

# Discrete Time Approximation and Monte-Carlo Simulation of Backward Stochastic Differential Equations

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## Abstract

We suggest a discrete-time approximation for decoupled forward-backward stochastic differential equations. The  $L^p$  norm of the error is shown to be of the order of the time step. Given a simulation-based estimator of the conditional expectation operator, we then suggest a backward simulation scheme, and we study the induced  $L^p$  error. This estimate is more investigated in the context of the Malliavin approach for the approximation of conditional expectations. Extensions to the reflected case are also considered.

**Key words:** Monte-Carlo methods for (reflected) forward-backward SDE's, Malliavin calculus, regression estimation.

**MSC 1991 subject classifications:** 65C05, 60H07, 62G08.

# 1 Introduction

In this paper, we are interested in the problem of discretization and simulation of the (decoupled) forward-backward stochastic differential equation (SDE, hereafter) on the time interval  $[0, 1]$  :

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad dY_t = f(t, X_t, Y_t, Z_t)dt - Z_t \cdot dW_t$$

$$X_0 = x \text{ and } Y_1 = g(X_1),$$

where  $W$  is a standard Brownian motion,  $b$ ,  $\sigma$  and  $f$  are valued respectively in  $\mathbb{R}^n$ ,  $\mathbb{M}^n$  and  $\mathbb{R}$ . The analysis of this paper extends easily to the case of reflected backward SDE's with  $z$ -independent generator  $f$ . This extension is presented in the last section of this paper.

Notice that the problem of discretization and simulation of the forward components  $X$  is well-understood, see e.g. [18], and we are mainly interested in the backward component  $Y$ . Given a partition  $\pi : 0 = t_0 < \dots < t_n = 1$  of the interval  $[0, 1]$ , we consider the first naive Euler discretization of the backward SDE :

$$\tilde{Y}_i^\pi - \tilde{Y}_{i-1}^\pi = f(t_{i-1}, X_{t_{i-1}}^\pi, \tilde{Y}_{i-1}^\pi, \tilde{Z}_{i-1}^\pi)(t_i - t_{i-1}) - \tilde{Z}_{i-1}^\pi \cdot (W_{t_i} - W_{t_{i-1}}),$$

together with the final data  $\tilde{Y}_{t_n}^\pi = g(X_{t_n}^\pi)$ . Of course, given  $(\tilde{Y}_{t_i}, \tilde{Z}_{t_i})$ , there is no  $\mathcal{F}_{t_{i-1}}$ -measurable random variables  $(\tilde{Y}_{t_{i-1}}, \tilde{Z}_{t_{i-1}})$  which satisfy the above equation. A workable backward induction scheme is obtained by taking conditional expectations. This suggests naturally the following backward procedure for the definition of the discrete-time approximation  $(Y^\pi, Z^\pi)$  :

$$\begin{aligned} Y_{t_n}^\pi &= g(X_{t_n}^\pi), \quad Z_{t_{i-1}}^\pi = (t_i - t_{i-1})^{-1} E[Y_{t_i}^\pi(W_{t_i} - W_{t_{i-1}}) | \mathcal{F}_{t_{i-1}}] \\ Y_{t_{i-1}}^\pi &= E[Y_{t_i}^\pi | \mathcal{F}_{t_{i-1}}] + f(t_{i-1}, X_{t_{i-1}}^\pi, Y_{t_{i-1}}^\pi, Z_{t_{i-1}}^\pi)(t_i - t_{i-1}), \end{aligned}$$

for all  $i = 1, \dots, n$ . Here  $\{\mathcal{F}_t\}$  is the completed filtration of the Brownian motion  $W$ . Our first main result, Theorem 3.1, is an estimate of the error  $Y^\pi - Y$  of the order of  $|\pi|^{-1}$ . A similar error estimate was obtained by [27], but with a slightly different, and less natural, discretization scheme.

The key-ingredient for the simulation of the backward component  $Y$  is the following well-known result : under standard Lipschitz conditions, the backward component and the associated control  $(Y, Z)$ , which solves the backward SDE, can be expressed as a function of  $X$ , i.e.  $(Y_t, Z_t) = (u(t, X_t), v(t, X_t))$ ,  $t \leq 1$ , for some deterministic functions  $u$  and  $v$ . Then, the conditional expectations, involved in the above discretization scheme, reduce to the regression of  $Y_{t_i}^\pi$  and  $Y_{t_i}^\pi(W_{t_i} - W_{t_{i-1}})$  on the random variable  $X_{t_{i-1}}^\pi$ . For instance, one can use the classical kernel regression estimation, as in [9], the basis projection method suggested by [21], see also [11], or the Malliavin approach introduced in [15], and further developed in [7], see also [19].

Given a simulation-based approximation  $\hat{E}_{i-1}^\pi$  of  $E[\cdot | \mathcal{F}_{t_{i-1}}]$ , we then analyse the backward simulation scheme

$$\begin{aligned} \hat{Y}_{t_n}^\pi &= g(X_{t_n}^\pi), \quad \hat{Z}_{t_{i-1}}^\pi = (t_i - t_{i-1})^{-1} \hat{E}_{i-1}^\pi [\hat{Y}_{t_i}^\pi(W_{t_i} - W_{t_{i-1}})] \\ \hat{Y}_{t_{i-1}}^\pi &= \hat{E}_{i-1}^\pi [\hat{Y}_{t_i}^\pi] + f(t_{i-1}, X_{t_{i-1}}^\pi, \hat{Y}_{t_{i-1}}^\pi, \hat{Z}_{t_{i-1}}^\pi)(t_i - t_{i-1}), \end{aligned}$$

Let  $\eta$  denote the maximum simulation error of  $(\hat{E}_{i-1} - E_{i-1}) \left[ \hat{Y}_{t_i}^\pi \right]$  and  $(\hat{E}_{i-1} - E_{i-1}) \left[ \hat{Y}_{t_i}^\pi(W_{t_i} - W_{t_{i-1}}) \right]$ . Observe that  $\eta$  depends both on the number of simulated paths and the time step  $|\pi|$ . Also, given a number  $N$  of simulated paths for the regression approximation, the best estimate that one can expect for  $\eta$  is  $N^{1/2}$ , the classical Monte Carlo error deduced from the Central Limit Theorem. Our second main result, Theorem 4.1, states that the  $L^p$ -norm of the error due to the regression estimation is of the order  $|\pi|^{-1}\eta$ . This rate of convergence is easily understood in the case of a regular grid, as the scheme involves  $|\pi|^{-1}$  steps, each of them requiring some regression approximation. As a consequence of this result, for  $|\pi| = n^{-1}$ , we see that in order to achieve the rate  $n^{-1/2}$ , one needs to use at least  $N = n^3$  simulated paths for the regression estimation.

We next investigate in more details the error  $(\hat{E}_{i-1} - E_{i-1}) \left[ \hat{Y}_{t_i}^\pi \right]$  and  $(\hat{E}_{i-1} - E_{i-1}) \left[ \hat{Y}_{t_i}^\pi(W_{t_i} - W_{t_{i-1}}) \right]$ . More precisely, we examine a common difficulty to the kernel and the Malliavin regression estimation methods : in both methods the regression estimator is the ratio of two statistics, which is not guaranteed to be integrable. We solve this difficulty by introducing a truncation procedure along the above backward simulation scheme. In Theorem 5.1, we show that this reduces the error to the analysis of the "integrated standard deviation" of the regression estimator. This quantity is estimated for the Malliavin regression estimator in §6. The results of this section imply an estimate of the  $L^p$ -error  $\hat{Y}^\pi - Y^\pi$  of the order of  $|\pi|^{-1-d/4p}N^{-1/2p}$ , where  $N$  is the number of simulated paths for the regression estimation, see Theorem 6.2. In order to better understand this result, let  $\pi = n^{-1}$  ( $n$  time-steps), then in order to achieve an error estimate of the order  $n^{-1/2}$ , one needs to use  $N = n^{3p+d/2}$  simulated paths for the regression estimation at each step. In the limit case  $p = 1$ , this reduces to  $N = n^{3+d/2}$ . Unfortunately, we have not been able to obtain the best expected  $N = n^3$  number of simulated paths.

We conclude this introductory section by some references to the existing alternative numerical methods for backward SDE's. First, the four step algorithm was developed by [23] to solve a class of more general forward-backward SDE's, see also [13]. Their method is based on the finite difference approximation of the associated PDE, which unfortunately can not be managed in high dimension. Recently, a quantization technique was suggested by [3] and [4] for the resolution of reflected backward SDE's when the generator  $f$  does not depend on the control variable  $z$ . This method is based on the approximation of the continuous time processes on a finite grid, and requires a further estimation of the transition probabilities on the grid. Discrete-time scheme based on the approximation of the Brownian motion by some discrete process have been considered in [10], [12], [8], [1] and [22]. This technique allows to simplify the computation of the conditional expectations involved at each time step. However, the implementation of these schemes in high dimension is questionable. We finally refer to [2] for a random time schemes, which requires a further approximation of conditional expectations to give an implementation.

**Notations :** We shall denote by  $\mathbb{M}^{n,d}$  the set of all  $n \times d$  matrices with real coefficients. We simply denote  $\mathbb{R}^n := \mathbb{M}^{n,1}$  and  $\mathbb{M}^n := \mathbb{M}^{n,n}$ . We shall denote by  $|a| := \left( \sum_{i,j} a_{i,j}^2 \right)^{1/2}$  the Euclidian norm on  $\mathbb{M}^{n,d}$ ,  $a^*$  the transpose of  $a$ ,  $a^k$  the  $k$ -th column of  $a$ , or the  $k$ -th

component if  $a \in \mathbb{R}^d$ . Finally, we denote by  $x \cdot y := \sum_i x_i y_i$  the scalar product on  $\mathbb{R}^n$ .

## 2 The simulation and discretization problem

Let  $(\Omega, \{\mathcal{F}(t)\}_{0 \leq t \leq 1}, P)$  be a filtered probability space equipped with a  $d$ -dimensional standard Brownian motion  $\{W(t)\}_{0 \leq t \leq 1}$ .

Consider two functions  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\sigma : \mathbb{R}^d \rightarrow \mathbb{M}^d$  satisfying the Lipschitz condition :

$$|b(u) - b(v)| + |\sigma(u) - \sigma(v)| \leq K|u - v| \quad (2.1)$$

for some constant  $K$  independent of  $u, v \in \mathbb{R}^d$ . Then, it is well-known that, for any initial condition  $x \in \mathbb{R}^d$ , the (forward) stochastic differential equation

$$X_t = x + \int_0^t b(X_s) ds + \sigma(X_s) dW_s \quad (2.2)$$

has a unique  $\{\mathcal{F}_t\}$ -adapted solution  $\{X_t\}_{0 \leq t \leq 1}$  satisfying

$$E \left\{ \sup_{0 \leq t \leq 1} |X_t|^2 \right\} < \infty ,$$

see e.g. [17]. Next, let  $f : [0, 1] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  and  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  be two functions satisfying the Lipschitz condition

$$|g(u) - g(v)| + |f(\xi) - f(\zeta)| \leq K(|u - v| + |\xi - \zeta|) \quad (2.3)$$

for some constant  $K$  independent of  $u, v \in \mathbb{R}^d$  and  $\xi, \zeta \in [0, 1] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ . Consider the backward stochastic differential equation :

$$Y_t = g(X_1) + \int_t^1 f(s, X_s, Y_s, Z_s) ds - \int_t^1 Z_s \cdot dW_s , \quad t \leq 1 . \quad (2.4)$$

The Lipschitz condition (2.3) ensures the existence and uniqueness of an adapted solution  $(Y, Z)$  to (2.4) satisfying

$$E \left\{ \sup_{0 \leq t \leq 1} |Y_t|^2 + \int_0^1 |Z_t|^2 dt \right\} < \infty ,$$

see e.g. [24]. Equations (2.2)-(2.4) define a decoupled system of forward-backward stochastic differential equations. The purpose of this paper is to study the problem of discretization and simulation of the components  $(X, Y)$  of the solution of (2.2)-(2.4).

