on $\{t > \tau\}$ with positive Lebesgue$\otimes P$ measure. Hence $J_\infty(s; \tilde{\nu}) > J_\infty(s; \nu)$. □

5.3 Upper bound on the value function

We now derive an upper bound on the value function V, which expresses that there is no way for the investor to take advantage of tax credits in order to do better than in the tax-free financial market. Recall that V^0 denotes the value function (3.3) of the optimal investment problem in the tax-free financial market.

Proposition 5.5 *For $s = (x, y, k)$ in $\bar{\mathcal{S}}$, we have $V(s) \leq V^0(x + (1-\mu)\alpha k, (1-\alpha)y)$.*

Proof. Let $s = (x, y, k)$ be in $\bar{\mathcal{S}}$. Consider some consumption-investment strategy $\nu = (C, L, M)$ in $\mathcal{A}(s)$. Define a consumption-investment strategy $\tilde{\nu} = (C, (1-\alpha)L, M)$ and denote by $(\tilde{X}, \tilde{Y})$ the corresponding tax-free bank and risky assets account processes with the initial endowment $(x + (1-\mu)\alpha k, (1-\alpha)y)$. Clearly

$$\tilde{Y}_t \; = \; (1-\alpha) Y_t^{s,\nu} \; \geq \; 0 \qquad \mathbb{P} - \text{a.s.}$$

We next prove that $\tilde{\nu}$ is admissible in the tax-free financial market by showing that $\tilde{Z}_t := \tilde{X}_t + (1-\mu)\tilde{Y}_t$ is $\mathbb{P}-$a.s. nonnegative. Since $\tilde{Z}_{0-} - Z^{s,\nu}_{0-} = 0$, we have

$$\tilde{Z}_t - Z^{s,\nu}_t \;\geq\; e^{rt}\int_0^t e^{-ru} r(1-\mu)\alpha K^{s,\nu}_u du + e^{rt}\int_0^t e^{-ru}\alpha(\lambda+\mu)dL_u \;\geq\; 0\,.$$

Hence, $\tilde{Z}^{s,\nu}_t \geq Z^{s,\nu}_t \geq 0$ $\mathbb{P}-$a.s. and $V^0(x+(1-\mu)\alpha k, (1-\alpha)y) \geq J_\infty(s,\nu)$. By arbitrariness of ν in $\mathcal{A}(s)$, we may conclude that $V^0(x+(1-\mu)\alpha k,(1-\alpha)y) \geq V(s)$.

□

5.4 Lower bound in the absence of transaction costs

The following lower bound on the value function V is only valid in the absence of transaction costs. Recall the function γ defined in Theorem 3.1.

Proposition 5.6 *Let $\lambda = \mu = 0$. Then, for $s = (x,y,k)$ in $\bar{\mathcal{S}}$ and $z = x + (1-\alpha)y + \alpha k$, there exists a sequence of admissible strategies $(\nu^n)_{n\geq 1} \subset \mathcal{A}(s)$ such that*

$$V(s) \;\geq\; \gamma\left(r, \tilde{\theta}^\alpha\right)\frac{z^p}{p} \;=\; \lim_{n\to\infty} J_\infty(s,\nu^n) \quad \text{where} \quad \tilde{\theta}^\alpha \;:=\; \theta - \frac{r\alpha}{\sigma(1-\alpha)}\,,$$

i.e. the value function of the Merton frictionless problem with the smaller risk premium $\tilde{\theta}^\alpha$ can be approached as close as possible in the context of the financial market with taxes.

This result is proved by producing a sequence $(C_n, L_n, M_n)_{n\geq 1} \subset \mathcal{A}(s)$ of admissible strategies which approximates the Merton's value function with the smaller risk premium $\tilde{\theta}^\alpha$. To give an intuitive justification of this result, we re-write (2.11) as

$$dZ_t \;=\; (rZ_t - C_t)\,dt \;+\; Y_t\,\tilde{\sigma}^\alpha\left(dW_t + \tilde{\theta}^\alpha dt\right) \;+\; r\alpha\,(Y_t - K_t)\,dt\,, \qquad (5.2)$$

where $\tilde{\theta}^\alpha$ is defined in the statement of Proposition 5.6 and $\tilde{\sigma}^\alpha := (1-\alpha)\sigma$; recall that $\lambda = \mu = 0$ in the present analysis. The above equation shows that the dynamics of

Z differs from the dynamics (3.1) of the wealth process $\bar{Z}$, in the classical frictionless model with modified parameters $(\tilde{\sigma}^\alpha, \tilde{\theta}^\alpha)$, only by the term $r\alpha(Y-K)$. In view of Proposition 5.4, we expect this term to be non-negative for the optimal strategy (if exists). This hints that the liquidation value process Z is larger than the wealth process in the fictitious tax-free financial market with modified risk premium, and therefore justifies the inequality of Proposition 5.6.

Proposition 5.6 states a stronger result than the lower bound as we claim that the lower bound can be approached as close as desired by admissible strategies. Indeed, the proof reported in Appendix B exhibits an explicit sequence of strategies which mimics the optimal consumption-investment strategy in the Merton frictionless model, while keeping the difference $Y-K$ small, or equivalently, the tax basis close to the spot price of risky asset.

Remark 5.1 Let $b := r + \theta\sigma$ be the instantaneous mean return coefficient in our financial market. Then, the modified risk premium θ^α can be easily interpreted in terms of the modified volatility coefficient $\sigma^\alpha = (1-\alpha)\sigma$ and a similarly modified instantaneous mean return coefficient $b^\alpha := (1-\alpha)b$, as

$$\theta^\alpha \quad = \quad \frac{b^\alpha - r}{\sigma^\alpha} .$$

This fictitious financial market with such modified coefficients corresponds to the situation where the investor is forced to realize the capital gains or losses, at each time t, before adjusting the portfolio.

5.5 First order expansion in the absence of transaction costs

When the financial market is not subject to transaction costs, Propositions 5.5 and 5.6 provide the following bounds on the value function V

$$\gamma(r,\tilde{\theta}^\alpha)\,\frac{(x+(1-\alpha)y+\alpha k)^p}{p} \;\leq\; V(x,y,k) \;\leq\; \gamma\,(r,\theta)\;\frac{(x+(1-\alpha)y+\alpha k)^p}{p}\,, \quad (5.3)$$

where $\tilde{\theta}^\alpha$ is defined in the statement of Proposition 5.6, and γ is defined in Theorem 3.1. Observe that $\tilde{\theta}^\alpha = \theta$ whenever $\alpha = 0$ or $r = 0$. Therefore, we might expect that these bounds are tight for small interest rate or tax parameter. This is the object of the following first main result of this paper.

Proposition 5.7 *Let $\lambda = \mu = 0$. Then, for $s = (x, y, k) \in \mathcal{S}$, we have*

$$V(s) \quad = \quad V^{\mathrm{app}}(s) + \mathrm{o}(\alpha + r)\,,$$

where $\mathrm{o}(\xi)$ is a function on $\mathbb{R}$ with $\mathrm{o}(\xi)/\xi \longrightarrow 0$ as $\xi \to 0$, and

$$V^{\mathrm{app}}(s) \quad := \quad \left(\gamma(0,\theta) + r\,\frac{\partial\gamma}{\partial r}(0,\theta)\right)\,\frac{(x+y)^p}{p} + \alpha\,\gamma(0,\theta)(k-y)(x+y)^{p-1}\,.$$

Before turning to the proof of this result, let us make some comments.

1. Observe that the function γ defined in Theorem 3.1 is decreasing in the r variable. Then, the above first order expansion shows that the value function V is decreasing in the interest rate variable (for small interest rate and tax parameters).

2. The variation of the value function in terms of the tax rate α depends on the initial position of the tax basis. If the initial tax basis is larger than the spot price, i.e. in a situation of capital gain loss, the investor takes advantage immediately of the tax credit, as stated in Proposition 5.4, and the value function V is increasing in α (for small α). In the opposite situation, i.e. when the initial tax basis is smaller than the spot price, the value function is decreasing in α. Finally, when the initial tax basis coincides with the spot price, the value function is not sensitive to the tax rate in the first order.

This variation of the value function (up to the first order) in terms of the tax rate α is somehow surprising. Indeed, in a capital loss situation, an increase of the tax parameter implies

- on one hand, a increase of the tax credit received initially by the agent,

- on the other hand, a larger amount of tax to be paid during the infinite lifetime of the agent.

Our first order expansion shows that, for small interest rate and tax parameters, the increase of initial tax credit is never compensated by the increase of tax over the infinite lifetime.

Proof of Proposition 5.7 It is sufficient to observe that the bounds on the value function V in (5.3) are smooth functions with identical partial gradient with respect to (r, α) at the origin. This follows from the fact that

$$\left.\frac{\partial \tilde{\theta}^\alpha}{\partial \alpha}\right|_{(r,\alpha)=(0,0)} \;=\; \left.\frac{\partial \tilde{\theta}^\alpha}{\partial r}\right|_{(r,\alpha)=(0,0)} \;=\; 0\,.$$

$\square$

Remark 5.2 Since the lower bound in (5.3) has the same first order Taylor expansion than the value function V, we can view the corresponding strategy as nearly optimal. From the discussion following Proposition 5.6, the portfolio allocation defining the lower bound is by definition an approximation of the constant portfolio allocation

$$\bar{\pi}^\alpha \;:=\; \frac{\tilde{\theta}^\alpha}{(1-p)\tilde{\sigma}^\alpha} \;=\; \frac{1}{(1-p)\sigma^2}\left[\frac{\rho}{1-\alpha} - \frac{r}{(1-\alpha)^2}\right],$$

where $\rho := \sigma\theta + r$ is the instantaneous mean return of the risky asset. Direct computation shows that $\bar{\pi}^\alpha \le \bar{\pi}^0$ if and only if $r \ge (1-\alpha)(\rho - r)$. Using the data set of Dammon, Spatt and Zhang [11] ($r = 6\%$, $\rho = 9\%$, $\alpha = 36\%$), we see that $\bar{\pi}^\alpha \le \bar{\pi}^0$. Hence the portfolio π^α is consistent with the numerical findings of [11].

6 Characterization by the dynamic programming equation

The goal of this section is to provide a characterization of V by means of a second order partial differential equation for which we shall provide a numerical solution in the subsequent section. Unfortunately, we are unable to obtain a characterization of V by the corresponding dynamic programming equation. Nevertheless, we shall obtain in paragraph 6.1 the continuity of the value function V by techniques from viscosity theory. In paragraph 6.2 below, we exhibit a consistent approximation V^ε as the unique solution of an approximating second order partial differential equation. This uniqueness property together with the convergence of V^ε towards V, as $\varepsilon \to 0$, involve heavy technical arguments. Therefore, this section is only intended to report the necessary ingredients for the numerical results of Section 7. We refer the interested reader to the accompanying paper [4] for the technical proofs.