**Remark 2.1** Under the Lipschitz conditions (2.1)-(2.3), it is easily checked that :

$$|Y_t| \leq a_0 + a_1 |X_t| , \quad 0 \leq t \leq 1 ,$$

for some parameters  $a_0$  and  $a_1$  depending on  $K, b(0), \sigma(0), g(0)$  and  $f(0)$ . In the subsequent paragraph, we shall derive a similar bound on the discrete-time approximation of  $Y$ . The *a priori* knowledge of such a bound will be of crucial importance for the simulation scheme suggested in this paper.  $\square$

### 3 Discrete-time approximation error

In order to approximate the solution of the above BSDE, we introduce the following discretized version. Let  $\pi : 0 = t_0 < t_1 < \dots < t_n = 1$  be a partition of the time interval  $[0, 1]$  with mesh

$$|\pi| := \max_{1 \leq i \leq n} |t_i - t_{i-1}|.$$

Throughout this paper, we shall use the notations :

$$\Delta_i^\pi = t_i - t_{i-1} \quad \text{and} \quad \Delta^\pi W_i = W_{t_i} - W_{t_{i-1}}, \quad i = 1, \dots, n.$$

The forward component  $X$  will be approximated by the classical Euler scheme :

$$\begin{aligned} X_{t_0}^\pi &= X_{t_0} \\ X_{t_i}^\pi &= X_{t_{i-1}}^\pi + b(X_{t_{i-1}}^\pi) \Delta_i^\pi + \sigma(X_{t_{i-1}}^\pi) \Delta^\pi W_i, \quad \text{for } i = 1, \dots, n, \end{aligned} \quad (3.1)$$

and we set

$$X_t^\pi := X_{t_{i-1}}^\pi + b(X_{t_{i-1}}^\pi) (t - t_{i-1}) + \sigma(X_{t_{i-1}}^\pi) (W_t - W_{t_{i-1}}) \quad \text{for } t \in (t_{i-1}, t_i).$$

We shall denote by  $\{\mathcal{F}_i^\pi\}_{0 \leq i \leq n}$  the associated discrete-time filtration :

$$\mathcal{F}_i^\pi := \sigma(X_{t_j}^\pi, j \leq i).$$

Under the Lipschitz conditions on  $b$  and  $\sigma$ , the following  $L^p$  estimate for the error due to the Euler scheme is well-known :

$$\limsup_{|\pi| \rightarrow 0} |\pi|^{-1/2} E \left[ \sup_{0 \leq t \leq 1} |X_t - X_t^\pi|^p + \max_{1 \leq i \leq n} \sup_{t_{i-1} \leq t \leq t_i} |X_t - X_{t_{i-1}}^\pi|^p \right]^{1/p} < \infty, \quad (3.2)$$

for all  $p \geq 1$ , see e.g. [18]. We next consider the following natural discrete-time approximation of the backward component  $Y$  :

$$\begin{aligned} Y_1^\pi &= g(X_1^\pi), \\ Y_{t_{i-1}}^\pi &= E_{i-1}^\pi [Y_{t_i}^\pi] + f(t_{i-1}, X_{t_{i-1}}^\pi, Y_{t_{i-1}}^\pi, Z_{t_{i-1}}^\pi) \Delta_i^\pi, \quad 1 \leq i \leq n, \end{aligned} \quad (3.3)$$

$$Z_{t_{i-1}}^\pi = \frac{1}{\Delta_i^\pi} E_{i-1}^\pi [Y_{t_i}^\pi \Delta^\pi W_i] \quad (3.4)$$

where  $E_i^\pi[\cdot] = E[\cdot | \mathcal{F}_i^\pi]$ . The above conditional expectations are well-defined at each step of the algorithm. Indeed, by a backward induction argument, it is easily checked that  $Y_{t_i}^\pi \in L^2$  for all  $i$ .

**Remark 3.1** Using an induction argument, it is easily seen that the random variables  $Y_{t_i}^\pi$  and  $Z_{t_i}^\pi$  are deterministic functions of  $X_{t_i}^\pi$  for each  $i = 0, \dots, n$ . From the Markov feature of the process  $X^\pi$ , it then follows that the conditional expectations involved in (3.3)-(3.4) can be replaced by the corresponding regressions :

$$E_{i-1}^\pi [Y_{t_i}^\pi] = E [Y_{t_i}^\pi | X_{t_{i-1}}^\pi] \quad \text{and} \quad E_{i-1}^\pi [Y_{t_i}^\pi \Delta^\pi W_i] = E [Y_{t_i}^\pi \Delta^\pi W_i | X_{t_{i-1}}^\pi].$$

For later use, we observe that the same argument shows that :

$$E_{i-1} [Y_{t_i}^\pi] = E [Y_{t_i}^\pi | X_{t_{i-1}}^\pi] \quad \text{and} \quad E_{i-1} [Y_{t_i}^\pi \Delta^\pi W_i] = E [Y_{t_i}^\pi \Delta^\pi W_i | X_{t_{i-1}}^\pi],$$

where  $E_i[\cdot] := E[\cdot | \mathcal{F}_{t_i}]$  for all  $0 \leq i \leq n$ .  $\square$

Notice that  $(Y^\pi, Z^\pi)$  differs from the approximation scheme suggested in [27] which involves the computation of  $(2d+1)$  conditional expectations at each step.

For later use, we need to introduce a continuous-time approximation of  $(Y, Z)$ . Since  $Y_{t_i}^\pi \in L^2$  for all  $1 \leq i \leq n$ , we deduce, from the classical martingale representation theorem, that there exists some square integrable process  $Z^\pi$  such that :

$$\begin{aligned} Y_{t_{i+1}}^\pi &= E [Y_{t_{i+1}}^\pi | \mathcal{F}_{t_i}] + \int_{t_i}^{t_{i+1}} Z_s^\pi \cdot dW_s \\ &= E_i^\pi [Y_{t_{i+1}}^\pi] + \int_{t_i}^{t_{i+1}} Z_s^\pi \cdot dW_s. \end{aligned} \quad (3.5)$$

We then define :

$$Y_t^\pi := Y_{t_i}^\pi - (t - t_i) f(t_i, X_{t_i}^\pi, Y_{t_i}^\pi, Z_{t_i}^\pi) + \int_{t_i}^t Z_s^\pi \cdot dW_s, \quad t_i < t \leq t_{i+1}.$$

The following property of the  $Z^\pi$  is needed for the proof of the main result of this section.

**Lemma 3.1** *For all  $1 \leq i \leq n$ , we have :*

$$Z_{t_{i-1}}^\pi \Delta_i^\pi = E_{i-1} \left[ \int_{t_{i-1}}^{t_i} Z_s^\pi ds \right].$$

**Proof.** Since  $Y_{t_i}^\pi \in L^2$ , there is a sequence  $(\xi^k)_k$  of random variables in  $\mathbb{D}^{1,2}$  converging to  $Y_{t_i}^\pi$  in  $L^2$ . Then, it follows from the Clark-Ocone formula that, for all  $k$  :

$$\xi^k = E_{i-1} [\xi^k] + \int_{t_{i-1}}^{t_i} \zeta_s^k \cdot dW_s \quad \text{where} \quad \zeta_s^k := E [D_s \xi^k | \mathcal{F}_s], \quad t_{i-1} \leq s \leq t_i.$$

Using Remark 3.1, we now compute that :

$$\begin{aligned} Z_{t_{i-1}}^\pi \Delta_i^\pi &= E_{i-1} [Y_{t_i}^\pi \Delta^\pi W_i] = \lim_{k \rightarrow \infty} E_{i-1} [\xi^k \Delta^\pi W_i] \\ &= \lim_{k \rightarrow \infty} E_{i-1} \left[ \int_{t_{i-1}}^{t_i} D_s \xi^k ds \right] \\ &= \lim_{k \rightarrow \infty} E_{i-1} \left[ \int_{t_{i-1}}^{t_i} \zeta_s^k ds \right], \end{aligned} \quad (3.6)$$

by the Malliavin integration by parts formula and the tower property for conditional expectations. We then estimate that :

$$\left| E_{i-1} \left[ \int_{t_{i-1}}^{t_i} (\zeta_s^k - Z_s^\pi) ds \right] \right| \leq \left| E_{i-1} \left[ \int_{t_{i-1}}^{t_i} (\zeta_s^k - Z_s^\pi)^2 ds \right] \right|^{1/2}$$

$$\begin{aligned}
&= \left| E_{i-1} \left[ \xi^k - E_{i-1}[\xi^k] - (Y_{t_i}^\pi - E_{i-1}[Y_{t_i}^\pi]) \right] \right|^2 \Big|^{1/2} \\
&\leq 2 \left| E_{i-1} \left[ Y_{t_i}^\pi - \xi^k \right] \right|^2 \Big|^{1/2}.
\end{aligned}$$

Since  $\xi^k$  converges to  $Y_{t_i}^\pi$  in  $L^2$ , the last inequality together with (3.6) provide the required result.  $\square$

We also need the following estimate proved in Theorem 3.4.3 of [27].

**Lemma 3.2** *For each  $1 \leq i \leq n$ , define*

$$\bar{Z}_{t_{i-1}}^\pi := \frac{1}{\Delta t_i} E \left[ \int_{t_{i-1}}^{t_i} Z_s ds \mid \mathcal{F}_{t_{i-1}} \right].$$

Then :

$$\limsup_{|\pi| \rightarrow 0} |\pi|^{-1} \left\{ \max_{1 \leq i \leq n} \sup_{t_{i-1} \leq t < t_i} E |Y_t - Y_{t_{i-1}}| + \sum_{i=1}^n E \left[ \int_{t_{i-1}}^{t_i} |Z_t - \bar{Z}_{t_{i-1}}^\pi|^2 dt \right] \right\} < \infty.$$

We are now ready to state our first result, which provides an error estimate of the approximation scheme (3.3)-(3.4) of the same order than [27].

**Theorem 3.1**

$$\limsup_{|\pi| \rightarrow 0} |\pi|^{-1} \left\{ \sup_{0 \leq t \leq 1} E |Y_t^\pi - Y_t|^2 + E \left[ \int_0^1 |Z_t^\pi - Z_t|^2 dt \right] \right\} < \infty.$$

**Proof.** In the following,  $C > 0$  will denote a generic constant independent of  $i$  and  $n$  that may take different values from line to line. Let  $i \in \{0, \dots, n-1\}$  be fixed, and set

$$\delta Y_t := Y_t - Y_t^\pi, \quad \delta Z_t := Z_t - Z_t^\pi \quad \text{and} \quad \delta f_t := f(t, X_t, Y_t, Z_t) - f(t_i, X_{t_i}^\pi, Y_{t_i}^\pi, Z_{t_i}^\pi)$$

for  $t \in [t_i, t_{i+1})$ . By Itô's Lemma, we compute that

$$A_t := E |\delta Y_t|^2 + \int_t^{t_{i+1}} E |\delta Z_s|^2 ds - E |\delta Y_{t_{i+1}}|^2 = \int_t^{t_{i+1}} E [2\delta Y_s \delta f_s] ds, \quad t_i \leq t \leq t_{i+1}.$$

1. Let  $\alpha > 0$  be a constant to be chosen later on. From the Lipschitz property of  $f$ , together with the inequality  $ab \leq \alpha a^2 + b^2/\alpha$ , this provides :

$$\begin{aligned}
A_t &\leq E \left[ C \int_t^{t_{i+1}} |\delta Y_s| (|\pi| + |X_s - X_{t_i}^\pi| + |Y_s - Y_{t_i}^\pi| + |Z_s - Z_{t_i}^\pi|) ds \right] \\
&\leq \int_t^{t_{i+1}} \alpha E |\delta Y_s|^2 ds \\
&\quad + \frac{C}{\alpha} \int_t^{t_{i+1}} E \left[ |\pi|^2 + |X_s - X_{t_i}^\pi|^2 + |Y_s - Y_{t_i}^\pi|^2 + |Z_s - Z_{t_i}^\pi|^2 \right] ds \quad (3.7)
\end{aligned}$$

Now observe that :

$$E |X_s - X_{t_i}^\pi|^2 \leq C|\pi| \quad (3.8)$$

$$E |Y_s - Y_{t_i}^\pi|^2 \leq 2 \left\{ E |Y_s - Y_{t_i}|^2 + E |\delta Y_{t_i}|^2 \right\} \leq C \left\{ |\pi| + E |\delta Y_{t_i}|^2 \right\}, \quad (3.9)$$

by (3.2) and the estimate of Lemma 3.2. Also, with the notation of Lemma 3.2, it follows from Lemma 3.1 that :

$$\begin{aligned} E |Z_s - Z_{t_i}^\pi|^2 &\leq 2 \left\{ E |Z_s - \bar{Z}_{t_i}^\pi|^2 + E |Z_{t_i}^\pi - \bar{Z}_{t_i}^\pi|^2 \right\} \\ &= 2 \left\{ E |Z_s - \bar{Z}_{t_i}^\pi|^2 + E \left| \frac{1}{\Delta_{i+1}^\pi} \int_{t_i}^{t_{i+1}} E [\delta Z_r | \mathcal{F}_{t_i}] dr \right|^2 \right\} \\ &\leq 2 \left\{ E |Z_s - \bar{Z}_{t_i}^\pi|^2 + \frac{1}{\Delta_{i+1}^\pi} \int_{t_i}^{t_{i+1}} E |\delta Z_r|^2 dr \right\} \end{aligned} \quad (3.10)$$

by Jensen's inequality. We now plug (3.8)-(3.9)-(3.10) into (3.7) to obtain :

$$\begin{aligned} A_t &\leq \int_t^{t_{i+1}} \alpha E |\delta Y_s|^2 ds + \frac{C}{\alpha} \int_t^{t_{i+1}} E \left[ |\pi| + |\delta Y_{t_i}|^2 + |Z_s - \bar{Z}_{t_i}^\pi|^2 \right] ds \\ &\quad + \frac{C}{\alpha \Delta_{i+1}^\pi} \int_t^{t_{i+1}} \int_{t_i}^{t_{i+1}} E |\delta Z_r|^2 dr ds \end{aligned} \quad (3.11)$$

$$\begin{aligned} &\leq \int_t^{t_{i+1}} \alpha E |\delta Y_s|^2 ds + \frac{C}{\alpha} \int_t^{t_{i+1}} E \left[ |\pi| + |\delta Y_{t_i}|^2 + |Z_s - \bar{Z}_{t_i}^\pi|^2 \right] ds \\ &\quad + \frac{C}{\alpha} \int_{t_i}^{t_{i+1}} E |\delta Z_r|^2 dr. \end{aligned} \quad (3.12)$$

**2.** From the definition of  $A_t$  and (3.12), we see that, for  $t_i \leq t < t_{i+1}$ ,

$$E |\delta Y_t|^2 \leq E |\delta Y_{t_i}|^2 + \int_t^{t_{i+1}} E |\delta Z_s|^2 ds \leq \alpha \int_t^{t_{i+1}} E |\delta Y_s|^2 ds + B_i \quad (3.13)$$

where

$$B_i := E |\delta Y_{t_{i+1}}|^2 + \frac{C}{\alpha} \left\{ |\pi|^2 + |\pi| E |\delta Y_{t_i}|^2 + \int_{t_i}^{t_{i+1}} E |\delta Z_r|^2 dr + \int_{t_i}^{t_{i+1}} E |Z_s - \bar{Z}_{t_i}^\pi|^2 ds \right\}.$$

By Gronwall's Lemma, this shows that  $E |\delta Y_t|^2 \leq B_i e^{\alpha|\pi|}$  for  $t_i \leq t < t_{i+1}$ , which plugged in the second inequality of (3.13) provides :

$$E |\delta Y_t|^2 + \int_t^{t_{i+1}} E |\delta Z_s|^2 ds \leq B_i \left( 1 + \alpha|\pi| e^{\alpha|\pi|} \right) \leq B_i (1 + C\alpha|\pi|) \quad (3.14)$$

for small  $|\pi|$ . For  $t = t_i$  and  $\alpha$  sufficiently larger than  $C$ , we deduce from this inequality that :

$$E |\delta Y_{t_i}|^2 + \frac{1}{2} \int_{t_i}^{t_{i+1}} E |\delta Z_s|^2 ds \leq (1 + C|\pi|) \left\{ E |\delta Y_{t_{i+1}}|^2 + |\pi|^2 + \int_{t_i}^{t_{i+1}} |Z_s - \bar{Z}_{t_i}^\pi|^2 ds \right\},$$

for small  $|\pi|$ .



3. Iterating the last inequality, we get :

$$\begin{aligned} E|\delta Y_{t_i}|^2 + \frac{1}{2} \int_{t_i}^{t_{i+1}} E|\delta Z_s|^2 ds \\ \leq (1 + C|\pi|)^{1/|\pi|} \left\{ E|\delta Y_1|^2 + |\pi| + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} E \left| Z_s - \bar{Z}_{t_{i-1}}^\pi \right|^2 ds \right\}. \end{aligned}$$

Using the estimate of Lemma 3.2, together with the Lipschitz property of  $g$  and (3.2), this provides :

$$E|\delta Y_{t_i}|^2 + \frac{1}{2} \int_{t_i}^{t_{i+1}} E|\delta Z_s|^2 ds \leq C(1 + C|\pi|)^{1/|\pi|} \left\{ E|\delta Y_1|^2 + C|\pi| \right\} \leq C|\pi|, \quad (3.15)$$

for small  $|\pi|$ . Summing up the inequality (3.14) with  $t = t_i$ , we get :

$$\begin{aligned} \int_0^1 E|\delta Z_s|^2 ds \leq E|\delta Y_1|^2 - E|\delta Y_0|^2 + \frac{C}{\alpha} \int_0^1 E|\delta Z_s|^2 ds \\ + \frac{C}{\alpha} \sum_{i=0}^{n-1} \left\{ |\pi|^2 + |\pi| E|\delta Y_{t_i}|^2 + \int_{t_i}^{t_{i+1}} E \left| Z_s - Z_{t_i}^{\pi_0,1} \right|^2 ds \right\}. \end{aligned}$$

For  $\alpha$  sufficiently larger than  $C$ , it follows from (3.15) and Lemma 3.2 that :

$$\int_0^1 E|\delta Z_s|^2 ds \leq C|\pi|.$$

Together with Lemma 3.2 and (3.15), this shows that  $B_i \leq C|\pi|$ , and therefore :

$$\sup_{0 \leq t \leq 1} E|\delta Y_t|^2 \leq C|\pi|,$$

by taking the supremum over  $t$  in (3.14). This completes the proof of the theorem.  $\square$

We end up this section with the following bound on the  $Y_{t_i}^\pi$ 's which will be used in the simulation based approximation of the discrete-time conditional expectation operators  $E_i^\pi$ ,  $0 \leq i \leq n-1$ .

**Lemma 3.3** *Assume that*

$$|g(0)| + |f(0)| + \|b\|_\infty + \|\sigma\|_\infty \leq K, \quad (3.16)$$

for some  $K \geq 1$ , and define the sequence

$$\begin{aligned} \alpha_n^\pi &:= 2K, \quad \beta_n^\pi := K \\ \alpha_i^\pi &:= (1 - K|\pi|)^{-1} \left\{ (1 + K|\pi|)^{1/2} (\alpha_{i+1}^\pi + \beta_{i+1}^\pi 4K^2|\pi|) + 3K^2|\pi| \right\} \\ \beta_i^\pi &:= (1 - K|\pi|)^{-1} \left\{ (1 + K|\pi|)^{1/2} K|\pi|\beta_{i+1}^\pi + 3K^2|\pi| \right\}, \quad 0 \leq i \leq n-1. \end{aligned}$$

Then, for all  $0 \leq i \leq n$

$$|Y_{t_i}^\pi| \leq \alpha_i^\pi + \beta_i^\pi |X_{t_i}^\pi|^2, \quad (3.17)$$

$$E_{i-1}^\pi |\hat{Y}_{t_i}^\pi| \leq \left\{ E_{i-1}^\pi |\hat{Y}_{t_i}^\pi|^2 \right\}^{1/2} \leq \alpha_i^\pi + \beta_i^\pi \left[ (1 + 2K|\pi|) |X_{t_{i-1}}^\pi|^2 + 4K^2|\pi| \right], \quad (3.18)$$

$$E_{i-1}^\pi |\hat{Y}_{t_i}^\pi \Delta^\pi W_i| \leq \sqrt{|\pi|} \left\{ \alpha_i^\pi + \beta_i^\pi \left[ (1 + 2K|\pi|) |X_{t_{i-1}}^\pi|^2 + 4K^2|\pi| \right] \right\}. \quad (3.19)$$

Moreover,

$$\limsup_{|\pi| \rightarrow 0} \max_{0 \leq i \leq n} \{ \alpha_i^\pi + \beta_i^\pi \} < \infty.$$

**Proof.** We first observe that the bound (3.18) is a by-product of the proof of (3.17). The bound (3.19) follows directly from (3.18) together with the Cauchy-Schwartz inequality. In order to prove (3.17), we use a backward induction argument. First, since  $g$  is  $K$ -Lipschitz and  $g(0)$  is bounded by  $K$ , we have :

$$|Y_1^\pi| \leq K(1 + |X_1^\pi|) \leq K(2 + |X_1^\pi|) = \alpha_n^\pi + \beta_n^\pi |X_1^\pi|^2. \quad (3.20)$$

We next assume that

$$|Y_{t_{i+1}}^\pi| \leq \alpha_{i+1}^\pi + \beta_{i+1}^\pi |X_{t_{i+1}}^\pi|^2, \quad (3.21)$$

for some fixed  $0 \leq i \leq n-1$ . From the Lipschitz property of  $f$ , there exists an  $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ -valued  $\mathcal{F}_{t_i}$ -measurable random variable  $(\tau_i, \xi_i, \nu_i, \zeta_i)$ , essentially bounded by  $K$ , such that :

$$\begin{aligned} & f\left(t_i, X_{t_i}^\pi, Y_{t_i}^\pi, (\Delta_{i+1}^\pi)^{-1} E_i^\pi \left[ Y_{t_{i+1}}^\pi \Delta^\pi W_{i+1} \right] \right) - f(0) \\ &= \tau_i t_i + \xi_i X_{t_i}^\pi + \nu_i Y_{t_i}^\pi + (\Delta_{i+1}^\pi)^{-1} \zeta_i \cdot E_i^\pi \left[ Y_{t_{i+1}}^\pi \Delta^\pi W_{i+1} \right]. \end{aligned}$$

By the definition of  $Y^\pi$  in (3.3), this provides

$$\begin{aligned} Y_{t_i}^\pi &= E_i^\pi \left[ Y_{t_{i+1}}^\pi \right] + \Delta_{i+1}^\pi f(0) \\ &+ \Delta_{i+1}^\pi \left\{ \tau_i t_i + \xi_i X_{t_i}^\pi + \nu_i Y_{t_i}^\pi + (\Delta_{i+1}^\pi)^{-1} \zeta_i \cdot E_i^\pi \left[ Y_{t_{i+1}}^\pi \Delta^\pi W_{i+1} \right] \right\}. \end{aligned}$$

Then, it follows from the Cauchy-Schwartz inequality and the inequality  $|x| \leq 1 + |x|^2$  that, for  $|\pi| \leq 1$ ,

$$\begin{aligned} (1 - K|\pi|) |Y_{t_i}^\pi| &\leq E_i^\pi \left| Y_{t_{i+1}}^\pi (1 + \zeta_i \cdot \Delta^\pi W_{i+1}) \right| + K|\pi| (2 + |X_{t_i}^\pi|) \\ &\leq \left[ E_i^\pi |Y_{t_{i+1}}^\pi|^2 \right]^{1/2} \left[ E_i^\pi |1 + \zeta_i \cdot \Delta^\pi W_{i+1}|^2 \right]^{1/2} + K|\pi| (3 + |X_{t_i}^\pi|^2) \end{aligned} \quad (3.22)$$

Now, since  $\zeta_i$  is  $\mathcal{F}_{t_i}$ -measurable and bounded by  $K$ , observe that :

$$E_i^\pi |1 + \zeta_i \cdot \Delta^\pi W_{i+1}|^2 \leq 1 + K|\pi|.$$

We then get from (3.22) :

$$(1 - K|\pi|) |Y_{t_i}^\pi| \leq (1 + K|\pi|)^{1/2} \left[ E_i^\pi |Y_{t_{i+1}}^\pi|^2 \right]^{1/2} + K|\pi| \left( 3 + |X_{t_i}^\pi|^2 \right) . \quad (3.23)$$

Using (3.21), we now write that :

$$\left[ E_i^\pi |Y_{t_{i+1}}^\pi|^2 \right]^{1/2} \leq \alpha_{i+1}^\pi + \beta_{i+1}^\pi \left\{ E_i^\pi \left( X_{t_i}^\pi + |\pi|b(X_{t_i}^\pi) + \sigma(X_{t_i}^\pi) \Delta^\pi W_{i+1} \right)^4 \right\}^{1/2} \quad (3.24)$$

where, by (3.16) and the assumption  $K \geq 1$ , direct computation leads to :

$$\begin{aligned} \left\{ E_i^\pi \left( X_{t_i}^\pi + |\pi|b(X_{t_i}^\pi) + \sigma(X_{t_i}^\pi) \Delta^\pi W_{i+1} \right)^4 \right\}^{1/2} &\leq (|X_{t_i}^\pi| + K|\pi|)^2 + 3K|\pi|^2 \\ &\leq (1 + 2K|\pi|) |X_{t_i}^\pi|^2 + 4K^2|\pi| . \end{aligned} \quad (3.25)$$

Together with (3.23)-(3.24)-(3.25), this implies that :

$$\begin{aligned} (1 - K|\pi|) |Y_{t_i}^\pi| &\leq (1 + K|\pi|)^{1/2} \left\{ \alpha_{i+1}^\pi + \beta_{i+1}^\pi \left( (1 + 2K|\pi|) |X_{t_i}^\pi|^2 + 4K^2|\pi| \right) \right\} \\ &\quad + K|\pi| \left( 3 + K|\pi| |X_{t_i}^\pi|^2 \right) \\ &\leq (1 + K|\pi|)^{1/2} \left\{ \alpha_{i+1}^\pi + \beta_{i+1}^\pi \left( (1 + 2K|\pi|) |X_{t_i}^\pi|^2 + 4K^2|\pi| \right) \right\} \\ &\quad + 3K^2|\pi| \left( 1 + |X_{t_i}^\pi|^2 \right) . \end{aligned}$$

It follows that :

$$|Y_{t_i}^\pi| \leq \alpha_{i+1}^\pi + \beta_{i+1}^\pi |X_{t_i}^\pi|^2 .$$

Now observe that for all  $0 \leq i \leq n$  :

$$\begin{aligned} \beta_i^\pi &= 7K^2|\pi| \sum_{j=i}^{n-1} (1 + K|\pi|)^{(j-i)/2} (1 + 2K|\pi|)^{j-i} + (1 + K|\pi|)^{(n-i)/2} (1 + 2K|\pi|)^{n-i} \beta_n^\pi \\ &\leq 8K^2 (1 + K|\pi|)^{\frac{1}{2|\pi|}} (1 + 2K|\pi|)^{\frac{1}{|\pi|}} , \end{aligned}$$

where the last term is uniformly bounded in  $|\pi|$ . The same argument shows that  $\max_{0 \leq i \leq n} \alpha_i^\pi$  is uniformly bounded in  $|\pi|$ .  $\square$

## 4 Error due to the regression approximation

In this section, we focus on the problem of simulating the approximation  $(X^\pi, Y^\pi)$  of the components  $(X, Y)$  of the solution of the decoupled forward-backward stochastic differential equation (2.2)-(2.4). The forward component  $X^\pi$  defined by (3.1) can of course be simulated on the time grid defined by the partition  $\pi$  by the classical Monte-Carlo method. Then, we are reduced to the problem of simulating the approximation  $Y^\pi$  defined in (3.3)-(3.4), given the approximation  $X^\pi$  of  $X$ .

Notice that each step of the backward induction (3.3)-(3.4) requires the computation of  $(d+1)$  conditional expectations. In practice, one can only hope to have an approximation  $\hat{E}_i^\pi$  of the conditional expectation operator  $E_i^\pi$ . Therefore, the main idea for the definition of an approximation of  $Y^\pi$ , and therefore of  $Z^\pi$ , is to replace the conditional expectation  $E_i^\pi$  by  $\hat{E}_i^\pi$  in the backward scheme (3.3)-(3.4).

However, we would like to improve the efficiency of the approximation scheme of  $(Y^\pi, Z^\pi)$  when it is known to lie in some given domain. Let then  $\wp^\pi = \left\{ \left( \underline{\wp}_i^\pi, \overline{\wp}_i^\pi \right) \right\}_{0 \leq i \leq n}$  be a sequence of pairs of maps from  $\mathbb{R}^d$  into  $\mathbb{R} \cup \{-\infty, +\infty\}$  satisfying :

$$\underline{\wp}_i^\pi(X_{t_i}^\pi) \leq Y_{t_i}^\pi \leq \overline{\wp}_i^\pi(X_{t_i}^\pi) \quad \text{for all } i = 0, \dots, n, \quad (4.1)$$

i.e.  $\underline{\wp}_i^\pi \leq \overline{\wp}_i^\pi$  are some given *a priori* known bounds on  $Y_i^\pi$  for each  $i$ . For instance, one can define  $\wp^\pi$  by the bounds derived in Lemma 3.3. When no bounds on  $Y^\pi$  are known, one may take  $\underline{\wp}_i^\pi = -\infty$  and  $\overline{\wp}_i^\pi = +\infty$ .