6.1 The dynamic programming equation

For s in $\bar{\mathcal{S}}$ and $\nu = (C, L, M)$ in $\mathcal{A}$, the jumps of the state processes S are given by

$$\Delta S_t^{s,\nu} \;=\; -\Delta L_t \, \mathbf{g^b} \;-\; \Delta M_t \;\left[(1-\alpha) Y_{t-}^{s,\nu} + \alpha K_{t-}^{s,\nu}\right] \; \mathbf{g^s}\left(S_{t-}^{s,\nu}\right)$$

where the vector fields $\mathbf{g^b}$ and $\mathbf{g^s}(x, y, k)$ are defined by

$$\mathbf{g^b} := \begin{pmatrix} 1+\lambda \\ -1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{g^s}(s) := \begin{pmatrix} -(1-\mu) \\ \frac{1}{1-\alpha} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{-\alpha}{1-\alpha} \\ 1 \end{pmatrix} \frac{k}{(1-\alpha)\, y + \alpha k} \, \mathbf{1}_{(y,k)\neq 0} \,.$$

The dynamic programming equation of our problem is then given by

$$\min\left[-\mathcal{L}V \;,\; \mathbf{g^b}\cdot DV \;,\; \mathbf{g^s}\cdot DV\right] \;=\; 0 \text{ on } \bar{\mathcal{S}} \setminus \partial^z \mathcal{S} \quad \text{and} \quad V = 0 \;\text{ on } \partial^z \mathcal{S} \qquad (6.1)$$

where $\mathcal{L}$ is the second order differential operator defined in (3.5). Observe that we have no information on the regularity of the value function V, hence we cannot prove that V is a classical solution to (6.1). Moreover the value function V is only known on the boundary $\partial^z\mathcal{S}$, see Proposition 5.3, but there is no possible knowledge of V on $\partial^y\mathcal{S} \cup \partial^k\mathcal{S}$. We then need to use the notion of viscosity solutions which allows for a weak formulation of solutions to partial differential equations and boundary conditions. For a locally bounded function $v\ :\bar{\mathcal{S}} \longrightarrow \mathbb{R}$, we shall use the classical notations in viscosity theory

$$v^*(s) \ :=\ \limsup_{\mathcal{S}\ni s'\to s}\ v(s') \quad \text{and} \quad v_*(s)\ :=\ \liminf_{\mathcal{S}\ni s'\to s}\ v(s')$$

for the corresponding upper and lower semi-continuous envelopes.

Definition 6.1 (i) *A locally bounded function v is a constrained viscosity subsolution of* (6.1) *if $v^* \leq 0$ on $\partial^z\mathcal{S}$, and for all $s \in \bar{\mathcal{S}} \setminus \partial^z\mathcal{S}$ and $\varphi \in C^2(\bar{\mathcal{S}})$ with* $0 = (v^* - \varphi)(s) = \max_{\bar{\mathcal{S}}}(v^* - \varphi)$ *we have* $\min\left[-\mathcal{L}\varphi\ ,\ \mathbf{g^b}\cdot D\varphi\ ,\ \mathbf{g^s}\cdot D\varphi\right] \leq 0$.
(ii) *A locally bounded function v is a constrained viscosity supersolution of* (6.1) *if $v_* \geq 0$ on $\partial^z\mathcal{S}$, and for all $s \in \mathcal{S}$ and $\varphi \in C^2(\mathcal{S})$ with* $0 = (v_* - \varphi)(s) = \min_{\mathcal{S}}(v_* - \varphi)$ *we have* $\min\left[-\mathcal{L}\varphi\ ,\ \mathbf{g^b}\cdot D\varphi\ ,\ \mathbf{g^s}\cdot D\varphi\right] \geq 0$.
(iii) *A locally bounded function v is a constrained viscosity solution of* (6.1) *if it is a constrained viscosity subsolution and supersolution.*

In the above definition, Observe that there is no boundary value assigned to the value function on $\partial^y\mathcal{S}\cup\partial^k\mathcal{S}$. Instead, the subsolution property holds on this boundary. Notice that the supersolution property is satisfied only in the interior of the domain $\mathcal{S}$. The notion of constrained viscosity solution was introduced in [28]. For later use, we report the following results [4].

Proposition 6.1 (Viscosity property) *The value function V is a constrained viscosity solution of* (6.1).

Proposition 6.2 (Comparison result) *Let $\lambda+\mu > 0$. Let u be an upper- semicontinuous viscosity subsolution of* (6.1)*, and v be a lower-semicontinuous viscosity supersolution of* (6.1) *with $(u-v)^+ \in \mathrm{USC}_p(\bar{\mathcal{S}})$. Assume further that $(u-v)(x,0,0) \leq 0$ for all $x \geq 0$. Then $u \leq v$ on $\bar{\mathcal{S}}$.*

Unfortunately, this comparison result does not provide uniqueness of a constrained viscosity solution for the PDE (6.1), since we have no a priori comparison of two possible solutions on $\{(x,0,0) \ : \ x \in \mathbb{R}_+\}$.

Remark 6.1 The main difficulty to prove a stronger comparison result, which would not require a priori comparison on $\{(x,0,0) \ : \ x \geq 0\}$, is that the mapping $\mathbf{g}^{\mathbf{s}}$ is not continuous on this axis. In the subsequent paragraph, we define an approximation V^ε of V by means of a convenient approximation of $\mathbf{g}^{\mathbf{s}}$ by a sequence of locally Lipschitz-continuous functions $(\mathbf{g}^{\mathbf{s}}_\varepsilon)_{\varepsilon>0}$ on $\bar{\mathcal{S}} \setminus \partial^z\mathcal{S}$.

Nevertheless, the above viscosity property and comparison result can be used to obtain the continuity (and even more) of value function V, see Remark 2.4.

Proposition 6.3 *Let $\lambda + \mu > 0$. For $s = (x,y,k) \in \bar{\mathcal{S}}$ and $z := x + (1-\mu)[(1-\alpha)y + \alpha k]$, we have $V(s) = z^p \ \mathcal{V}\left(\frac{y}{z}, \frac{k}{z}\right)$, where $\mathcal{V}$ is a Lipschitz-continuous function on $\mathbb{R}^2_+$.*

The proof of this result is reported in Appendix C.

6.2 Characterization by approximation

For every $\varepsilon > 0$, we define the function

$$f^\varepsilon(x,y,k) \ := \ 1 \wedge \left(\frac{k}{\varepsilon z} - 1\right)^+ \quad \text{with} \quad z := x + (1-\mu)\left[(1-\alpha)y + \alpha k\right] \ ,$$

together with the approximation of $\mathbf{g^s}$:

$$\mathbf{g}^{\mathbf{s}}_{\varepsilon}(x,y,k) \;:=\; \mathbf{g^s}\left(x,y,kf^{\varepsilon}(s)\right) \quad \text{for all} \quad s \in \mathcal{S}\setminus\partial^z\mathcal{S}\;.$$

Notice that $\mathbf{g}^{\mathbf{s}}_{\varepsilon}$ is locally Lipschitz-continuous on $\bar{\mathcal{S}}\setminus\partial^z\mathcal{S}$, and $\mathbf{g}^{\mathbf{s}}_{\varepsilon}(s) = \mathbf{g^s}(s)$ whenever $k \geq 2\varepsilon z$. We now state the main result for the numerical application of the subsequent paragraph.

Theorem 6.1 ([4]) *Let $\lambda+\mu>0$. For each $\varepsilon>0$, the boundary value problem*

$$\min\left\{-\mathcal{L}\varphi\;,\;\mathbf{g^b}\cdot D\varphi\;,\;\mathbf{g}^{\mathbf{s}}_{\varepsilon}\cdot D\varphi\right\} \;=\; 0 \text{ on } \bar{\mathcal{S}}\setminus\partial^z\mathcal{S} \quad \text{and} \quad \varphi \;=\; 0 \quad \text{on } \partial^z\mathcal{S} \quad (6.2)$$

has a unique constrained viscosity solution V^{ε} in the class $C^0_p\left(\bar{\mathcal{S}}\right)$. Moreover,
(i) *the family $(V^{\varepsilon})_{\varepsilon>0}$ is non-increasing and converges to the value function V uniformly on compact subsets of $\bar{\mathcal{S}}$ as $\varepsilon\searrow 0$,*
(ii) *for every $s\in\bar{\mathcal{S}}$ and $\delta\geq 0$, we have $V^{\varepsilon}(\delta s) = \delta^p\, V^{\varepsilon}(s)$.*

In the above statement, a constrained viscosity solution of (6.2) is understood in the sense of Definition 6.1 with $\mathbf{g}^{\mathbf{s}}_{\varepsilon}$ substituted to $\mathbf{g^s}$.

In the accompanying paper, the existence result is proved by constructing explicitly a constrained viscosity solution V^{ε} of (6.2). This is achieved by considering a new control problem with convenient modification of the dynamics of the state processes.

Finally, the uniqueness statement is a consequence of the following comparison result, which only requires a priori comparison on the boundary $\partial^z\mathcal{S}$.

Proposition 6.4 ([4]) *Let $\lambda+\mu>0$. Let u be an upper-semicontinuous constrained viscosity subsolution of* (6.2)*, and v be a constrained viscosity supersolution of* (6.2) *such that $(u-v)^+\in \mathrm{USC}_p\left(\bar{\mathcal{S}}\right)$. Then $u\leq v$ in the entire domain $\bar{\mathcal{S}}$.*

7 Numerical results

We have stated in the previous section that the value function V is approximated by the functions $(V^\varepsilon)_{\varepsilon>0}$, where for each $\varepsilon > 0$, V^ε can be computed as the unique viscosity solution to the boundary value problem (6.2). In this section we provide a numerical estimate for V, based on a numerical schemes for (6.2) defined by the finite-difference discretization, and the classical Howard algorithm.

This section is organized as follows. We first exploit the homotheticity property of V^ε (Theorem 6.1 (ii)) to reduce the dimension and the domain of the state space to $[0,1] \times [0,1]$. We next describe the numerical scheme based on finite differences and the Howard algorithm.