Given a random variable  $\zeta$  valued in  $\mathbb{R}$ , we shall use the notation :

$$\mathbf{T}_i^{\wp^\pi}(\zeta) := \underline{\wp}_i^\pi(X_{t_i}^\pi) \vee \zeta \wedge \overline{\wp}_i^\pi(X_{t_i}^\pi),$$

where  $\vee$  and  $\wedge$  denote respectively the binary maximum and minimum operators. Since the backward scheme (3.3)-(3.4) involves the computation of the conditional expectations  $E_{i-1}^\pi[Y_{t_i}^\pi]$  and  $E_{i-1}^\pi[Y_{t_i}^\pi \Delta^\pi W_i]$ , we shall also need to introduce the sequences  $\mathfrak{R}^\pi = \left\{ \left( \underline{\mathfrak{R}}_i^\pi, \overline{\mathfrak{R}}_i^\pi \right) \right\}_{0 \leq i \leq n}$  and  $\mathfrak{S}^\pi = \left\{ \left( \underline{\mathfrak{S}}_i^\pi, \overline{\mathfrak{S}}_i^\pi \right) \right\}_{0 \leq i \leq n}$  of pairs of maps from  $\mathbb{R}^d$  into  $\mathbb{R} \cup \{-\infty, +\infty\}$  satisfying :

$$\begin{aligned} \underline{\mathfrak{R}}_{i-1}^\pi(X_{t_{i-1}}^\pi) &\leq E_{i-1}^\pi[Y_{t_i}^\pi] \leq \overline{\mathfrak{R}}_{i-1}^\pi(X_{t_{i-1}}^\pi) \\ \underline{\mathfrak{S}}_{i-1}^\pi(X_{t_{i-1}}^\pi) &\leq E_{i-1}^\pi[Y_{t_i}^\pi \Delta^\pi W_i] \leq \overline{\mathfrak{S}}_{i-1}^\pi(X_{t_{i-1}}^\pi) \end{aligned}$$

for all  $i = 1, \dots, n$ . The corresponding operators  $\mathbf{T}_i^{\mathfrak{R}^\pi}$  and  $\mathbf{T}_i^{\mathfrak{S}^\pi}$  are defined similarly to  $\mathbf{T}_i^{\wp^\pi}$ . An example of such sequences is given by Lemma 3.3.

Now, given an approximation  $\hat{E}_i^\pi$  of  $E_i^\pi$ , we define the process  $(\hat{Y}^\pi, \hat{Z}^\pi)$  by the backward induction scheme :

$$\begin{aligned} \hat{Y}_1^\pi &= Y_1^\pi = g(X_1^\pi), \\ \check{Y}_{t_{i-1}}^\pi &= \hat{E}_{i-1}^\pi[\hat{Y}_{t_i}^\pi] + \Delta_i^\pi f\left(t_{i-1}, X_{t_{i-1}}^\pi, \check{Y}_{t_{i-1}}^\pi, \hat{Z}_{t_{i-1}}^\pi\right) \end{aligned} \quad (4.2)$$

$$\hat{Y}_{t_{i-1}}^\pi = \mathbf{T}_{i-1}^{\wp^\pi}(\check{Y}_{t_{i-1}}^\pi), \quad (4.3)$$

$$\hat{Z}_{t_{i-1}}^\pi = \frac{1}{\Delta_i^\pi} \hat{E}_{i-1}^\pi[\hat{Y}_{t_i}^\pi \Delta^\pi W_i], \quad (4.4)$$

for all  $1 \leq i \leq n$ . Recall from Remark 3.1 that the conditional expectations involved in (3.3)-(3.4) are in fact regression functions. This simplifies considerably the problem of approximating  $E_i^\pi$ .

**Example 4.1** (*Non-parametric regression*) Let  $\zeta$  be an  $\mathcal{F}_{t_{i+1}}$ -measurable random variable, and  $(X^{\pi(j)}, \zeta^{(j)})_{j=1}^N$  be  $N$  independent copies of  $(X_{t_1}^\pi, \dots, X_{t_n}^\pi, \zeta)$ . The non-parametric

kernel estimator of the regression operator  $E_i^\pi$  is defined by :

$$\tilde{E}_i^\pi[\zeta] := \frac{\sum_{j=1}^N \zeta^{(j)} \kappa \left( (h_N)^{-1} \left( X_{t_i}^{\pi(j)} - X_{t_i}^\pi \right) \right)}{\sum_{j=1}^N \kappa \left( (h_N)^{-1} \left( X_{t_i}^{\pi(j)} - X_{t_i}^\pi \right) \right)},$$

where  $\kappa$  is a kernel function and  $h_N$  is a bandwidth matrix converging to 0 as  $N \rightarrow \infty$ . We send the reader to [5] for details on the analysis of the error  $\tilde{E}_i^\pi - E_i^\pi$ .

The above regression estimator can be improved in our context by using the *a priori* bounds  $\wp^\pi$  on  $Y^\pi$  :

$$\hat{E}_{i-1}^\pi \left[ \hat{Y}_{t_i} \right] = \mathbf{T}_{i-1}^{\mathfrak{R}^\pi} \left( \tilde{E}_{i-1}^\pi \left[ \hat{Y}_{t_i} \right] \right) \quad \text{and} \quad \hat{E}_{i-1}^\pi \left[ \hat{Y}_{t_i} \Delta^\pi W_i \right] = \mathbf{T}_{i-1}^{\mathfrak{S}^\pi} \left( \tilde{E}_{i-1}^\pi \left[ \hat{Y}_{t_i} \Delta^\pi W_i \right] \right).$$

**Example 4.2** (*Malliavin regression approach*) Let  $\phi$  be a mapping from  $\mathbb{R}^d$  into  $\mathbb{R}$ , and  $(X^{\pi(j)})_{j=1}^N$  be  $N$  independent copies of  $(X_{t_1}^\pi, \dots, X_{t_n}^\pi)$ . The Malliavin regression estimator of the operator  $E_i^\pi$  is defined by :

$$\tilde{E}_i^\pi \left[ \phi(X_{t_{i+1}}^\pi) \right] := \frac{\sum_{j=1}^N \phi \left( X_{t_{i+1}}^{\pi(j)} \right) H_{X_{t_i}^\pi} \left( X_{t_i}^{\pi(j)} \right) S^{(j)}}{\sum_{j=1}^N H_{X_{t_i}^\pi} \left( X_{t_i}^{\pi(j)} \right) S^{(j)}},$$

where  $H_x$  is the Heaviside function,  $H_x(y) = \prod_{i=1}^d \mathbf{1}_{x^i \leq y^i}$ , and  $S^{(j)}$  are independent copies of some random variable  $S$  whose precise definition is given in §6 below. Notice that the practical implementation of this approximation procedure in the backward induction (4.2)-(4.4) requires a slight extension of this estimator. This issue will be discussed precisely in §6.

As in the previous example, we use the bounds  $\wp^\pi$  on  $Y$  to define the approximations. The above regression estimator can be improved in our context by using the *a priori* bounds  $\wp^\pi$  on  $Y^\pi$  :

$$\hat{E}_{i-1}^\pi \left[ \hat{Y}_{t_i} \right] = \mathbf{T}_{i-1}^{\mathfrak{R}^\pi} \left( \tilde{E}_{i-1}^\pi \left[ \hat{Y}_{t_i} \right] \right) \quad \text{and} \quad \hat{E}_{i-1}^\pi \left[ \hat{Y}_{t_i} \Delta^\pi W_i \right] = \mathbf{T}_{i-1}^{\mathfrak{S}^\pi} \left( \tilde{E}_{i-1}^\pi \left[ \hat{Y}_{t_i} \Delta^\pi W_i \right] \right).$$

**Remark 4.1** The use of *a priori* bounds on the conditional expectation to be computed is a crucial step in our analysis. This is due to the fact that, in general, natural estimators  $\tilde{E}_i^\pi$ , as in Examples 4.1 and 4.2, produce random variables which are not necessarily integrable.  $\square$

We now turn to the main result of this section, which provides an  $L^p$  estimate of the error  $\hat{Y}^\pi - Y^\pi$  in terms of the regression errors  $\hat{E}_i^\pi - E_i^\pi$ .

**Theorem 4.1** *Let  $p > 1$  be given, and  $\wp^\pi$  be a sequence of pairs of maps valued in  $\mathbb{R} \cup \{-\infty, \infty\}$  satisfying (4.1). Then, there is a constant  $C > 0$  which only depends on  $(K, p)$  such that :*

$$\left\| \hat{Y}_{t_i}^\pi - Y_{t_i}^\pi \right\|_{L^p} \leq \frac{C}{|\pi|} \max_{0 \leq j \leq n-1} \left\{ \left\| (\hat{E}_j^\pi - E_j^\pi) \left[ \hat{Y}_{t_{j+1}}^\pi \right] \right\|_{L^p} + \left\| (\hat{E}_j^\pi - E_j^\pi) \left[ \hat{Y}_{t_{j+1}}^\pi \Delta^\pi W_{j+1} \right] \right\|_{L^p} \right\}$$

for all  $0 \leq i \leq n$ .

**Proof.** In the following,  $C > 0$  will denote a generic constant, which only depends on  $(K, p)$ , that may change from line to line. Let  $0 \leq i \leq n - 1$  be fixed. We first compute that :

$$\begin{aligned} Y_{t_i}^\pi - \check{Y}_{t_i}^\pi &= \epsilon_i + E_i^\pi \left[ Y_{t_{i+1}}^\pi - \hat{Y}_{t_{i+1}}^\pi \right] \\ &\quad + \Delta_{i+1}^\pi \left\{ f \left( t_i, X_{t_i}^\pi, Y_{t_i}^\pi, (\Delta_{i+1}^\pi)^{-1} E_i^\pi \left[ Y_{t_{i+1}}^\pi \Delta^\pi W_{i+1} \right] \right) \right. \\ &\quad \left. - f \left( t_i, X_{t_i}^\pi, \check{Y}_{t_i}^\pi, (\Delta_{i+1}^\pi)^{-1} E_i^\pi \left[ \hat{Y}_{t_{i+1}}^\pi \Delta^\pi W_{i+1} \right] \right) \right\} \end{aligned} \quad (4.5)$$

where

$$\begin{aligned} \epsilon_i &:= \left( E_i^\pi - \hat{E}_i^\pi \right) \left[ \hat{Y}_{t_{i+1}}^\pi \right] + \Delta_{i+1}^\pi \left\{ f \left( t_i, X_{t_i}^\pi, \check{Y}_{t_i}^\pi, (\Delta_{i+1}^\pi)^{-1} \hat{E}_i^\pi \left[ \hat{Y}_{t_{i+1}}^\pi \Delta^\pi W_{i+1} \right] \right) \right. \\ &\quad \left. - f \left( t_i, X_{t_i}^\pi, \check{Y}_{t_i}^\pi, (\Delta_{i+1}^\pi)^{-1} E_i^\pi \left[ \hat{Y}_{t_{i+1}}^\pi \Delta^\pi W_{i+1} \right] \right) \right\} . \end{aligned}$$

From the Lipschitz property of  $f$ , we have :

$$\|\epsilon_i\|_{L^p} \leq \eta_i := C \left\{ \left\| (\hat{E}_i^\pi - E_i^\pi) \left[ \hat{Y}_{t_{i+1}}^\pi \right] \right\|_{L^p} + \left\| (\hat{E}_i^\pi - E_i^\pi) \left[ \hat{Y}_{t_{i+1}}^\pi \Delta^\pi W_{i+1} \right] \right\|_{L^p} \right\} .$$

Again, from the Lipschitz property of  $f$ , there exists an  $\mathbb{R} \times \mathbb{R}^d$ -valued  $\mathcal{F}_{t_i}$ -measurable random variable  $(\nu_i, \zeta_i)$ , essentially bounded by  $K$ , such that :

$$\begin{aligned} &f \left( t_i, X_{t_i}^\pi, Y_{t_i}^\pi, (\Delta_{i+1}^\pi)^{-1} E_i^\pi \left[ Y_{t_{i+1}}^\pi \Delta^\pi W_{i+1} \right] \right) - f \left( t_i, X_{t_i}^\pi, \check{Y}_{t_i}^\pi, (\Delta_{i+1}^\pi)^{-1} E_i^\pi \left[ \hat{Y}_{t_{i+1}}^\pi \Delta^\pi W_{i+1} \right] \right) \\ &= \nu_i (Y_{t_i}^\pi - \check{Y}_{t_i}^\pi) + (\Delta_{i+1}^\pi)^{-1} \zeta_i \cdot E_i^\pi \left[ (Y_{t_{i+1}}^\pi - \hat{Y}_{t_{i+1}}^\pi) \Delta^\pi W_{i+1} \right] . \end{aligned}$$

Then, it follows from (4.5) and the Hölder inequality that :

$$\begin{aligned} (1 - K|\pi|) |Y_{t_i}^\pi - \check{Y}_{t_i}^\pi| &\leq |\epsilon_i| + E_i^\pi \left| (Y_{t_{i+1}}^\pi - \hat{Y}_{t_{i+1}}^\pi) (1 + \zeta_i \cdot \Delta^\pi W_{i+1}) \right| \\ &\leq |\epsilon_i| + \left[ E_i^\pi \left| Y_{t_{i+1}}^\pi - \hat{Y}_{t_{i+1}}^\pi \right|^p \right]^{1/p} \left[ E_i^\pi |1 + \zeta_i \cdot \Delta^\pi W_{i+1}|^q \right]^{1/q} \\ &\leq |\epsilon_i| + \left[ E_i^\pi \left| Y_{t_{i+1}}^\pi - \hat{Y}_{t_{i+1}}^\pi \right|^p \right]^{1/p} \left[ E_i^\pi |1 + \zeta_i \cdot \Delta^\pi W_{i+1}|^{2k} \right]^{1/2k} \end{aligned}$$

where  $q$  is the conjugate of  $p$  and  $k \geq q/2$  is an arbitrary integer. Recalling that  $\hat{Y}_{t_i}^\pi = \mathbf{T}_i^{\varphi^\pi}(\check{Y}_{t_i}^\pi)$  and  $Y_{t_i}^\pi = \mathbf{T}_i^{\varphi^\pi}(Y_{t_i}^\pi)$ , by (4.1), this provides

$$(1 - K|\pi|) \left| Y_{t_i}^\pi - \hat{Y}_{t_i}^\pi \right| \leq |\epsilon_i| + \left[ E_i^\pi \left| Y_{t_{i+1}}^\pi - \hat{Y}_{t_{i+1}}^\pi \right|^p \right]^{1/p} \left[ E_i^\pi |1 + \zeta_i \cdot \Delta^\pi W_{i+1}|^{2k} \right]^{1/2k} \quad (4.6)$$

by the 1-Lipschitz property of  $\mathbf{T}_i^{\varphi^\pi}$ . Now, since  $\zeta_i$  is  $\mathcal{F}_{t_i}$ -measurable and bounded by  $K$ , observe that :

$$\begin{aligned} E_i^\pi |1 + \zeta_i \cdot \Delta^\pi W_{i+1}|^{2k} &= \sum_{j=0}^{2k} \binom{2k}{j} E_i^\pi (\zeta_i \cdot \Delta^\pi W_{i+1})^j \\ &= \sum_{j=0}^k \binom{2k}{2j} E_i^\pi (\zeta_i \cdot \Delta^\pi W_{i+1})^{2j} \\ &\leq 1 + C|\pi| . \end{aligned}$$

We then get from (4.6) :

$$\begin{aligned} (1 - K|\pi|) \left\| Y_{t_i}^\pi - \hat{Y}_{t_i}^\pi \right\|_{L^p} &\leq \|\epsilon_i\|_{L^p} + (1 + C|\pi|)^{1/2k} \left\| Y_{t_{i+1}}^\pi - \hat{Y}_{t_{i+1}}^\pi \right\|_{L^p} \\ &\leq \eta_i + (1 + C|\pi|)^{1/2k} \left\| Y_{t_{i+1}}^\pi - \hat{Y}_{t_{i+1}}^\pi \right\|_{L^p} . \end{aligned} \quad (4.7)$$

For small  $|\pi|$ , it follows from this inequality that :

$$\begin{aligned} \left\| Y_{t_{i+1}}^\pi - \hat{Y}_{t_{i+1}}^\pi \right\|_{L^p} &\leq \frac{1}{|\pi|} (1 - K|\pi|)^{-1/|\pi|} (1 + C|\pi|)^{1/(2k|\pi|)} \max_{0 \leq j \leq n-1} \eta_j \\ &\leq \frac{C}{|\pi|} \max_{0 \leq j \leq n-1} \eta_j . \end{aligned}$$

□

**Remark 4.2** In the particular case where the generator  $f$  does not depend on the control variable  $z$ , Theorem 4.1 is valid for  $p = 1$ . This is easily checked by noticing that, in this case,  $\zeta_i = 0$  in the above proof. □

## 5 Regression error estimate

In this section, we focus on the regression procedure. Let  $(R, S)$  be a pair of random variables. In both Examples 4.1 and 4.2, the regression estimator is based on the observation that the regression function can be written in

$$r(x) := E[R|S = x] = \frac{q^R(x)}{q^1(x)} \quad \text{where} \quad q^R(x) := E[R \varepsilon_x(S)] ,$$

and  $\varepsilon_x$  denotes the Dirac measure at the point  $x$ . Then, the regression estimation problem is reduced to the problem of estimating separately  $q^R(x)$  and  $q^1(x)$ , and the main difficulty lies in the presence of the Dirac measure inside the expectation operator.

While the kernel estimator is based on approximating the Dirac measure by a kernel function with bandwidth shrinking to zero, the Malliavin estimator is suggested by an alternative representation of  $q^R(x)$  obtained by integrating up the Dirac measure to the Heaviside function, see §6. In both cases, one defines an estimator :

$$\tilde{r}_N(x, \omega) := \frac{\hat{q}_N^R(x, \omega)}{\hat{q}_N^1(x, \omega)} , \quad \omega \in \Omega ,$$

where  $\hat{q}_N^R(x, \omega)$  and  $\hat{q}_N^1(x, \omega)$  are defined as the means on a sample of  $N$  independent copies  $\{A^{(i)}(x, \omega), B^{(i)}(x, \omega)\}_{1 \leq i \leq N}$  of some corresponding random variables  $\{A(x, \omega), B(x, \omega)\}$  :

$$\hat{q}_N^R(x, \omega) := \frac{1}{N} \sum_{i=1}^N A^{(i)}(x, \omega) \quad \text{and} \quad \hat{q}_N^1(x, \omega) := \frac{1}{N} \sum_{i=1}^N B^{(i)}(x, \omega) .$$

In the Malliavin approach these random variables  $\{A(x, \omega), B(x, \omega)\}$  have expectation equal to  $\{q^R(x), q^1(x)\}$ , see Theorem 6.1 below. Using the above definitions, it follows that :

$$\begin{aligned} V_R(x) &:= N \operatorname{Var} [\hat{q}_N^R(x)] = \operatorname{Var}[A(x)] , \\ V_1(x) &:= N \operatorname{Var} [\hat{q}_N^1(x)] = \operatorname{Var}[B(x)] . \end{aligned} \quad (5.1)$$

In order to prepare for the results of §6, we shall now concentrate on the case where  $E(A(x), B(x)) = (q^R(x), q^1(x))$ , so that (5.1) holds. A similar analysis can be performed for the kernel approach, see Remark 5.1 below.

In view of Theorem 4.1, the  $L^p$  error estimate of  $|\hat{Y}^\pi - Y^\pi|$  is related to the  $L^p$  error on the regression estimator. As mentioned in Remark 4.1, the regression error  $\tilde{r}_N(S) - r(S)$  is not guaranteed to be integrable. Indeed, the classical central limit theorem indicates that the denominator  $\hat{q}_N[1](x)$  of  $\tilde{r}_N(x)$  has a gaussian asymptotic distribution, which induces integrability problems on the ratio  $\hat{q}_N^R(x)/\hat{q}_N^1(x)$ .

We solve this difficulty by introducing the *a priori* bounds  $\{\underline{\rho}(x), \bar{\rho}(x)\}$  on  $r(x)$  :

$$\underline{\rho}(x) \leq r(x) \leq \bar{\rho}(x) \quad \text{for all } x ,$$

and we define the truncated estimators :

$$\hat{r}_N(x) := \mathbf{T}^\rho(\tilde{r}_N(x)) := \underline{\rho}(x) \vee \tilde{r}_N(x) \wedge \bar{\rho}(x) .$$

We are now ready for the main result of this section.

**Theorem 5.1** *Let  $R$  and  $S$  be random variables valued respectively in  $\mathbb{R}$  and  $\mathbb{R}^d$ . Assume that  $S$  has a density  $q^1 > 0$  with respect to the Lebesgue measure on  $\mathbb{R}^d$ . For all  $x \in \mathbb{R}^d$ , let  $\{A^{(i)}(x)\}_{1 \leq i \leq N}$  and  $\{B^{(i)}(x)\}_{1 \leq i \leq N}$  be two families of independent and identically distributed random variables on  $\mathbb{R}$  satisfying*

$$E[A^{(i)}(x)] = q^R(x) \quad \text{and} \quad E[B^{(i)}(x)] = q^1(x) \quad \text{for all } 1 \leq i \leq N .$$

Set  $\gamma(x) := |\bar{\rho} - \underline{\rho}|(x)$ , and assume that

$$\Gamma(r, \gamma, V_1, V_R) := \int_{\mathbb{R}^d} \gamma(x)^{p-1} \left[ V_R(x)^{1/2} + (|r(x)| + \gamma(x)) V_1(x)^{1/2} \right] dx < \infty \quad (5.2)$$

for some  $p \geq 1$ . Then :

$$\|\hat{r}_N(S) - r(S)\|_{L^p} \leq \left( 2 \Gamma(r, \gamma, V_1, V_R) N^{-1/2} \right)^{1/p} .$$

**Proof.** We first estimate that :

$$\begin{aligned} \|\hat{r}_N(S) - r(S)\|_{L^p}^p &\leq E \left[ |\tilde{r}_N(S) - r(S)|^p \wedge \gamma(S)^p \right] \\ &= E \left[ \left| \frac{\varepsilon_N^R(S) - r(S) \varepsilon_N^1(S)}{\hat{q}_N^1(S)} \right|^p \wedge \gamma(S)^p \right] \end{aligned} \quad (5.3)$$

where

$$\varepsilon_N^R(x, \omega) := \hat{q}_N^R(x, \omega) - q^R(x) \quad \text{and} \quad \varepsilon_N^1(x, \omega) := \hat{q}_N^1(x, \omega) - q^1(x) .$$



For later use, we observe that :

$$\begin{aligned}\|\varepsilon_N^R(x)\|_{L^1} &\leq \|\varepsilon_N^R(x)\|_{L^2} \leq N^{-1/2}V_R(x)^{1/2}, \\ \|\varepsilon_N^1(x)\|_{L^1} &\leq \|\varepsilon_N^1(x)\|_{L^2} \leq N^{-1/2}V_1(x)^{1/2},\end{aligned}$$

by (5.1). Next, for all  $x \in \mathbb{R}^d$ , we consider the event set

$$\mathcal{M}(x) := \{\omega \in \Omega : |\hat{q}_N^1(x, \omega) - q^1(x)| \leq 2^{-1}q^1(x)\},$$

and observe that, for a.e.  $\omega \in \Omega$ ,

$$\begin{aligned}\left| \frac{\varepsilon_N^R(x, \omega) - r(x)\varepsilon_N^1(x)}{\hat{q}_N^1(x, \omega)} \right|^p \wedge \gamma(x)^p &\leq \left| 2 \frac{\varepsilon_N^R(x, \omega) - r(x)\varepsilon_N^1(x)}{q^1(x)} \right|^p \wedge \gamma(x)^p \mathbf{1}_{\mathcal{M}(x)}(\omega) \\ &\quad + \gamma(x)^p \mathbf{1}_{\mathcal{M}(x)^c}(\omega).\end{aligned}\tag{5.4}$$

As for the first term on the right hand-side, we directly compute that for  $p \geq 1$  :

$$\left| 2 \frac{\varepsilon_N^R(x) - r(x)\varepsilon_N^1(x)}{q^1(x)} \right|^p \wedge \gamma(x)^p \leq 2 \left| \frac{\varepsilon_N^R(x) - r(x)\varepsilon_N^1(x)}{q^1(x)} \right| \gamma(x)^{p-1}$$

so that

$$\begin{aligned}&E \left[ \left| 2 \frac{\varepsilon_N^R(S) - r(S)\varepsilon_N^1(S)}{q^1(S)} \right|^p \wedge \gamma(S)^p \mathbf{1}_{\mathcal{M}(S)} \right] \\ &\leq 2 \int_{\mathbb{R}^d} E |\varepsilon_N^R(x) - r(x)\varepsilon_N^1(x)| \gamma(x)^{p-1} dx \\ &\leq 2 \int_{\mathbb{R}^d} (\|\varepsilon_N^R(x)\|_{L^2} + \|r(x)\varepsilon_N^1(x)\|_{L^2}) \gamma(x)^{p-1} dx \\ &= 2N^{-1/2} \int_{\mathbb{R}^d} \left( V_R(x)^{1/2} + |r(x)|V_1(x)^{1/2} \right) \gamma(x)^{p-1} dx.\end{aligned}\tag{5.5}$$

The second term on the right hand-side of (5.4) is estimated by means of the Tchebychev inequality :

$$\begin{aligned}E [\gamma(S)^p \mathbf{1}_{\mathcal{M}(S)^c}] &= E [E (\gamma(S)^p \mathbf{1}_{\mathcal{M}(S)^c} | S)] \\ &= E [\gamma(x)^p P(\mathcal{M}(S)^c | S)] \\ &= E [\gamma(x)^p P(2|\hat{q}_N^1(S) - q^1(S)| > q^1(S) | S)] \\ &\leq 2 E [\gamma(x)^p q^1(S)^{-1} E |\hat{q}_N^1(S) - q^1(S)|] \\ &\leq 2 E [\gamma(x)^p q^1(S)^{-1} \text{Var}[\hat{q}_N^1(S)]^{1/2}] \\ &= 2N^{-1/2} \int_{\mathbb{R}^d} \gamma(x)^p V_1(x)^{1/2} dx.\end{aligned}\tag{5.6}$$

The required result follows by plugging inequalities (5.4), (5.5) and (5.6) into (5.3).  $\square$

**Remark 5.1** Observe from the above proof, that the error estimate of Theorem 5.1 could have been written in terms of  $\|\varepsilon_N^R(x)\|_{L^1}$  and  $\|\varepsilon_N^1(x)\|_{L^1}$  instead of  $N^{-1/2}V_R(x)^{1/2}$  and  $N^{-1/2}V_1(x)^{1/2}$ . In that case, the estimate of Theorem 5.1 reads :

$$\|\hat{r}_N(S) - r(S)\|_{L^p}^p \leq 2 \int_{\mathbb{R}^d} \gamma(x)^{p-1} [\|\varepsilon_N^R(x)\|_{L^1} + (|r(x)| + \gamma(x)) \|\varepsilon_N^1(x)\|_{L^1}] dx,$$

a result which does not require the assumption  $E(A(x), B(x)) = (q^R(x), q^1(x))$ . In the kernel approach,  $\|\varepsilon_N^R(x)\|_{L^1}$  and  $\|\varepsilon_N^R(x)\|_{L^1}$  will typically go to zero as  $N$  tends to infinity. A detailed study of the above quantity is left to future researches.  $\square$

## 6 Malliavin Calculus based regression approximation

In this section, we concentrate on the Malliavin approach for the approximation of the conditional expectation  $E_i^\pi$  as introduced in Example 4.2. We shall assume throughout this section that

$$b, \sigma \in C_b^\infty \quad \text{and} \quad \inf \left\{ \xi^* \sigma(x) \xi : \xi \in \mathbb{R}^d \text{ and } |\xi| = 1 \right\} > 0 \text{ for all } x \in \mathbb{R}^d. \quad (6.1)$$

For sake of simplicity, we shall also restrict the presentation to the case of regular sampling :

$$\Delta_i^\pi = t_i - t_{i-1} = |\pi| \quad \text{for all } 1 \leq i \leq n.$$

### 6.1 Alternative representation of conditional expectation

We start by introducing some notations. Throughout this section, we shall denote by  $\mathcal{J}_k$  the subset of  $\mathbb{N}^k$  whose elements  $I = (i_1, \dots, i_k)$  satisfy  $1 \leq i_1 < \dots < i_k \leq d$ . We extend this definition to  $k = 0$  by setting  $\mathcal{J}_0 = \emptyset$ .