7.1 Change of variables and reduction of the state dimension

By the homotheticity property of V^ε (Theorem 6.1(ii)) we have for $s = (x, y, k) \in \bar{\mathcal{S}} \setminus \partial^z \mathcal{S}$ and $z := x + (1-\mu)\left[(1-\alpha)y + \alpha k\right]$

$$V^\varepsilon(s) = z^p \mathcal{V}^\varepsilon\left(\frac{y}{z}, \frac{k}{z}\right) \quad \text{where} \quad \mathcal{V}^\varepsilon(\xi_1, \xi_2) := V^\varepsilon\left(1 - (1-\mu)\left((1-\alpha)\xi_1 + \alpha\xi_2\right), \xi_1, \xi_2\right).$$

Next, for a vector $\xi \in \mathbb{R}^2_+$, we define the vector $\zeta \in [0,1]^2$ by

$$\zeta_i := \xi_i/(1+\xi_i), \ i = 1, 2, \quad \text{and} \quad \Psi^\varepsilon(\zeta) := \mathcal{V}^\varepsilon(\xi).$$

This reduces the domain of $\mathcal{V}^\varepsilon$ from $\mathbb{R}^2_+$ to the compact $[0,1]^2$. By changing variables, it is immediately checked that Ψ^ε is a continuous constrained viscosity solution on $[0,1) \times [0,1)$ of

$$\min_{a\in\mathbb{A}} \left\{ \beta(a)\Psi^\varepsilon(\zeta) - \sum_{i=1}^{2} b_i(a,\zeta) \cdot D_i\Psi^\varepsilon(\zeta) - \frac{1}{2}\sum_{i,j=1}^{2} \eta_{ij}(a,\zeta) D^2_{ij}\Psi^\varepsilon(\zeta) - g(a) \right\} = 0 \quad (7.1)$$

where the definition of the control set $\mathbb{A}$ and the expressions of β, $(b_i)_{i=1,2}$ $(\eta_{i,j})_{i,j=1,2}$ and g are provided in Appendix A.

7.2 Numerical scheme for (7.1)

We adopt a classical finite difference discretization in order to obtain a numerical scheme for (7.1).

Let N be a positive integer, and set $h := \frac{1}{N}$, the finite difference step, we set $e_1 := (1,0)$, $e_2 := (0,1)$, and we define the uniform grid $\bar{\mathcal{S}}_h := [0,1]^2 \cap (h\mathbb{Z})^2$. We denote by $\zeta^h := (\zeta_1^h, \zeta_2^h)$ a point of the grid $\bar{\mathcal{S}}_h$, and we set $\mathcal{S}_h := (0,1) \times [0,1) \cap (h\mathbb{Z})^2$. In order to define a discretization of (7.1) , we approximate the partial derivatives of Ψ^ε by the corresponding backward and forward finite differences

$$\begin{aligned} b_i(a,\zeta)\partial_i \Psi^\varepsilon(\zeta) &\approx \begin{cases} b_i(a,\zeta) D_i^+ \Psi^\varepsilon(\zeta) & \text{if } b_i(a,\zeta) \ge 0\,, \\ b_i(a,\zeta) D_i^- \Psi^\varepsilon(\zeta) & \text{if } b_i(a,\zeta) < 0\,, \end{cases} \\ \partial_{ii}\Psi^\varepsilon(\zeta) &\approx D_i^2 \Psi^\varepsilon(\zeta)\,, \\ \eta_{ij}(a,\zeta)\partial_{ij}\Psi^\varepsilon(\zeta) &\approx \begin{cases} \eta_{ij}(a,\zeta) D_{ij}^+ \Psi^\varepsilon(\zeta) & \text{if } \eta_{ij}(a,\zeta) \ge 0\,, \\ \eta_{ij}(a,\zeta) D_{ij}^- \Psi^\varepsilon(\zeta) & \text{if } \eta_{ij}(a,\zeta) < 0\,, \end{cases} \end{aligned}$$

where the finite difference operators are defined for $i \neq j \in \{1,2\}$ by

$$\begin{aligned} D_i^+ \Psi^\varepsilon(\zeta) &= \frac{\Psi^\varepsilon(\zeta + he_i) - \Psi^\varepsilon(\zeta)}{h}\,, \\ D_i^- \Psi^\varepsilon(\zeta) &= \frac{\Psi^\varepsilon(\zeta) - \Psi^\varepsilon(\zeta - he_i)}{h}\,, \\ D_i^2 \Psi^\varepsilon(\zeta) &= \frac{\Psi^\varepsilon(\zeta + he_i) - 2\Psi^\varepsilon(\zeta) + \Psi^\varepsilon(\zeta - he_i)}{h^2}\,, \\ D_{ij}^\pm \Psi^\varepsilon(\zeta) &= \frac{1}{2h^2}\{2\Psi^\varepsilon(\zeta) + \Psi^\varepsilon(\zeta + he_i \pm he_j) + \Psi^\varepsilon(\zeta - he_i \mp he_j) \\ &\quad - \Psi^\varepsilon(\zeta + he_i) - \Psi^\varepsilon(\zeta - he_i) - \Psi^\varepsilon(\zeta + he_j) - \Psi^\varepsilon(\zeta - he_j)\}\,. \end{aligned}$$

In order to compute these differences at every point of $\mathcal{S}_h$, we extend Ψ^ε as follows

$$\Psi^\varepsilon\left(\zeta_0^h\right) = \Psi^\varepsilon\left(\zeta_0^h + he_1\right)\,, \qquad \Psi^\varepsilon\left(\zeta_1^h\right) = \Psi^\varepsilon\left(\zeta_1^h - he_1\right)\,,$$

for $\zeta_0^h \in \{0\} \times [0,1]$, $\zeta_1^h \in \{1\} \times [0,1]$, and

$$\Psi^\varepsilon\left(\zeta_0^h - he_2\right) = \Psi^\varepsilon\left(\zeta_0^h\right)\,, \quad \Psi^\varepsilon\left(\zeta_1^h\right) = \Psi^\varepsilon\left(\zeta_1^h - he_2\right)$$

for $\zeta_0^h \in [0,1]\times\{0\}$, $\zeta_1^h \in [0,1]\times\{1\}$. This provides a system of $(N-1)N$ non linear equations with the $(N-1)N$ unknowns $\Psi_h^\varepsilon(\zeta^h)$, $\zeta^h \in \mathcal{S}_h$:

$$\min_{a\in\mathbb{A}} \{ A_h^a \Psi_h^\varepsilon - g(a) \} \;=\; 0 \,. \tag{7.2}$$

7.3 The classical Howard algorithm

In order to solve (7.2) we adopt the classical Howard algorithm which can be described as follows

$$\begin{array}{rl} \text{step } 0 \;: & \text{start from an initial value for the control } : a^0 \in \mathbb{A} \,, \\ & \Psi_h^0 \;\text{ solution of }\; A_h^{a^0}\varphi - g(a^0) \;=\; 0 \,, \\ \text{step } k+1 \,, k \,\geq\, 0 \;: & \text{find } a^{k+1} \;\in\; \text{argmin}_{a\in\mathbb{A}} \left\{ A_h^a \Psi_h^k - g(a) \right\} \,, \\ & \Psi_h^{k+1} \;\text{ solution of }\; A_h^{a^{k+1}}\varphi - g(a^{k+1}) \;=\; 0 \,. \end{array}$$

7.4 Accuracy of the first order Taylor expansion

We implement the above numerical algorithm with the following parameters

$$p \,=\, 0.3 \,,\; \sigma \,=\, 0.3 \,,\; \text{and } \beta \,=\, 0.1 \,.$$

We also fix the instantaneous mean return of the risky asset to

$$b \;:=\; \theta\sigma + r \;\;=\;\; 0.11 \,,$$

In order to compare the results of our numerical algorithm to the explicit first order Taylor expansion V^{app} of Theorem 5.7, which is valid in the absence of transaction costs, we fix very small transaction cost parameters

$$\lambda \,=\, 0.001 \quad \text{and} \quad \mu \,=\, 0 \,,$$

see Remark 2.1.

In Figures 1, 2 and 3 we have plotted the relative error

$$\frac{\left|\mathcal{V}_\varepsilon^h\left(\zeta_{ij}^h\right) - \mathcal{V}^{\rm app}\left(\zeta_{ij}^h\right)\right|}{\mathcal{V}^{\rm app}\left(\zeta_{ij}^h\right)} .$$

on the grid for various fixed values of r an α. The right hand-side of all these figures reports the same plot than the left hand-side concentrated on $[(y/z),(k/z)] \in [0,1]^2$. We observe large errors near the boundary of the grid which can explode up to 50 %. In these regions, we can draw no conclusions as the numerical scheme based on the finite differences typically exhibits large approximation errors near the boundary. However, we observe that the relative error is remarkably small for points of the grid which are located far from the boundary.

We next examine the accuracy of the approximation for different sets of parameters r and α :

$$r \in \{0.001\,,\,.01\,,\,.07\} \quad \text{and} \quad \alpha \in \{.001\,,\,.01\,,\,.05\,,\,.1\,,\,.2\,,\,.3\,,\,.36\} .$$

Figure 7 plots the mean relative error between the results of the first order expansion and the numerical algorithm over all points of the grid :

$$\frac{1}{N(N-1)} \sum_{i,j} \frac{\left|\mathcal{V}_\varepsilon^h\left(\zeta_{ij}^h\right) - \mathcal{V}^{\rm app}\left(\zeta_{ij}^h\right)\right|}{\mathcal{V}^{\rm app}\left(\zeta_{ij}^h\right)} ,$$

where $N(N-1)$ is the total number of points in the grid, $\mathcal{V}_\varepsilon^h$ is the approximation of $\mathcal{V}_\varepsilon$ obtained by our numerical scheme, and

$$\mathcal{V}^{\rm app}(\xi_1,\xi_2) \;:=\; V^{\rm app}(1-(1-\mu)[(1-\alpha)\xi_1+\alpha\xi_2],\xi_1,\xi_2) .$$

As expected, the relative error is zero at the origin, and increases when the values of the parameters r and α increase. For realistic market values of r and α, the average relative error is of the order of 40 %.

In order examine further the error, we concentrate on the points of the grid which are located far from the boundary. The corresponding average relative error is plotted in Figure 8. We observe that the average relative error is remarkably small, and is of the order of 4 % for realistic values of r and α. This figure is our main numerical result as it shows the high accuracy of the first order Taylor approximation V^{app} of the value function V.

7.5 Optimal investment strategies

Throughout this subsection we implement our numerical algorithm with the following parameters

$$p = .3\,,\ \beta = .1\,,\ b := r+\theta\sigma = 0.11\,,\ \sigma = 0.3\,,\ \text{ and }\ r = .07\,.$$

The tax-free model. When $\alpha = 0$, our model reduces to the optimal consumption-investment problem under transaction costs of Constantinides and Magill [6]. We first implement our numerical scheme in this context in order to verify that our algorithm produces the well-known results of this problem. The transaction cost parameters are set to

$$\lambda = 0.04 \quad \text{and} \quad \mu = 0\,,$$

which are the values considered in [1]. Given the above values of the parameters, the Merton's optimal portfolio proportion is given by

$$\bar{\pi} \quad = \quad .6349\,.$$

In the present context, the value function $V = V^0$ is characterized by the PDE (3.6), and the domain $\bar{\mathcal{S}}^0$ defined by (3.4) is partitioned into three convex cones

$$\bar{\mathcal{S}}_0 \;=\; \mathbf{NT} \cup \mathbf{Sell} \cup \mathbf{Buy}\;, \tag{7.3}$$

corresponding to

- the region of no transaction (**NT**), where no position adjustment is considered by the investor,

- the **Sell** region, where the investor immediately sells risky assets so as to attain the region **NT** by moving along the ray $(1-\mu,-1)$,

- the **Buy** region, where the investor immediately purchases risky assets so as to attain the region **NT** by moving along the ray $(-(1+\lambda),1)$.

Notice the no-transaction region **NT** corresponds to the positions (x, y) such that the proportion of wealth allocated to the risky asset $(y/(x+(1-\mu)y)$ lies in some interval $[\pi^-, \pi^+]$ containing the Merton optimal portfolio proportion $\bar{\pi}$.

Figure 9 reports the numerical solution for the function $\mathcal{V}_\varepsilon^h$. We verify that the function $\mathcal{V}_\varepsilon^h$ in this tax-free context does not depend on the variable ξ_2, so that the value function V_ε^h does not depend on the k component. We also see that the value function is concave. Figure 10 reports the optimal investment strategy, and produces the expected partition of the state space into the regions **NT**, **Sell**, and **Buy**. We verify again that this partition is independent of the variable ξ_2, and we obtain the same bounds π^- and π^+ for the no-transaction region as in [1].

The value function approximation with taxes. We next concentrate on the case where the tax coefficient is positive. Figures 11, 12, 13 and 14 report the numerical solution for the function $\mathcal{V}_\varepsilon^h$ for $\alpha = .01$, .1, .2, .3, .36, and .4. The main observation out of these numerical results is that, for a positive tax parameter, the value function is no longer concave. This surprising feature lead to mathematical difficulties as we had to derive the dynamic programming equation without any *a priori* regularity of the value function.

Optimal investment strategy under taxes and transaction costs. Figures 15, 16, 17 and 18 show that, for positive α, the domain is again partitioned into three non-intersecting regions :

- the no-transaction region **NT**, where no portfolio adjustment is performed by the optimal investor,

- the **Sell** region, where the investor immediately sells risky assets so as to attain the region **NT** by moving towards the origin along the ray $-[(1-\alpha)y + \alpha k]\mathbf{g^s} = ((1-\mu)[(1-\alpha)y + \alpha k], -y, -k)$,

- the **Buy** region, where the investor immediately purchases risky assets so as to attain the region **NT** by moving along the ray $(-(1+\lambda), 1, 1) = -\mathbf{g^b}$.

For positive α, the boundaries of the no-transaction region depend on the tax-basis, and the range of the proportion of wealth allocated to the risky asset, (y/z), for which no-transaction is optimal is very sensible to the values of the tax basis (k/z). Notice that according to our numerical results, when the capital loss is large, i.e $k >> y$, wash sales are optimal. This suggests that the statement of Proposition 5.4 has a convenient extension for positive transaction costs parameters.

Finally, we observe that, for small values of the k variable, the no-transaction region **NT** contains the Merton optimal portfolio proportions $\bar{\pi}$ and $\bar{\pi}^\alpha$ corresponding respectively to our financial market and to the fictitious financial market with modified parameters.

Optimal investment strategy under taxes with no transaction costs. We now consider very small values of the transaction costs parameters

$$\lambda = .001 \quad \text{and} \quad \mu = 0 \,,$$

see Remark 2.1. Figures 19, 20 and 21 report the partition of the domain into the three regions **NT**, **Sell**, and **Buy**. In this context, we observe that the **Buy** region is limited from the left side by the *wash-sales* region which is part of the **Sell** region, exactly according to the statement of Proposition 5.4.

We also observe that, for small values of the k variable, the no-transaction region **NT** contains the Merton optimal portfolio proportions $\bar{\pi}$ and $\bar{\pi}^\alpha$ corresponding respectively to our financial market and to the fictitious financial market with modified parameters.

The other parameters are set to

$$p = 0.3 \; , \; \sigma = 0.3 \; , \; \text{and} \; \beta = 0.1 \; , \; \theta\sigma + r = 0.11 \; , r = 0.05 \; .$$

8 Conclusion

In this paper, we formulated a continuous-time version of the optimal investment problem under capital gains taxes, which was introduced by Dammon, Spatt and Zhang [11] in the context of the binomial model. As a main result, we derived an explicit first order Taylor expansion of the value function for small tax and interest rate parameters. Our numerical results show that the error induced by this approximation is remarkably small for reasonable values of market data. The first order approximation is decreasing in the interest rate parameter, and exhibits a surprising sensitivity with respect to the tax parameter : in a situation of capital loss, an increase of the tax parameter implies an increase of the value function. This suggests that the initial tax credit is never compensated by the increases of taxes through the lifetime of the agent.

The expansion was obtained from explicit bounds on the value function. The lower bound is obtained as the limit of the expected utility of a sequence of strategies which are built so as to mimic the Merton optimal strategy in a frictionless financial market with tax-deflated parameters. Then, this sequence can be viewed as a "first order maximizing sequence" for the problem of optimal investment under capital gains taxes.

The optimal strategies produced by our numerical results are however different in nature from the "first order" optimal strategy, as it exhibits three non-intersecting regions of no transaction, immediate selling and immediate buying.

The bounds on the value function were obtained in the context of the Black and Scholes model and the power utility function. We shall investigate in future work whether similar bounds are still valid in a multiple asset problem with more general dynamics for underlying risky asset, and whether such bounds still induce a first order Taylor expansion of the value function. Another interesting question is whether these results are valid in the corresponding finite horizon problem.

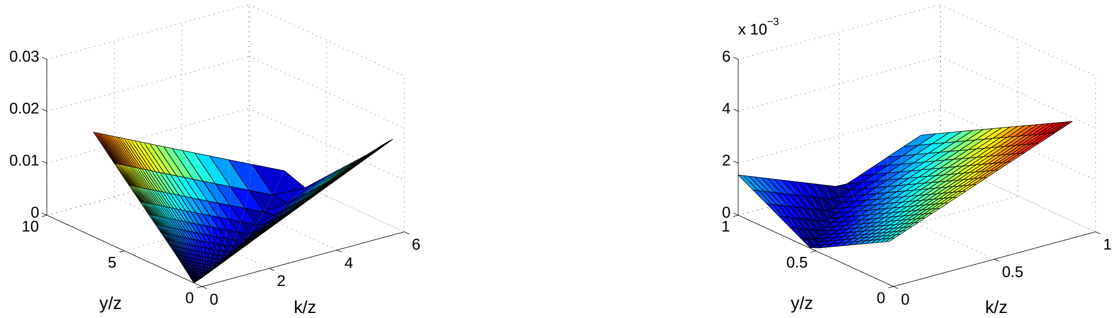

Figure 1: Relative error for $r = 0.001$ and $\alpha = 0.01$. $p = .3\,, b = .11\,, \sigma = .3\,, \beta = .1\,,$ $\lambda = .001\,, \mu = 0$.

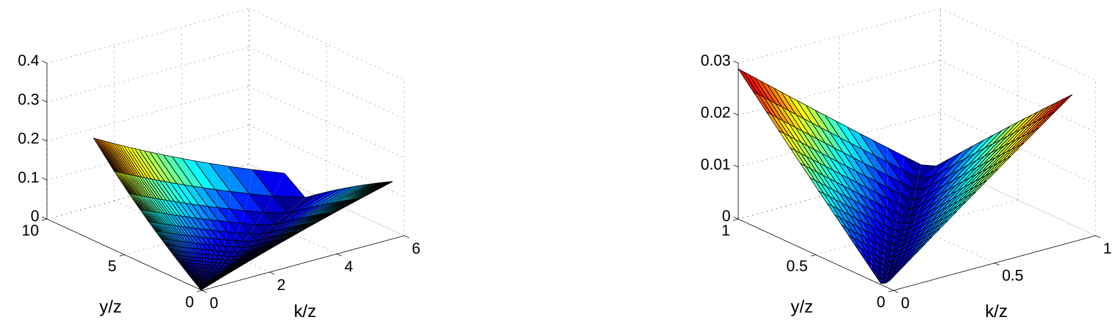

Figure 2: Relative error for $r = 0.001$ and $\alpha = 0.1$. $p = .3\,, b = .11\,, \sigma = .3\,, \beta = .1\,,$ $\lambda = .001\,, \mu = 0$.

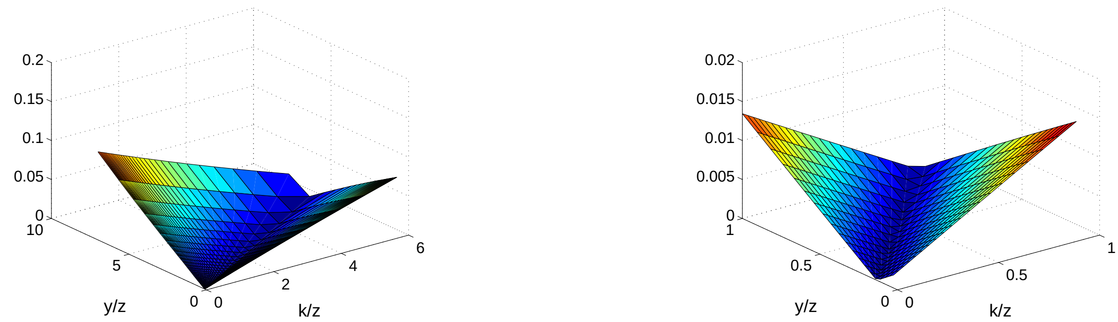

Figure 3: Relative error for $r = 0.01$ and $\alpha = 0.05$. $p = .3\,, b = .11\,, \sigma = .3\,, \beta = .1\,,$ $\lambda = .001\,, \mu = 0$.