Let  $I = (i_1, \dots, i_m)$  and  $J = (j_1, \dots, j_n)$  be two arbitrary elements in  $\mathcal{J}_m$  and  $\mathcal{J}_n$ . Then  $\{i_1, \dots, i_m\} \cup \{j_1, \dots, j_n\} = \{k_1, \dots, k_p\}$  for some  $\max\{m, n\} \leq p \leq \min\{d, m+n\}$ , and  $1 \leq k_1 < \dots < k_p \leq d$ . We then denote  $I \vee J := (k_1, \dots, k_p) \in \mathcal{J}_p$ .

Given a matrix-valued process  $h$ , with columns denoted by  $h^i$ , and a random variable  $F$ , we denote

$$S_i^h[F] := \int_0^\infty F h_t^i \cdot dW_t \quad \text{for } i = 1, \dots, k, \text{ and } S_I^h[F] := S_{i_1}^h \circ \dots \circ S_{i_k}^h[F]$$

for  $I = (i_1, \dots, i_k) \in \mathcal{J}_k$ , whenever these stochastic integrals exist in the Skorohod sense. We extend this definition to  $k = 0$  by setting  $S_\emptyset^h[F] := F$ . Similarly, for  $I \in \mathcal{J}_k$ , we set :

$$S_{-I}^h[F] := S_{\bar{I}}^h[F] \quad \text{where } \bar{I} \in \mathcal{J}_{d-k} \text{ and } I \vee \bar{I} \text{ is the unique element of } \mathcal{J}_d.$$

Let  $\varphi$  be a  $C_b^0(\mathbb{R}_+^d)$ , i.e. continuous and bounded, mapping from  $\mathbb{R}_+^d$  into  $\mathbb{R}$ . We say that  $\varphi$  is a *smooth localizing function* if

$$\varphi(0) = 1 \quad \text{and} \quad \partial_I \varphi \in C_b^0(\mathbb{R}_+^d) \quad \text{for all } k = 0, \dots, d \text{ and } I \in \mathcal{J}_k.$$

Here,  $\partial_I \varphi = \partial^k \varphi / \partial x_{i_1} \dots \partial x_{i_k}$ . For  $k = 0$ ,  $\mathcal{J}_k = \emptyset$ , and we set  $\partial_\emptyset \varphi := \varphi$ . We denote by  $\mathcal{L}$  the collection of all smooth localization functions.

With these notations, we introduce the set  $\mathbf{H}(X_{t_i}^\pi)$  ( $1 \leq i \leq n-1$ ) as the collection of all matrix-valued  $L^2(\mathcal{F}_1)$  processes  $h$  satisfying

$$\int_0^\infty D_t X_{t_i}^\pi h_t dt = I_d \quad \text{and} \quad \int_0^\infty D_t X_{t_{i+1}}^\pi h_t dt = 0 \quad (6.2)$$

(here  $I_d$  denotes the identity matrix of  $\mathbb{M}^d$ ) and such that, for any affine function  $a : \mathbb{R}^d \longrightarrow \mathbb{R}$ ,

$$S_I^h [a(\Delta^\pi W_{i+1}) \varphi(X_{t_i}^\pi)] \text{ is well-defined in } \mathbb{D}^{1,2} \text{ for all } I \in \mathcal{J}_k, k \leq d \text{ and } \varphi \in \mathcal{L}. \quad (6.3)$$

For later use, we observe that by a straightforward extension of Remark 3.4 in [7], we have the representation :

$$S^{h_i} [a(\Delta^\pi W_{i+1}) \varphi(X_{t_i}^\pi - x)] = \sum_{j=0}^d (-1)^j \sum_{J \in \mathcal{J}_j} \partial_J \varphi(X_{t_i}^\pi - x) S_{-J}^{h_i} [a(\Delta^\pi W_{i+1})], \quad (6.4)$$

for any affine function  $a : \mathbb{R}^d \longrightarrow \mathbb{R}$ .

Moreover, it follows from that (6.1), that  $X_{t_i}^\pi \in \mathbb{D}^\infty$  for each  $i \in \{1, \dots, n\}$ , where the Malliavin derivatives can be computed recursively as follows :

$$\begin{aligned} D_t X_{t_1}^\pi &= \sigma(X_0^\pi) \mathbf{1}_{t \leq t_1} \\ D_t X_{t_i}^\pi &= D_t X_{t_{i-1}}^\pi + |\pi| \nabla b(X_{t_{i-1}}^\pi) (D_t X_{t_{i-1}}^\pi) \\ &\quad + \sum_{j=1}^d \nabla \sigma^j(X_{t_{i-1}}^\pi) (D_t X_{t_{i-1}}^\pi) \Delta^\pi W_i^j + \sigma(X_{t_{i-1}}^\pi) \mathbf{1}_{t \in (t_{i-1}, t_i]}. \end{aligned}$$

In particular, for  $t \in (t_{i-1}, t_i)$ , we obtain that  $D_t X_{t_i}^\pi = \sigma(X_{t_{i-1}}^\pi)$ . Let  $h_i$  be the  $\mathbb{M}^d$ -valued process defined :

$$\begin{aligned} &h_{i,t} |\pi| \\ := &\begin{cases} \sigma(X_{t_{i-1}}^\pi)^{-1} & \text{on } t \in [t_{i-1}, t_i) \\ -\sigma(X_{t_i}^\pi)^{-1} \left( I_d + |\pi| \nabla b(X_{t_i}^\pi) + \sum_{j=1}^d \nabla \sigma^j(X_{t_i}^\pi) \Delta^\pi W_{i+1}^j \right) & \text{on } t \in [t_i, t_{i+1}) \\ 0 & \text{elsewhere.} \end{cases} \quad (6.5) \end{aligned}$$

Since  $b, \sigma, \sigma^{-1} \in C_b^\infty$  by (6.1), one easily checks that  $h_i$  satisfies (6.2)-(6.3) and therefore lies in  $\mathbf{H}(X_{t_i}^\pi)$ .

**Remark 6.1** For later use, let us observe that, for all  $s_1, \dots, s_\ell \in [t_{i-1}, t_{i+1}]$ , it follows from (6.1) that :

$$\sup_{t_{i-1} \leq t \leq t_{i+1}} \{ \|h_{i,t}\|_{L^p} + \|D_{s_1, \dots, s_\ell} h_{i,t}\|_{L^p} \} \leq C_p |\pi|^{-1}, \quad (6.6)$$

for some constant  $C_p$  which does not depend on  $|\pi|$ . Moreover, for any affine function  $a : \mathbb{R}^d \longrightarrow \mathbb{R}$ , and all  $1 \leq k, i \leq d$ ,

$$\begin{aligned} S_{\{k\}}^{h_i} [a(\Delta^\pi W_{i+1})] &= a(\Delta^\pi W_{i+1}) h_{i,t_{i-1}}^k \cdot \Delta^\pi W_i + a(\Delta^\pi W_{i+1}) h_{i,t_i}^k \cdot \Delta^\pi W_{i+1} \\ &\quad - a(\Delta^\pi W_{i+1}) \int_{t_i}^{t_{i+1}} \text{Trace} \left( D_t h_{i,t_i}^k \right) dt \\ &\quad - \int_{t_i}^{t_{i+1}} \nabla a \cdot h_{i,t_i}^k dt, \end{aligned}$$

so that we also deduce the estimates :

$$\left\| S_{\{k\}}^{h_i} [a(\Delta^\pi W_{i+1})] \right\|_{L^p} \leq C_p |\pi|^{-1/2}, \quad \left\| D_{s_1, \dots, s_\ell} S_{\{k\}}^{h_i} [a(\Delta^\pi W_{i+1})] \right\|_{L^p} \leq C_p |\pi|^{-1}. \quad (6.7)$$

□

We now provide a slight extension of Corollary 3.1 in [7] which will be the starting point for our conditional expectation estimator.

**Theorem 6.1** *Let  $\varrho$  be a real-valued mapping and  $\xi$  be a (vector) random variable independent of  $\sigma(X_{t_i}^\pi, 1 \leq i \leq n)$  with  $R := \varrho(X_{t_{i+1}}^\pi, \xi) a(\Delta^\pi W_{i+1}) \in L^2$ , for some affine function  $a : \mathbb{R}^d \rightarrow \mathbb{R}$ . Then, for all localizing functions  $\varphi, \psi \in \mathcal{L}$  :*

$$E[R | X_{t_i}^\pi = x] = \frac{E[Q^R[h_i, \varphi](x)]}{E[Q^1[h_i, \psi](x)]} \quad (6.8)$$

where

$$Q^R[h_i, \varphi](x) := H_x(X_{t_i}^\pi) \varrho(X_{t_{i+1}}^\pi, \xi) S^{h_i} [a(\Delta^\pi W_{i+1}) \varphi(X_{t_i}^\pi - x)],$$

and  $S^{h_i} = S_{(1, \dots, d)}^{h_i}$ . Moreover, if  $q_i^\pi$  denotes the density of  $X_{t_i}^\pi$ , then :

$$q_i^\pi(x) = E[Q^1[h_i, \varphi](x)].$$

**Remark 6.2** The above theorem holds for any random variable  $F \in \mathbb{D}^\infty$ , instead of the particular affine transformation of the Brownian increments  $a(\Delta^\pi W_{i+1})$ . One only has to change the definition of  $\mathcal{L}$  accordingly in order to ensure that the involved Skorohod integrals are well-defined. However, we shall see in Section 6.2 that we only need this characterization for the affine functions  $a_k(x) = \mathbf{1}_{k=0} + x^k \mathbf{1}_{k \geq 1}$ ,  $0 \leq k \leq d$ . Indeed, writing  $\hat{Y}_{t_{i+1}}^\pi$  as  $\varrho(X_{t_{i+1}}^\pi, \zeta)$ , we are interested in computing  $E_i^\pi[\hat{Y}_{t_{i+1}}^\pi] = E_i^\pi[\hat{Y}_{t_{i+1}}^\pi a_0(\Delta^\pi W_{i+1})]$  and  $E_i^\pi[\hat{Y}_{t_{i+1}}^\pi \Delta^\pi W_{i+1}^k] = E_i^\pi[\hat{Y}_{t_{i+1}}^\pi a_k(\Delta^\pi W_{i+1})]$ ,  $1 \leq k \leq d$ , see also the definition of  $R_i$  after (6.12). □

## 6.2 Application to the estimation of $\hat{E}^\pi$

The algorithm is inspired from the work of [9] and [21]. We consider  $nN$  copies  $(X^{\pi(1)}, \dots, X^{\pi(nN)})$  of the discrete-time process  $X^\pi$  on the grid  $\pi$ , where  $N$  is some positive integer. Set

$$\mathcal{N}_i := \{(i-1)N + 1, \dots, iN\}, \quad 1 \leq i \leq n.$$

For ease of notation, we write  $X^{\pi(0)}$  for  $X^\pi$ . We consider the approximation scheme (4.2)-(4.3)-(4.4) with an approximation of conditional expectation operator  $E_i^\pi$  suggested by Theorem 6.1. At each time step  $t_i$  of this algorithm, we shall make use of a the subset  $\mathcal{X}_i$

$:= (X^{\pi^{(j)}})$ ,  $j \in \mathcal{N}_i$ ). The independence of the  $\mathcal{X}_i$ 's is crucial as explained in Remark 6.3 below.

**Initialization :** For  $j \in \{0\} \cup \mathcal{N}_n$ , we set :

$$\hat{Y}_1^{\pi^{(j)}} := Y_1^{\pi^{(j)}} = g\left(X_1^{\pi^{(j)}}\right). \quad (6.9)$$

**Backward induction :** For  $i = n, \dots, 2$ , we set, for  $j \in \{0\} \cup \mathcal{N}_{i-1}$  :

$$\begin{aligned} \check{Y}_{t_{i-1}}^{\pi^{(j)}} &= \hat{E}_{i-1}^{\pi^{(j)}} \left[ \hat{Y}_{t_i}^{\pi^{(j)}} \right] + |\pi| f\left(t_{i-1}, X_{t_{i-1}}^{\pi^{(j)}}, \check{Y}_{t_{i-1}}^{\pi^{(j)}}, \hat{Z}_{t_{i-1}}^{\pi^{(j)}}\right) \\ \hat{Y}_{t_{i-1}}^{\pi^{(j)}} &:= \mathbf{T}_{i-1}^{\varphi^{\pi^{(j)}}} \left( \check{Y}_{t_{i-1}}^{\pi^{(j)}} \right) \end{aligned} \quad (6.10)$$

$$\hat{Z}_{t_{i-1}}^{\pi^{(j)}} = \frac{1}{|\pi|} \hat{E}_{i-1}^{\pi^{(j)}} \left[ \hat{Y}_{t_i}^{\pi^{(j)}} \Delta^\pi W_i^{(j)} \right] \quad (6.11)$$

The approximations of the conditional expectations  $\hat{E}_{i-1}^{\pi^{(j)}}[\cdot]$  are obtained as follows.