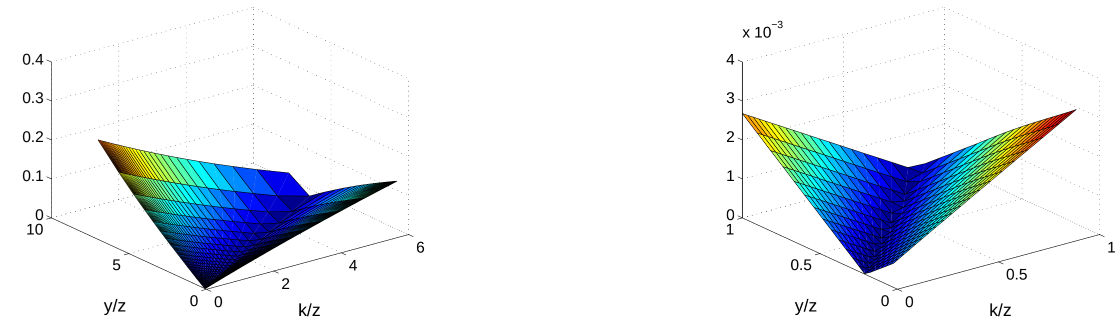

Figure 4: Relative error for $r = 0.01$ and $\alpha = 0.1$. $p = .3\,, b = .11\,, \sigma = .3\,, \beta = .1\,,$ $\lambda = .001\,, \mu = 0$.

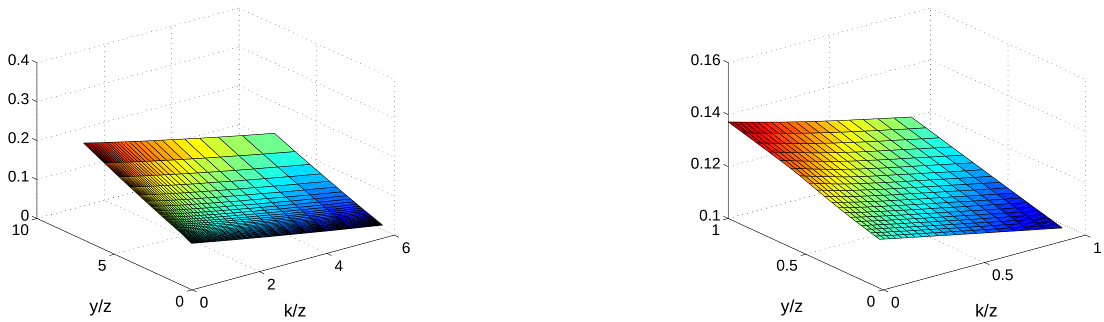

Figure 5: Relative error for $r = 0.07$ and $\alpha = 0.05$. $p = .3\,, b = .11\,, \sigma = .3\,, \beta = .1\,,$ $\lambda = .001\,, \mu = 0$.

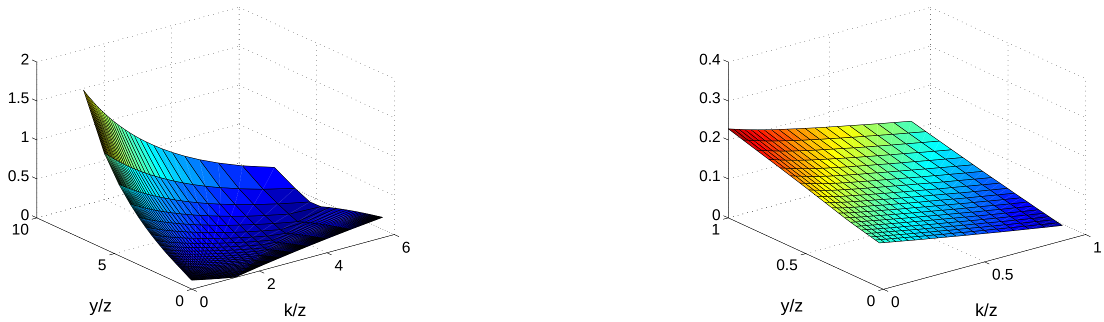

Figure 6: Relative error for $r = 0.07$ and $\alpha = 0.3$. $p = .3\,, b = .11\,, \sigma = .3\,, \beta = .1\,,$ $\lambda = .001\,, \mu = 0$.

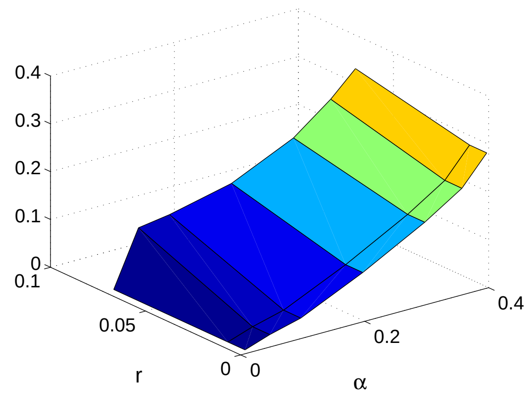

Figure 7: Mean relative error on $[0,\,10]^2$. $p = .3\,, b = .11\,, \sigma = .3\,, \beta = .1\,,\ \lambda = .001\,, \mu = 0$.

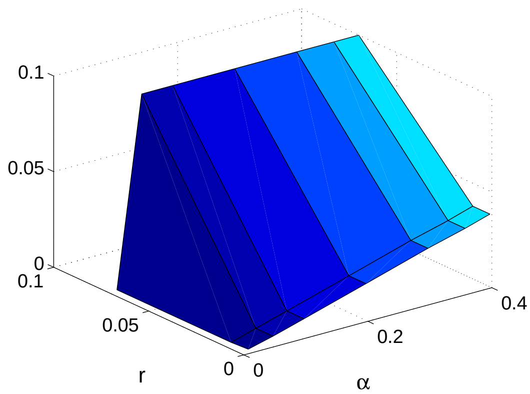

Figure 8: Mean relative error on $[0,\,1]^2$. $p = .3\,, b = .11\,, \sigma = .3\,, \beta = .1\,,\ \lambda = .001\,, \mu = 0$.

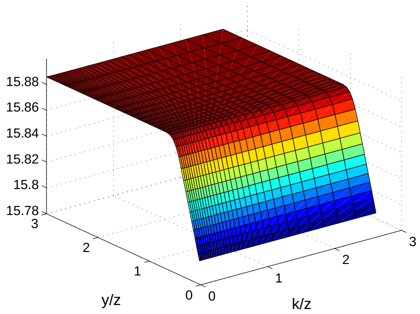

Figure 9: $\mathcal{V}_\varepsilon^h$ for $\alpha = 0.0$. $p = .3\,, r = .07\,, b = .11\,, \sigma = .3\,, \beta = .1\,,\ \lambda = .04\,, \mu = 0$.

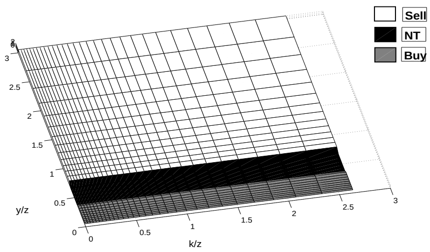

Figure 10: Partition of the domain $\mathbb{R}_+^2$ for $\alpha = 0.0$. $\bar{\pi} = .6349$ $p = .3\,, r = .07\,, b = .11\,, \sigma = .3\,, \beta = .1\,,\ \lambda = .04\,, \mu = 0$.

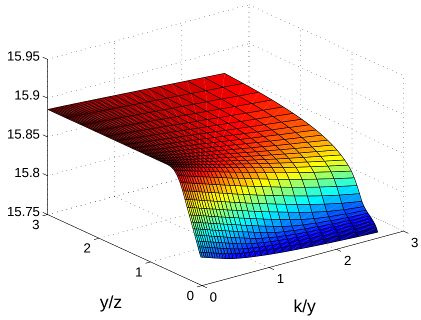

Figure 11: $\mathcal{V}_{\varepsilon}^{h}$ for $\alpha = 0.01$. $p = .3$, $r = .07$, $b = .11$, $\sigma = .3$, $\beta = .1$, $\lambda = .04$, $\mu = 0$.

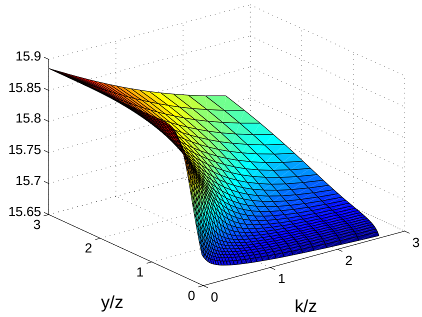

Figure 12: $\mathcal{V}_{\varepsilon}^{h}$ for $\alpha = 0.1$. $p = .3$, $r = .07$, $b = .11$, $\sigma = .3$, $\beta = .1$, $\lambda = .04$, $\mu = 0$.

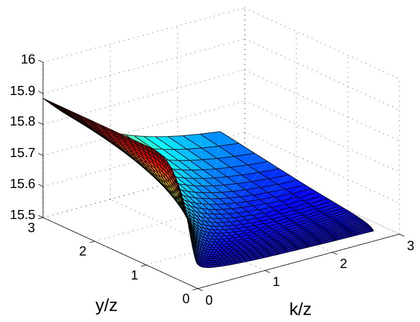

Figure 13: $\mathcal{V}_\varepsilon^h$ for $\alpha = 0.3$. $p = .3\,, r = .07\,, b = .11\,, \sigma = .3\,, \beta = .1\,,\ \lambda = .04\,, \mu = 0$.

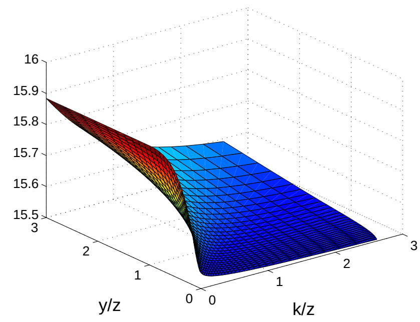

Figure 14: $\mathcal{V}_\varepsilon^h$ for $\alpha = 0.36$. $p = .3\,, r = .07\,, b = .11\,, \sigma = .3\,, \beta = .1\,,\ \lambda = .04\,, \mu = 0$.

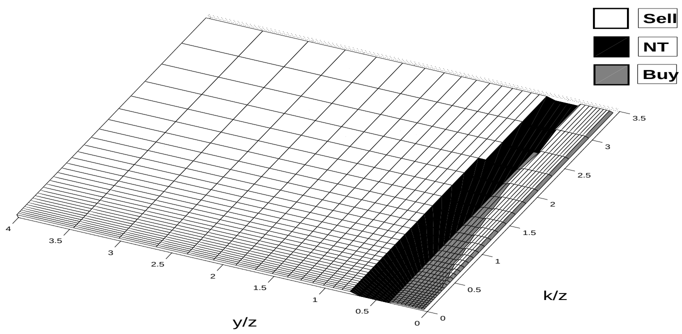

Figure 15: Partition of the domain $\mathbb{R}_+^2$ for $\alpha = 0.01$, $\bar{\pi} = .6349, \bar{\pi}^\alpha = .6300$. $p = .3\,, r = .07\,, b = .11\,, \sigma = .3\,, \beta = .1\,,\ \lambda = .04\,, \mu = 0$.