1. We first compute the estimator suggested by Theorem 6.1 :

$$\tilde{E}_{i-1}^{\pi^{(j)}} \left[ R_i^{(j)} \right] := \frac{\hat{Q}^{R_i^{(j)}} \left[ h_{i-1}^{(j)}, \varphi \right] \left( X_{t_{i-1}}^{\pi^{(j)}} \right)}{\hat{Q}^1 \left[ h_{i-1}^{(j)}, \varphi \right] \left( X_{t_{i-1}}^{\pi^{(j)}} \right)} \quad (6.12)$$

where, for  $R_i = \hat{Y}_{t_i}^{\pi^{(j)}} a(\Delta^\pi W_i)$  and  $a : \mathbb{R}^d \rightarrow \mathbb{R}$  is an affine function,

$$\begin{aligned} \hat{Q}^{R_i^{(j)}} \left[ h_{i-1}^{(j)}, \varphi \right] \left( X_{t_{i-1}}^{\pi^{(j)}} \right) &:= \frac{1}{N} \sum_{l \in \mathcal{N}_i} H_{X_{t_{i-1}}^{\pi^{(j)}}} \left( X_{t_{i-1}}^{\pi^{(l)}} \right) \hat{Y}_{t_i}^{\pi^{(l)}} S^{h_i^{(l)}} \left[ a \left( \Delta^\pi W_{i+1}^{(l)} \right) \varphi \left( X_{t_{i-1}}^{\pi^{(l)}} - X_{t_{i-1}}^{\pi^{(j)}} \right) \right] \\ \hat{Q}^1 \left[ h_{i-1}^{(j)}, \varphi \right] \left( X_{t_{i-1}}^{\pi^{(j)}} \right) &:= \frac{1}{N} \sum_{l \in \mathcal{N}_i} H_{X_{t_{i-1}}^{\pi^{(j)}}} \left( X_{t_{i-1}}^{\pi^{(l)}} \right) S^{h_i^{(l)}} \left[ \varphi \left( X_{t_{i-1}}^{\pi^{(l)}} - X_{t_{i-1}}^{\pi^{(j)}} \right) \right]. \end{aligned}$$

2. We next use the sequence  $\varphi^\pi$  of *a priori* bounds on  $Y$ , see Section 4, together with the induced sequences  $\mathfrak{R}^\pi$  and  $\mathfrak{S}^\pi$ , to improve the above estimator :

$$\hat{E}_{i-1}^{\pi^{(j)}} \left[ \hat{Y}_{t_i}^{\pi^{(j)}} \right] := \mathbf{T}_{i-1}^{\mathfrak{R}^{\pi^{(j)}}} \left( \tilde{E}_{i-1}^{\pi^{(j)}} \left[ \hat{Y}_{t_i}^{\pi^{(j)}} \right] \right)$$

and

$$\hat{E}_{i-1}^{\pi^{(j)}} \left[ \hat{Y}_{t_i}^{\pi^{(j)}} \Delta^\pi W_i^{(j)} \right] := \mathbf{T}_{i-1}^{\mathfrak{S}^{\pi^{(j)}}} \left( \tilde{E}_{i-1}^{\pi^{(j)}} \left[ \hat{Y}_{t_i}^{\pi^{(j)}} \Delta^\pi W_i^{(j)} \right] \right).$$

**Final step :** For  $i = 1$ , the conditional expectations  $\hat{E}_{i-1}^{\pi^{(j)}}[\cdot] = \hat{E}_0^\pi[\cdot]$  are computed by the usual empirical mean :

$$\hat{E}_0^\pi \left[ R_1^{(0)} \right] := \frac{1}{N} \sum_{l \in \mathcal{N}_1} R_1^{(l)}. \quad (6.13)$$

**Remark 6.3** Notice that by construction, for each  $i = 1, \dots, n-1$  and  $k \in \mathcal{N}_i$ ,  $(\hat{Y}_{t_i}^{\pi^{(k)}}, \hat{Z}_{t_i}^{\pi^{(k)}})$  can be written as a square integrable function of  $X_{t_i}^{\pi^{(k)}}$  and  $\zeta := \left( \Delta^\pi W^{(j)}, j \in \bigcup_{l \geq i+1} \mathcal{N}_l \right)$ . This is precisely the reason why our simulation scheme uses  $n$  independent sets of  $N$  simulated paths. Indeed, this ensures that the above random variable  $\zeta$  is independent of  $\mathcal{F}_1^{(k)}$ , and therefore we fall in the context of Theorem 6.1.  $\square$

The following provides an estimate of the simulation error in the above algorithm.

**Theorem 6.2** *Let  $p > 1$  and  $\varphi \in \mathcal{L}$  satisfying*

$$\sum_{k=0}^d \sum_{I \in \mathcal{J}_k} \int_{\mathbb{R}^d} |u|^{4p+2} \partial_I \varphi(u)^2 du < \infty .$$

*Consider the function  $\varphi^\pi(x) = \varphi(|\pi|^{-1/2}x)$  as a localizing function in (4.2)-(4.3)-(4.4)-(6.12). Let  $\mathfrak{R}^\pi$ ,  $\mathfrak{R}^\pi$ ,  $\mathfrak{S}^\pi$  be the bounds defined by Lemma 3.3. Then :*

$$\limsup_{|\pi| \rightarrow 0} \max_{0 \leq i \leq n} |\pi|^{p+d/4} N^{1/2} \left\| \hat{Y}_{t_i}^\pi - Y_{t_i}^\pi \right\|_{L^p}^p < \infty .$$

The above estimate is obtained in two steps. First, Theorem 4.1 reduces the problem to the analysis of the regression simulation error. Next, for  $1 \leq i \leq n$ , the result follows from Theorem 6.3 which is the main object of the subsequent paragraph. The case  $i = 0$  is trivial as the regression estimator (6.13) is the classical empirical mean.

**Remark 6.4** In the particular case where the generator  $f$  does not depend on the control variable  $z$ , the above proposition is valid with  $p = 1$ . This follows from Remark 4.2.  $\square$

**Remark 6.5** In the previous proposition, we have introduced the normalized localizing function  $\varphi^\pi(x) = \varphi(\pi^{-1/2}x)$ . This normalization is necessary for the control of the error estimate as  $|\pi|$  tends to zero. An interesting observation is that, in the case where  $R$  is of the form  $\varrho(X_{t_{i+1}}^\pi, \zeta)a(\Delta^\pi W_{i+1})$  for some affine function  $a : \mathbb{R}^d \rightarrow \mathbb{R}$ , this normalization is in agreement with [7] who showed that the minimal integrated variance within the class of separable localization functions is given by  $\hat{\varphi}(x) = \exp(-\hat{\eta} \cdot x)$  with

$$(\hat{\eta}^i)^2 = \frac{E \left[ \left\{ R \sum_{k=0}^{d-1} (-1)^k \sum_{i \notin I \in \mathcal{J}_k} S_{-I}^h [a(\Delta^\pi W_{i+1})] \prod_{j \in I} \hat{\eta}^j \right\}^2 \right]}{E \left[ \left\{ R \sum_{k=0}^{d-1} (-1)^k \sum_{i \notin I \in \mathcal{J}_k} S_{-(I \vee i)}^h [a(\Delta^\pi W_{i+1})] \prod_{j \in I} \hat{\eta}^j \right\}^2 \right]} , \quad 1 \leq i \leq d .$$

Indeed, we will show in Lemma 6.1 below that  $S_I^h [a(\Delta^\pi W_{i+1})]$  is of order  $|\pi|^{-|I|/2}$ , and therefore the above ratio is of order  $|\pi|^{-1}$ .  $\square$

### 6.3 Analysis of the regression error

According to Theorem 5.1, the  $L^p$  estimate of the regression error depends on the *integrated standard deviation*  $\Gamma$  defined in (5.2). In order to analyze this term, we start by

**Lemma 6.1** *For any integer  $m = 0, \dots, d$ ,  $I \in \mathcal{J}_m$ , and any affine function  $a : \mathbb{R}^d \rightarrow \mathbb{R}$ , we have :*

$$\limsup_{|\pi| \rightarrow 0} |\pi|^{m/2} \max_{1 \leq i \leq n-1} \left\| S_I^{h_i} [a(\Delta^\pi W_{i+1})] \right\|_{L^p} < \infty ,$$

for all  $p \geq 1$ .

**Proof.** Let  $d \geq j_1 > \dots > j_m \geq 1$  be  $m$  integers, and define  $I_k := (j_{m-k}, \dots, j_1)$ . For ease of presentation, we introduce process  $\tilde{h}_i := |\pi| h_i$ , and we observe that  $S_I^{h_i} = |\pi|^m S_I^{\tilde{h}_i}$ , by linearity of the Skorohod integral. We shall also write  $S_I^{\tilde{h}_i}$  for  $S_I^{\tilde{h}_i} [a(\Delta^\pi W_{i+1})]$ . In order to prove the required result, we intend to show that

$$\text{for all integers } 0 \leq \ell \leq k \leq m-1 \text{ and all } \tau := (s_1, \dots, s_\ell) \in [t_{i-1}, t_{i+1}]^\ell, \quad (6.14)$$

$$\left\| D_\tau S_{I_k}^{\tilde{h}_i} \right\|_{L^p} \leq C_p |\pi|^{(k-\ell)/2},$$

where  $C_p$  is a constant which does not depend on  $\pi$ , and,  $\tau = \emptyset$ ,  $D_\tau F = F$ , whenever  $\ell = 0$ . We shall use a backward induction argument on the variable  $k$ . First, for  $k = m-1$ , (6.14) follows from (6.7). We next assume that (6.14) holds for some  $1 \leq k \leq m-1$ , and we intend to extend it to  $k-1$ . We first need to introduce some notations. Let  $\mathcal{S}_\ell$  be the collection of all permutations  $\sigma$  of the set  $(1, \dots, \ell)$ . For  $\sigma \in \mathcal{S}_\ell$  and some integer  $u \leq \ell$ , we set  $\tau_u^\sigma := (s_{\sigma(1)}, \dots, s_{\sigma(u)})$  and  $\bar{\tau}_u^\sigma := (s_{\sigma(u+1)}, \dots, s_{\sigma(\ell)})$ , with the convention  $\tau_0^\sigma = \bar{\tau}_\ell^\sigma = \emptyset$ .

Let  $\ell \leq k-1$  be fixed. By direct computation, we see that :

$$\begin{aligned} \left\| D_\tau S_{I_{k-1}}^{\tilde{h}_i} \right\|_{L^p} &\leq \sum_{\sigma \in \mathcal{S}_\ell} \sum_{u=0}^{\ell} \left\| D_{\tau_u^\sigma} S_{I_k}^{\tilde{h}_i} \right\|_{L^{2p}} \left\| D_{\bar{\tau}_u^\sigma} S_{I_{m-k+1}}^{\tilde{h}_i} \right\|_{L^{2p}} \\ &\quad + \int_{t_{i-1}}^{t_{i+1}} \left\| D_{\tau_u^\sigma} D_t S_{I_k}^{\tilde{h}_i} \right\|_{L^{2p}} \left\| D_{\bar{\tau}_u^\sigma} \tilde{h}_t^{j_{m-k+1}} \right\|_{L^{2p}} dt \end{aligned}$$

by Cauchy-Schwarz inequality. By (6.7) and the induction hypothesis (6.14), this provides :

$$\begin{aligned} \left\| D_\tau S_{I_{k+1}}^{\tilde{h}_i} \right\|_{L^p} &\leq C \sum_{\sigma \in \mathcal{S}_\ell} \left( |\pi|^{(k-\ell)/2} |\pi|^{1/2} + \sum_{u=0}^{\ell-1} |\pi|^{(k-u)/2} + |\pi| \sum_{u=0}^{\ell} |\pi|^{(k-u-1)/2} \right) \\ &\leq C \sum_{\sigma \in \mathcal{S}_\ell} \left( |\pi|^{(k+1-\ell)/2} + \sum_{u=0}^{\ell-1} |\pi|^{(k+1-\ell)/2} + \sum_{u=0}^{\ell} |\pi|^{(k+1-\ell)/2} \right) \\ &= C |\pi|^{(k+1-\ell)/2}, \end{aligned}$$

where  $C$  is a generic constant, independent of  $|\pi|$  and  $i$ , with different values from line to line.  $\square$

**Lemma 6.2** *Let  $\mu$  be a map from  $\mathbb{R}^d$  into  $\mathbb{R}$  with polynomial growth :*

$$\sup_{x \in \mathbb{R}^d} \frac{|\mu(x)|}{1 + |x|^m} < \infty, \quad \text{for some } m \geq 1.$$

*Let  $\varphi \in \mathcal{L}$  be such that :*

$$\sum_{k=0}^d \sum_{I \in \mathcal{J}_k} \int_{\mathbb{R}^d} |u|^m \partial_I \varphi(u)^2 du < \infty.$$

*Let  $R_{i+1} := \varrho \left( X_{i+1}^\pi, \zeta \right) a(\Delta^\pi W_{i+1})$  for some deterministic function  $\varrho$ , some affine function  $a : \mathbb{R}^d \rightarrow \mathbb{R}$ , and some random variable  $\zeta$  independent of  $\mathcal{F}_1^\pi$ . Assume that  $R_{i+1} \in L^{2+\varepsilon}$*

for some  $\varepsilon > 0$ . Then,

$$\limsup_{|\pi| \rightarrow 0} \max_{1 \leq i \leq n} |\pi|^{d/2} \|R_{i+1}\|_{L^{2+\varepsilon}}^{-2} \int_{\mathbb{R}^d} \mu(x) V_{i,R_{i+1}}^\pi(x) dx < \infty,$$

where

$$V_{i,R_{i+1}}^\pi(x) = \text{Var} [Q^{R_{i+1}}[h_i, \varphi^\pi](x)] \quad \text{and} \quad \varphi^\pi(x) := \varphi(|\pi|^{-1/2}x).$$