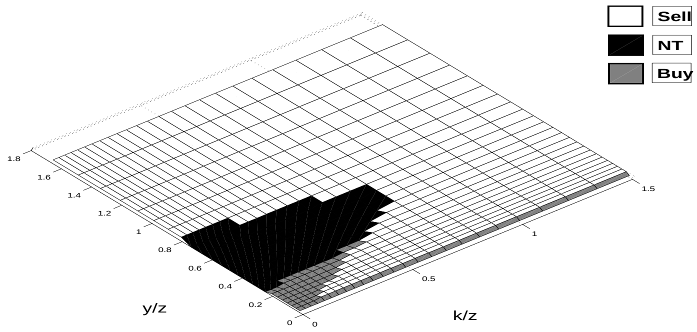

Figure 16: Partition of the domain $\mathbb{R}^2_+$ for$\alpha = 0.1$, $\bar{\pi} = .6349\,, \bar{\pi}^{\alpha} = .5683\,.$ $p = .3\,, r = .07\,, b = .11\,, \sigma = .3\,, \beta = .1\,,\ \lambda = .04\,, \mu = 0\,.$

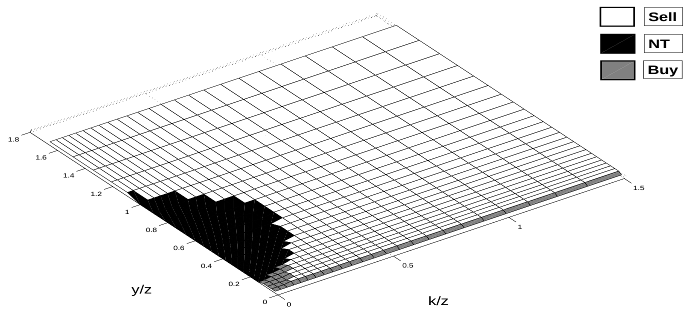

Figure 17: Partition of the domain $\mathbb{R}^2_+$ for $\alpha = 0.3$, $\bar{\pi} = .6349\,, \bar{\pi}^{\alpha} = .2268\,.$ $p = .3\,, r = .07\,, b = .11\,, \sigma = .3\,, \beta = .1\,,\ \lambda = .04\,, \mu = 0\,.$

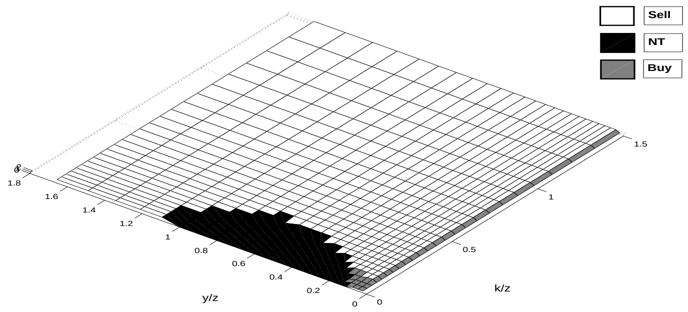

Figure 18: Partition of the domain $\mathbb{R}^2_+$ for $\alpha = 0.36$, $\bar{\pi} = .6349\,, \bar{\pi}^{\alpha} = .2268\,.$ $p = .3\,, r = .07\,, b = .11\,, \sigma = .3\,, \beta = .1\,,\ \lambda = .04\,, \mu = 0\,.$

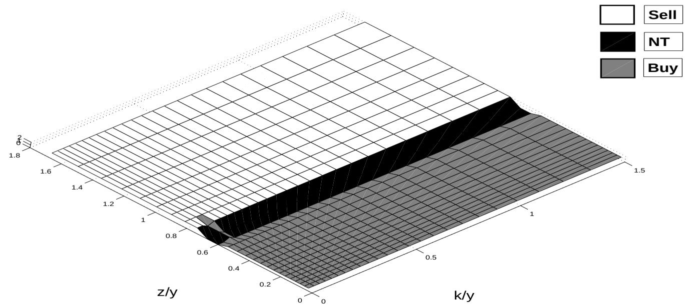

Figure 19: Partition of the domain $\mathbb{R}^2_+$ for $\alpha = 0.0$, $\bar{\pi} = .6349\,, \bar{\pi}^{\alpha} = .6349\,.$ $p = .3\,, r = .07\,, b = .11\,, \sigma = .3\,, \beta = .1\,,\ \lambda = .001\,, \mu = 0\,.$

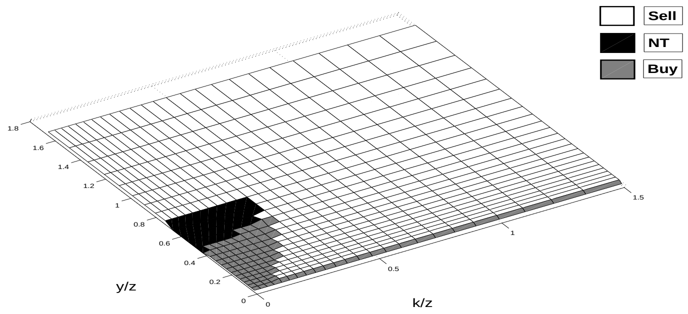

Figure 20: Partition of the domain $\mathbb{R}_+^2$ for $\alpha = 0.05$, $\bar{\pi} = .6349\,, \bar{\pi}^{\alpha} = .6068\,.$ $p = .3\,, r = .07\,, b = .11\,, \sigma = .3\,, \beta = .1\,,\ \lambda = .001\,, \mu = 0\,.$

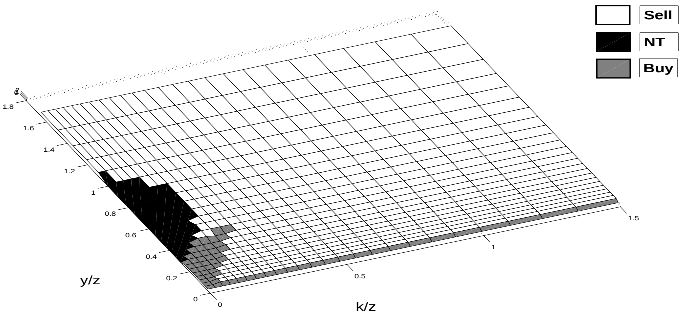

Figure 21: Partition of the domain $\mathbb{R}_+^2$ for $\alpha = 0.3$, $\bar{\pi} = .6349\,, \bar{\pi}^{\alpha} = .2268\,.$ $p = .3\,, r = .07\,, b = .11\,, \sigma = .3\,, \beta = .1\,,\ \lambda = .001\,, \mu = 0\,.$

Appendix A : Dynamic programming equation (7.1)

The expressions of the coefficients of equation (7.1) :

$$\min_{\substack{\upsilon\in\{0,1,2\}\\ c\geq 0}} \left\{ \beta(c,\upsilon)\Psi^{\varepsilon} - b(c,\upsilon)\cdot D\Psi^{\varepsilon} - \frac{1}{2}\sum_{i,j=1}^{2} a_{ij}(c,\upsilon)D^2_{ij}\Psi^{\varepsilon} \ - g(c,\upsilon) \right\} \ = \ 0 \ .$$

are given by

$$\begin{aligned}
\beta(c,\upsilon;\zeta_1,\zeta_2) &= \tilde{\beta}(c,\upsilon;\xi_1,\xi_2)\ , \\
b_i(c,\upsilon;\zeta_1,\zeta_2) &= (1-\zeta_i)^2 \left\{ \tilde{b}_i(c,\upsilon;\xi_1,\xi_2) - (1-\zeta_1)\tilde{a}_{ii}(c,\upsilon;\xi_1,\xi_2) \right\}\ ,\ i=1,2\ , \\
a_{ii}(c,\upsilon;\zeta_1,\zeta_2,t) &= (1-\zeta_i)^4\ ,\ i=1,2\ , \\
a_{ij}(c,\upsilon;\zeta_1,\zeta_2) &= (1-\zeta_i)(1-\zeta_j)\ ,\ i\neq j\ ,
\end{aligned}$$

where $\xi_1 := \zeta_1/(1-\zeta_1)$, $\xi_2 := \zeta_2/(1-\zeta_2)$, and

$$
\begin{aligned}
\tilde{\beta}(c,0;\xi_1,\xi_2) &= \beta - pr - pa(\delta - r)\xi_1 + pbr\xi_2 + a^2p(1-p)\frac{\sigma^2}{2}\xi_1^2 + cp \,, \\
\tilde{b}_1(c,0;\xi_1,\xi_2) &= \xi_1 \left\{(\delta - r)\,(1 - a\xi_1) + rb\xi_2 - a(1-p)\sigma^2\xi_1(1-a\xi_1) \,+ c \,\right\}, \\
\tilde{b}_2(c,0;\xi_1,\xi_2) &= \xi_2 \left\{-\,(\delta - r)\,\xi_1 - r(1-b\xi_2) + a^2(1-p)\sigma^2\xi_1^2 \,+ c\right\};, \\
\tilde{a}_{11}(c,u;\xi_1,\xi_2) &= \sigma^2\xi_1^2(1-a\xi_1)^2 \,, \\
\tilde{a}_{22}(c,0;\xi_1,\xi_2) &= \sigma^2\xi_1^2a^2\xi_2^2 \,, \\
\tilde{a}_{12}(c,0;\xi_1,\xi_2) &= -\sigma^2\xi_1^2a\xi_2(1-a\xi_1)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\beta}(c,1;\xi_1,\xi_2) &= p\,(\lambda + \mu)\,, \\
\tilde{b}_1(c,1;\xi_1,\xi_2) &= (\lambda + \mu)\xi_1 + 1 \,, \\
\tilde{b}_2(c,1;\xi_1,\xi_2) &= (\lambda + \mu)\xi_2 + 1 \,, \\
\tilde{a}_{ij}(c,1\,;\xi_1,\xi_2) &= 0 \quad \text{for} \;\; i,j \in \{1,2\} \,,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\beta}(c,2;\xi_1,\xi_2) &= 0 \,, \\
\tilde{b}_1(c,2;\xi_1,\xi_2) &= -1 \,, \\
\tilde{b}_2(c,2;\xi_1,\xi_2) &= -f^{\varepsilon}(\xi_1 + \xi_2)\,\frac{\xi_2}{\xi_1} \,, \\
\tilde{a}_{ij}(c,2;\xi_1,\xi_2) &= 0 \;\; \text{for} \;\; i,j \,\in\, \{1,2\} \,,
\end{aligned}
$$

$$
g(c,0) \;=\; \frac{c^p}{p} \;\; \text{and} \; g(c,1) \,=\, g(c,2) \,=\, 0 \,.
$$

Appendix B : Proof of Proposition 5.6

Preliminaries and notations

Since $\lambda = \mu = 0$, for $s \in \bar{\mathcal{S}}$ and $\nu \in \mathcal{A}(s)$, the process $Z^{s,\nu}$ is defined by the initial condition $Z_0^{s,\nu} = z := x + (1-\alpha)y + \alpha k$ and the dynamics

$$dZ_t^{s,\nu} \;=\; (rZ_t^{s,\nu} - C_t)\,dt + Y_t^{s,\nu}\tilde{\sigma}^\alpha\left(\tilde{\theta}^\alpha dt + dW_t\right) + r\alpha Y_t^{s,\nu}\left(1 - \frac{B_t^{s,\nu}}{P_t}\right)dt\,,$$

where

$$\tilde{\sigma}^\alpha \;:=\; (1-\alpha)\sigma \quad \text{and} \quad \tilde{\theta}^\alpha \;:=\; \theta - \frac{r\alpha}{\tilde{\sigma}^\alpha}\,.$$

Our purpose is to show that the value function V outperforms the maximal utility achieved in a frictionless financial market consisting of one bank account with the constant interest rate r, and one risky asset with price process P^α given by

$$dP_t^\alpha \;=\; P_t^\alpha\left[rdt + \tilde{\sigma}^\alpha(\tilde{\theta}^\alpha dt + dW_t)\right] \quad \text{and} \quad \tilde{P}_0^\alpha \;=\; P_0\,.$$

From Theorem 3.1, the solution of the optimal consumption-investment problem with price process P^α is given by the constant controls

$$\bar{\pi}^\alpha \;:=\; \frac{\tilde{\theta}^\alpha}{(1-p)\tilde{\sigma}^\alpha} \quad \text{and} \quad \bar{c}^\alpha \;:=\; \gamma\left(r, \tilde{\theta}^\alpha\right)\,,$$

and the corresponding optimal wealth process is defined by

$$\bar{Z}^\alpha = z \quad \text{and} \quad d\bar{Z}_t^\alpha \;=\; \bar{Z}^\alpha\left[(r-\bar{c})dt + \bar{\pi}^\alpha\tilde{\sigma}^\alpha\left(\tilde{\theta}^\alpha dt + dW_t\right)\right]\,.$$

In order to prove the required result, we shall fix an arbitrary maturity $T > 0$, and construct a sequence of admissible strategies $\hat{\nu}^{T,n}$ such that

$$V(s) \;\geq\; \lim_{n\to\infty} J_T\left(s, \hat{\nu}^{T,n}\right) \;=\; \mathbb{E}\left[\int_0^T e^{-\beta t}U\left(\bar{c}\bar{Z}_t^\alpha\right)dt\right]\,.$$

Then, the required result follows by sending T to infinity in this inequality.

A sequence of strategies tracking the Merton optimal policy

Let $T > 0$ be a fixed maturity. We construct a sequence of consumption-investment strategies $\hat{\nu}^{T,n}$ by forcing the tax basis B to be close to the spot price, and by *tracking* Merton's optimal strategy, i.e. keeping the proportion of wealth invested in the risky asset and the proportion of wealth dedicated for consumption :

$$\pi_t \ := \ \frac{Y_t}{Z_t}\,\mathbf{1}_{\{Z_t \neq 0\}} \quad \text{and} \quad c_t \ := \ \frac{C_t}{Z_t}\,\mathbf{1}_{\{Z_t \neq 0\}}\ , \quad 0 \le t \le T\, ,$$

close to the pair $(\bar{\pi}^\alpha, \bar{c}^\alpha)$.

To do this, we define a convenient sequence $(\nu^{T,n})_{n\ge 1} := (C^{T,n}, L^{T,n}, M^{T,n})_{n\ge1}$ for all $s = (x, y, k) \in \bar{\mathcal{S}}$. We shall denote by $\left(Y^{T,n}, Z^{T,n}, B^{T,n}\right) = \left(Y^{\nu^{T,n}}, Z^{\nu^{T,n}}, B^{\nu^{T,n}}\right)$ the corresponding state processes. For each integer $n \ge 1$, the consumption-investment strategy $\nu^{T,n}$ is defined as follows.

1. At time 0, set $\Delta L_0^{T,n} := \bar{\pi}^\alpha z$ and $\Delta M_0^{T,n} := 1$, so that

$$K_0^{T,n} \ = \ Y_0^{T,n}\ , \quad \pi_0^{T,n} \ = \ \bar{\pi}^\alpha\ , \quad \text{and} \quad Z_0^{T,n} \ = \ z\ .$$

2. At the final time T, set $\Delta L_T^{T,n} := 0$ and $\Delta M_T^{T,n} = 1$, so that all the wealth is transferred to the bank :

$$Y_T^{T,n} \ = \ 0 \quad \text{and} \quad X_T^{T,n} \ = \ Z_T^{T,n}\ .$$

3. In Step 4 below, we shall construct a sequence of stopping times $(\tau_k^{T,n})_{k\ge1}$. Our consumption strategy is defined by

$$C_t^{T,n} \ := \ \bar{c}^\alpha Z_t^{T,n} \quad \text{for} \quad 0 \le t \le T\ ,$$

and the investment strategy is piecewise constant :

$$dL_t^{T,n} \ = \ dM_t^{T,n} \ = \ 0 \quad \text{for all} \quad t \in [0, T] \setminus \{\tau_k^{T,n},\ k \ge 1\}\ .$$

4. We now introduce the sequence of stopping times $\tau_k^{T,n}$ as the hitting times of the pair process $(\pi^{T,n}, B^{T,n})$ of some barrier close to $(\bar{\pi}, 1)$. Set

$$\tau_0^{T,n} \;:=\; 0 \quad \text{and} \quad \tau_k^{T,n} \;:=\; T \wedge \tau_k^{\pi} \wedge \tau_k^{B} \;, \; \text{for } k \geq 1 \,,$$

where

$$\begin{aligned} \tau_k^{\pi} &:= \inf\left\{ t \geq \tau_{k-1}^{T,n} \;:\; |\pi_t^{T,n} - \bar{\pi}^{\alpha}| \;>\; n^{-1}\bar{\pi}^{\alpha} \right\}, \\ \tau_k^{B} &:= \inf\left\{ t \geq \tau_{k-1}^{T,n} \;:\; \left| 1 - \frac{B_t^{T,n}}{P_t} \right| \;>\; n^{-1} \right\} . \end{aligned}$$

Clearly, the sequence $\left(\tau_k^{T,n}\right)_{k\geq 0}$ is increasing, and converges to T.
5. Finally, we specify the jumps $\left(\Delta L^{T,n}, \Delta M^{T,n}\right)$ at each time $\tau_k^{T,n}$ by :

$$\Delta L_t^{T,n} \;:=\; \bar{\pi}^{\alpha} Z_t^{T,n} \text{ and } \Delta M_t^{T,n} \;:=\; 1 \quad \text{for} \quad t \in \{\tau_k^{T,n}, \; k \geq 0\} \;,$$

so that

$$\pi_t^{T,n} \;=\; \bar{\pi}^{\alpha} \text{ and } B_t^{t,n} \;=\; P_t \quad \text{for} \quad t \in \{\tau_k^{t,n}, \; k \geq 0\} \;.$$

Lemma B.1 *For each integer n, we have $\nu^{t,n} \in \mathcal{A}(s)$.*

Proof. By (5.2), we have

$$dZ_t^{T,n} \;=\; Z_t^{T,n} \left[(r - \bar{c}^{\alpha})\, dt + \pi_t^{T,\nu} \tilde{\sigma}^{\alpha} \left(\tilde{\theta}^{\alpha} dt + dW_t \right) + r\alpha \pi_t^{T,n} \left(1 - B_t^{T,n} \right) dt \right] .$$

Also $0 \;<\; (1 - n^{-1})\, \bar{\pi}^{\alpha} \;\leq\; \pi_t^{T,n} \;\leq\; (1 + n^{-1})\, \bar{\pi}^{\alpha}$. In particular, the process $\pi^{T,n}$ is bounded, so that the above dynamics implies that the process $Z^{T,n}$ is positive, and $Y_t^{T,n} = \pi_t^{T,n} Z_t^{T,n} > 0$ $\mathbb{P}-$ a.s. □

The convergence result

Lemma B.2 *There is a constant A depending on T such that*

$$\mathbb{E}\left[\sup_{0\le s\le t}\left|Z_t^{T,n}-\bar{Z}_t^{\alpha}\right|^2\right] \;\le\; n^{-2}\,Ae^{AT}\;.$$

Proof. By definition of the sequence of consumption-investment strategies $\left(\nu^{T,n}\right)$, we have

$$\sup_{0\le t\le T}\left|\pi_t^{T,n}-\bar{\pi}^{\alpha}\right| \;\le\; \frac{1}{n}\bar{\pi}^{\alpha} \quad\text{and}\quad \sup_{0\le t\le T}\left|1-\frac{B_t^{T,n}}{P_t}\right| \;\le\; \frac{1}{n}\;. \tag{B.1}$$

By direct computation, we decompose the difference $Z^{T,n}-\bar{Z}^{\alpha}$ into :

$$D_t \;:=\; Z_t^{T,n}-\bar{Z}_t^{\alpha} \;=\; F_t+G_t+H_t\;,$$

where

$$\begin{aligned}
F_t &:= \int_0^t D_t\left[(r-\bar{c}^{\alpha})du+\pi_u^{T,n}\left(\tilde{\sigma}^{\alpha}\tilde{\theta}^{\alpha}du+\alpha r\left(1-\frac{B_u^{T,n}}{P_u}\right)du+\tilde{\sigma}^{\alpha}dW_u\right)\right]\;,\\
G_t &:= \int_0^t \bar{Z}_u^{\alpha}\tilde{\sigma}^{\alpha}\left(\pi_t^{T,n}-\bar{\pi}^{\alpha}\right)(\theta^{\alpha}du+dW_u)\;,\\
H_t &:= \alpha r\int_0^t \pi_u^{T,n}\bar{Z}_u^{\alpha}\left(1-\frac{B_u^{T,n}}{P_u}\right)du\,.
\end{aligned}$$

In the subsequent calculation, A will denote a generic ($T-$dependent) constant whose value may change from line to line. We shall also denote by $V_t^* := \sup_{0\le u\le t}|V_u|$ for all process $(V_t)_t$.

We first start by estimating the first component F. Observe that the process $\pi^{t,n}$ is bounded by $2\bar{\pi}$. Then

$$|F_t|^2 \;\le\; A\int_0^t |D_u^*|^2\,du+2\left(\int_0^t D_u\pi_u^{t,n}\tilde{\kappa}dW_u\right)^2\;.$$

By the Buckholder-Davis-Gundy inequality, this provides

$$\mathbb{E}\,|F_t^*|^2 \;\le\; A\int_0^t \mathbb{E}\,|D_u^*|^2\,du\;.$$

By a similar calculation, it follows from (B.1) that :

$$\mathbb{E}|G_t^*|^2 \;\leq\; \frac{A}{n^2} \quad\text{and}\quad \mathbb{E}|H_t^*|^2 \;\leq\; \frac{A}{n^2}\,.$$

Collecting the above estimates, we see that :

$$E|D_t^*|^2 \;\leq\; \frac{A}{n^2} + K\int_0^t \mathbb{E}|D_u^*|^2 du \quad\text{for all}\quad t \leq T\,,$$

and we obtain the required result by the Gronwall inequality. □

8.1 Proof of Proposition 5.6

For $s = (x, y, k) \in \bar{\mathcal{S}}$, and $T > 0$,

$$\begin{aligned}\left|J_T(s,\nu^{T,n}) - \int_0^T e^{-\beta t} U\left(\bar{c}^\alpha \bar{Z}_t^\alpha\right) dt\right| &= \left|\int_0^T e^{-\beta t}\left(U\left(\bar{c}^\alpha Z_t^{T,n}\right) - U\left(\bar{c}^\alpha \bar{Z}_t^\alpha\right)\right) dt\right)\right| \\ &\leq A\int_0^T e^{-\beta t}\left|Z_t^{T,n} - \bar{Z}_t^\alpha\right|^p dt\end{aligned}$$

for some positive constant A. Now, by the estimate of Lemma B.2, it follows that

$$\lim_{n\to\infty} J_T(s,\nu^{T,n}) \;=\; \int_0^T e^{-\beta t} U\left(\bar{c}^\alpha \bar{Z}_t^\alpha\right) dt\,.$$

Since $V(s) \geq J_T(s,\nu^{T,n})$ for every $T > 0$, this implies that

$$V(s) \;\geq\; \lim_{T\to\infty}\int_0^T e^{-\beta t} U\left(\bar{c}^\alpha \bar{Z}_t^\alpha\right) dt \;=\; \gamma\left(r,\tilde{\theta}^\alpha\right)\frac{z^p}{p}\,.$$

□

Appendix C : Proof of Proposition 6.3

We organize our arguments into three steps.

Step 1. We first show that V_* is continuous. Let v be the function defined on $\mathbb{R}_+^2$ by

$$\mathcal{V}(\xi,\zeta) \;:=\; V_*\left(1-(1-\mu)[(1-\alpha)\xi+\alpha\zeta],\xi,\zeta\right)\,, \tag{C.2}$$

so that

$$V_*(x,y,k) \;=\; z^p\, \mathcal{V}\left(\frac{y}{z},\frac{k}{z}\right) \quad \text{with} \quad z = x+(1-\mu)[(1-\alpha)y+\alpha k]\,, \tag{C.3}$$

by the homotheticity property of V stated in Proposition 5.2. By Proposition 6.1, we have $\mathbf{g^b}\cdot DV_* \geq 0$ and $\mathbf{g^s}\cdot DV_* \geq 0$ in the viscosity sense. By a direct change of variable, this implies that $\mathcal{V}$ is a lower semicontinuous viscosity supersolution of the equation

$$p\mathcal{V} - (\xi\mathcal{V}_\xi + \zeta\mathcal{V}_\zeta) - \frac{1}{\lambda+\mu}\,(\mathcal{V}_\xi+\mathcal{V}_\zeta) \;\geq\; 0 \quad \text{and} \quad \xi\mathcal{V}_\xi + \zeta\mathcal{V}_\zeta \;\geq\; 0\,.$$

Also, from the monotonicity of V in x, y and k, it follows that $\mathcal{V}$ is a lower semicontinuous viscosity supersolution of the equation

$$p\mathcal{V} - (\xi\mathcal{V}_\xi + \zeta\mathcal{V}_\zeta) + \frac{1}{1-\mu}\ \min\left\{0\ ,\ \frac{1}{1-\alpha}\,\mathcal{V}_\xi\ ,\ \frac{1}{\alpha}\,\mathcal{V}_\zeta\right\} \;\geq\; 0$$

Observe that $\mathcal{V}$ is bounded as a consequence of the upper bound provided in Proposition 5.5. We then deduce from the above viscosity supersolution properties that $-|\nabla\mathcal{V}| \;\geq\; -A$ on $(0,\infty)^2$, in the viscosity sense, for some constant A. Hence $\mathcal{V}$ is Lipschitz-continuous.

Step 2. We next prove that $V_* = V^*$ on the axis $\{(x,0,0)\ :\ x\geq 0\}$. Notice that for all $s=(x,y,k)\in\bar{\mathcal{S}}$ and $z := x+(1-\mu)\,[(1-\alpha)y+\alpha k]$

$$V(z,0,0) \;\leq\; V(s) \;\leq\; V\left(z+(1+\lambda)y,0,0\right)\ . \tag{C.4}$$

Before proving these inequalities, let us complete the proof of $V_* = V^*$ on $\{(x,0,0)\ :\ x\geq 0\}$. For an arbitrary $x\in\mathbb{R}_+$, let $\{s_n=(x_n,y_n,k_n), n\geq 1\}$, $\{s'_n=(x'_n,y'_n,k'_n), n\geq 1\}$ be two sequences in $\bar{\mathcal{S}}$ such that

$$s_n, s'_n \;\underset{n\to\infty}{\longrightarrow}\; (x,0,0)\ ,\quad V(s_n) \;\underset{n\to\infty}{\longrightarrow}\; V_*(x,0,0)\ ,\quad \text{and}\quad V(s'_n) \;\underset{n\to\infty}{\longrightarrow}\; V^*(x,0,0)\ .$$

By (C.4) together with the homotheticity property of Proposition 5.2, we see that

$$
\begin{aligned}
V(s_n') \;\leq\; V(z_n' + (1+\lambda)y_n') &= (z_n' + (1+\lambda)y_n')^p \, V(1,0,0) \;, \\
V(s_n) \;\geq\; V(z_n) &= (z_n)^p \, V(1,0,0) \;,
\end{aligned}
$$

where $z_n = x_n + (1-\mu)\,[(1-\alpha)y_n + \alpha k_n]$ and $z_n' = x_n' + (1-\mu)\,[(1-\alpha)y_n' + \alpha k_n']$. Letting $n \to \infty$ in the above inequalities, and recalling that z_n, $z_n' + (1+\lambda)y_n' \to x$, we get the required result.

We now turn to the proof of (C.4).

- The left hand side of (C.4) holds since for each consumption-investment strategy $\nu = (C, L, M) \in \mathcal{A}(z, 0, 0)$, the strategy $\bar{\nu} := \nu + \{1 - \Delta M_0\}\,(0,0,1) \in \mathcal{A}(s)$.
- The right-hand side of (C.4) holds since for each $\nu = (C, L, M) \in \mathcal{A}(s)$, the strategy $\bar{\nu} := \nu + \{y(1-\Delta M_0)\}\,(0,1,0) \in \mathcal{A}(\bar{s})$ where $\bar{s} := (z + (1+\lambda)y, 0, 0)$.

Indeed, since ν and $\bar{\nu}$ differ only by the jump at time $t = 0$ the dynamics of the state processes $S^{s,\nu}$ and $S^{\bar{s},\bar{\nu}}$ are such that $Y_0^{s,\nu} = Y_0^{\bar{s},\bar{\nu}}$ and therefore $Y_t^{s,\nu} = Y_t^{\bar{s},\bar{\nu}}$ for $t \geq 0$, and

$$
\begin{aligned}
K_t^{\bar{s},\bar{\nu}} - K_t^{s,\nu} &= \left(K_0^{\bar{s},\bar{\nu}} - K_0^{s,\nu}\right) e^{-M_t^c} \prod_{0 \leq s \leq t} (1 - \Delta M_s) \\
&\leq \left(K_0^{\bar{s},\bar{\nu}} - K_0^{s,\nu}\right)^+ \;=\; (y-k)^+(1-\Delta M_0) \;.
\end{aligned}
$$

Then the corresponding liquidation value processes $Z^{s,\nu}$ and $Z^{\bar{s},\bar{\nu}}$ are such that

$$
\begin{aligned}
Z_t^{\bar{s},\bar{\nu}} - Z_t^{s,\nu} &= e^{rt} \left\{ Z_0^{\bar{s},\bar{\nu}} - Z_0^{s,\nu} - \alpha(1-\mu) \int_0^t e^{-rs} \left(K^{s,\nu} - K^{\bar{s},\bar{\nu}}\right)_s ds \right\} \\
&\geq e^{rt} \left\{ Z_0^{\bar{s},\bar{\nu}} - Z_0^{s,\nu} - (1-\mu)(K_0^{s,\nu} - K_0^{\bar{s},\bar{\nu}})^+ \right\} \\
&= e^{rt} \left\{ (1-\mu)y - (1-\mu)(y-k)^+(1-\Delta M_0) \right\} \;\geq\; 0 \;.
\end{aligned}
$$

It follows that $Z^{\bar{s},\nu} \geq 0$, hence $\bar{\nu} \in \mathcal{A}(\bar{s})$.

Step 3. By Proposition 5.5, the semi-continuous envelopes V^* and V_* satisfy the polynomial growth condition $(V^* - V_*)^+ \in \mathrm{USC}_p(\bar{\mathcal{S}})$. We also know from Proposition

6.1 that they are respectively a constrained subsolution and supersolution of (6.1). Finally, we have proved in Steps 1 and 2 above that the function V_* is continuous, and $(V_* - V^*)(x, 0, 0) = 0$. We are then in the context of the comparison result of Proposition 6.2, and we conclude that $V_* \geq V^*$. Since the reverse inequality holds by definition, this shows that $V_* = V^*$. Then, the function $\mathcal{V}$ defined in (C.2), which was shown to be Lipschitz-continuous in Step 1, can be defined equivalently in terms of V as in the statement of the proposition. □

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