**Proof.** We shall write  $S_I^{h_i}$  for  $S_I^{h_i}[a(\Delta^\pi W_{i+1})]$ . We first estimate that :

$$\begin{aligned} V_{i,R_{i+1}}^\pi(x) &\leq E \left[ (Q^{R_{i+1}}[h_i, \varphi^\pi](x))^2 \right] \\ &= E \left[ H_x(X_{t_i}^\pi) R_{i+1}^2 \left\{ S^{h_i}[a(\Delta^\pi W_{i+1}) \varphi^\pi(X_{t_i}^\pi - x)] \right\}^2 \right] \\ &\leq 2 \sum_{j=0}^d \sum_{J \in \mathcal{J}_j} E \left[ H_x(X_{t_i}^\pi) R_{i+1}^2 \left\{ \partial_J \varphi^\pi(X_{t_i}^\pi - x) S_{-J}^{h_i} \right\}^2 \right] \end{aligned}$$

where we used (6.4). For ease of notation, we introduce the parameter  $\eta > 0$  such that  $2(1+\eta)^2 = 2+\varepsilon$  and  $\bar{\eta} := 1+1/\eta$  is the conjugate of  $1+\eta$ . Applying twice the Hölder inequality, we see that :

$$\begin{aligned} \int_{\mathbb{R}^d} \mu(x) V_{i,R_{i+1}}^\pi(x) dx &\leq 2 \sum_{j=0}^d \sum_{J \in \mathcal{J}_j} E \left[ R_{i+1}^2 \left\{ S_{-J}^{h_i} \right\}^2 \int_{\mathbb{R}^d} H_x(X_{t_i}^\pi) \mu(x) \partial_J \varphi^\pi(X_{t_i}^\pi - x)^2 dx \right] \\ &\leq 2 \|R_{i+1}\|_{L^{2+\varepsilon}}^2 \sum_{j=0}^d \sum_{J \in \mathcal{J}_j} \left\| S_{-J}^{h_i} \right\|_{L^{2\bar{\eta}(1+\eta)}}^2 A_i^J \end{aligned} \quad (6.15)$$

where

$$A_i^J := \left\| \int_{\mathbb{R}^d} H_x(X_{t_i}^\pi) \mu(x) \partial_J \varphi^\pi(X_{t_i}^\pi - x)^2 dx \right\|_{L^{\bar{\eta}}}.$$

By definition of  $\varphi^\pi$ , we observe that  $\partial_J \varphi^\pi(x) = |\pi|^{-|J|/2} \partial_J \varphi(|\pi|^{-1/2}x)$ . It then follows from a direct change of variable together with the polynomial growth condition on  $\mu$  that :

$$\begin{aligned} A_i^J &= |\pi|^{d/2-|J|} \left\| \int_{\mathbb{R}_+^d} \mu(X_{t_i}^\pi - |\pi|^{1/2}x) \partial_J \varphi(x)^2 dx \right\|_{L^{\bar{\eta}}} \\ &\leq C |\pi|^{d/2-|J|} \left\| \int_{\mathbb{R}_+^d} \left( 1 + \sum_{k=0}^m \binom{k}{m-k} |X_{t_i}^\pi|^k |\pi|^{1/2} x^{m-k} \right) \partial_J \varphi(x)^2 dx \right\|_{L^{\bar{\eta}}} \\ &\leq C |\pi|^{d/2-|J|} \left\{ \int_{\mathbb{R}_+^d} \partial_J \varphi(x)^2 dx + 2C \|X_{t_i}^\pi\|_{L^{m\bar{\eta}}}^m \int_{\mathbb{R}_+^d} |x|^m \partial_J \varphi(x)^2 dx \right\} \end{aligned}$$

Notice that the right hand-side term is finite by our assumption on the localizing function. Since  $\max_{1 \leq i \leq n} \|X_{t_i}^\pi\|_{L^{m\bar{\eta}}}$  is bounded uniformly in  $\pi$  by (3.2), this proves that  $|A_i^J| \leq C |\pi|^{d/2-|J|}$ . Plugging this into (6.15), we obtain :

$$\int_{\mathbb{R}^d} \mu(x) V_{i,R_{i+1}}^\pi(x) dx \leq C \|R_{i+1}\|_{L^{2+\varepsilon}}^2 \sum_{j=0}^d \sum_{J \in \mathcal{J}_j} |\pi|^{d/2-|J|} \left\| S_{-J}^{h_i} \right\|_{L^{2\bar{\eta}(1+\eta)}}^2,$$



and the required result follows from Lemma 6.1 (recall that  $|\cdot - J| = d - |J|$ ), see the definitions in §6.1.  $\square$

**Theorem 6.3** *Let  $R_{i+1} := \varrho \left( X_{t_{i+1}}^\pi, \zeta \right) a \left( \Delta^\pi W_{i+1} \right)$  for some deterministic function  $\varrho$ , some affine function  $a : \mathbb{R}^d \rightarrow \mathbb{R}$ , and some random variable  $\zeta$  independent of  $\mathcal{F}_1^\pi$ . Assume that*

$$\underline{\rho}_i(X_{t_i}^\pi) \leq r_i(X_{t_i}^\pi) := E_i^\pi [R_{i+1}] \leq \bar{\rho}_i(X_{t_i}^\pi)$$

for some  $\rho_i = (\underline{\rho}_i, \bar{\rho}_i)$  with polynomial growth :

$$\sup_{x \in \mathbb{R}^d} \max_{1 \leq i \leq n} \frac{|\underline{\rho}_i(x)| + |\bar{\rho}_i(x)|}{1 + |x|^m} < \infty, \quad \text{for some } m \geq 0.$$

Let  $p \geq 1$  be arbitrary, consider some localizing function  $\varphi \in \mathcal{L}$  satisfying :

$$\sum_{k=0}^d \sum_{I \in \mathcal{J}_k} \int_{\mathbb{R}^d} |u|^{2pm+2} \partial_I \varphi(u)^2 du < \infty,$$

and set  $\varphi^\pi(x) := \varphi(|\pi|^{-1/2}x)$ . Let  $\tilde{E}_i^\pi [R_{i+1}]$  be defined as in (6.12), with localizing function  $\varphi^\pi$ , and consider the truncated regression estimator  $\hat{E}_i^\pi [R_{i+1}] := \mathbf{T}_i^{\rho_i} \left( \tilde{E}_i^\pi [R_{i+1}] \right)$ . Then,

$$\limsup_{|\pi| \rightarrow 0} \max_{1 \leq i \leq n} |\pi|^{d/4} N^{1/2} \left\| \left( \hat{E}_i^\pi - E_i^\pi \right) [R_{i+1}] \right\|_{L^p}^p < \infty.$$

**Proof.** Set  $\gamma_i := \bar{\rho}_i - \underline{\rho}_i$  and observe that  $\gamma_i$  inherits the polynomial growth of  $\rho_i$ . With the notations of Lemma 6.2, it follows from Theorem 5.1 that :

$$\left\| \left( \hat{E}_i^\pi - E_i^\pi \right) [R_{i+1}] \right\|_{L^p}^p \leq 2N^{-1/2} \Gamma \left( r_i, \gamma_i, V_{i,1}^\pi, V_{i,R_{i+1}}^\pi \right),$$

provided that the right hand-side is finite. The rest of this proof is dedicated to the estimation of this term. From the polynomial growth condition on  $\rho_i$ , we estimate that :

$$\Gamma \left( r_i, \gamma_i, V_{i,1}^\pi, V_{i,R_{i+1}}^\pi \right) \leq C \int_{\mathbb{R}^d} (1 + |x|)^{mp} V_{i,R_{i+1}}^\pi(x)^{1/2} + C \int_{\mathbb{R}^d} (1 + |x|)^{mp} V_{i,1}^\pi(x)^{1/2}.$$

We only consider the first term on the right hand-side, as the second one is treated similarly. In order to prove the required result, it is sufficient to show that :

$$\limsup_{|\pi| \rightarrow 0} \max_{1 \leq i \leq n} |\pi|^{d/4} \int_{\mathbb{R}^d} (1 + |x|)^{mp} V_{i,R_{i+1}}^\pi(x)^{1/2} < \infty.$$

Let  $\phi(x) = C_\phi (1 + |x|^2)^{-1}$  with  $C_\phi$  such that  $\int_{\mathbb{R}^d} \phi(x) dx = 1$ . By the Jensen inequality, we get :

$$\begin{aligned} \int_{\mathbb{R}^d} (1 + |x|)^{mp} V_{i,R_{i+1}}^\pi(x)^{1/2} &\leq C \int_{\mathbb{R}^d} \phi(x) \left( \phi^{-2}(x) \left( 1 + |x|^{2mp} \right) V_{i,R_{i+1}}^\pi(x) \right)^{1/2} \\ &\leq C \left( \int_{\mathbb{R}^d} \left( 1 + |x|^{2mp+2} \right) V_{i,R_{i+1}}^\pi(x) \right)^{1/2}. \end{aligned}$$

The proof is completed by appealing to Lemma 6.2.  $\square$

## 7 Extension to reflected backward SDE's

The purpose of this section is to extend our analysis to reflected backward SDE's in the case where the generator  $f$  does not depend on the  $z$  variable. We then consider  $K$ - Lipschitz functions  $f : [0, 1] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R}^d \rightarrow \mathbb{R}$ , for some  $K > 0$ , and we let  $(Y, Z, A)$  be the unique solution of :

$$Y_t = g(X_1) + \int_t^1 f(s, X_s, Y_s) ds - \int_t^1 Z_s \cdot dW_s + A_1 - A_t \quad (7.1)$$

$$Y_t \geq g(X_t) \text{ , } 0 \leq t \leq 1 \text{ ,} \quad (7.2)$$

such that  $Y_t \in L^2$ , for all  $0 \leq t \leq 1$ ,  $Z \in L^2([0, 1])$  and  $A$  is a non-decreasing cadlag process satisfying :

$$\int_0^1 (Y_t - g(X_t)) dA_t = 0 \text{ .}$$

We refer to [14] for the existence and uniqueness issue.

### 7.1 Discrete-time approximation

It is well-known that  $Y$  admits a Snell envelope type representation. We therefore introduce the discrete-time counterpart of this representation :

$$Y_1^\pi = g(X_1^\pi) \text{ ,} \quad (7.3)$$

$$Y_{t_{i-1}}^\pi = \max \left\{ g(X_{t_{i-1}}^\pi) \text{ , } E_{t_{i-1}}^\pi [Y_{t_i}^\pi] + f(t_{i-1}, X_{t_{i-1}}^\pi, Y_{t_{i-1}}^\pi) \Delta_i^\pi \right\} \text{ , } 1 \leq i \leq n \text{ .} \quad (7.4)$$

Observe that our scheme differs for [4], who consider the backward scheme defined by

$$\tilde{Y}_{t_{i-1}}^\pi = \max \left\{ g(X_{t_{i-1}}^\pi) \text{ , } E_{t_{i-1}}^\pi \left[ \tilde{Y}_{t_i}^\pi + f(t_i, X_{t_i}^\pi, \tilde{Y}_{t_i}^\pi) \Delta_i^\pi \right] \right\} \text{ , } 1 \leq i \leq n \text{ ,}$$

instead of (7.4). By direct adaptation of the proofs of [4], we obtain the following estimate of the discretization error. Notice that it is of the same order than in the non-reflected case.

**Theorem 7.1** *For all  $p \geq 1$ ,*

$$\limsup_{|\pi| \rightarrow 0} |\pi|^{-1/2} \sup_{0 \leq i \leq n} \|Y_{t_i}^\pi - Y_{t_i}\|_{L^p} < \infty \text{ .}$$

**Proof.** For  $0 \leq i \leq n$ , we denote by  $\Theta_i^\pi$  the set of stopping times with values in  $\{t_i, \dots, t_n = 1\}$ , and we define :

$$R_{t_i}^\pi := \operatorname{ess\,sup}_{\tau \in \Theta_i^\pi} E_{t_i}^\pi \left[ g(X_\tau) + |\pi| \sum_{j=i}^{n-1} \mathbf{1}_{\tau > t_j} f(t_j, X_{t_j}, Y_{t_j}) \right]$$

$$L_{t_i}^\pi := \operatorname{ess\,sup}_{\tau \in \Theta_i} E_{t_i}^\pi \left[ g(X_\tau) + \int_{t_i}^\tau f(s, X_s, Y_s) ds \right] \text{ , } 0 \leq i \leq n \text{ .}$$

From Lemma 2 (a) in [4], we have :

$$\limsup_{|\pi| \rightarrow 0} |\pi|^{-1/2} \max_{0 \leq i \leq n} \|L_{t_i}^\pi - Y_{t_i}\|_{L^p} < \infty. \quad (7.5)$$

A straightforward adaptation of Lemma 5 in [4] leads to

$$\limsup_{|\pi| \rightarrow 0} |\pi|^{-1/2} \max_{0 \leq i \leq n} \|R_{t_i}^\pi - L_{t_i}^\pi\|_{L^p} < \infty. \quad (7.6)$$

In order to complete the proof, we shall show that :

$$\limsup_{|\pi| \rightarrow 0} |\pi|^{-1/2} \max_{0 \leq i \leq n} \|Y_{t_i}^\pi - R_{t_i}^\pi\|_{L^p} < \infty. \quad (7.7)$$

We first write  $Y_{t_i}^\pi$  in its Snell envelope type representation :

$$Y_{t_i}^\pi := \operatorname{ess\,sup}_{\tau \in \Theta_i^\pi} E_{t_i}^\pi \left[ g(X_\tau^\pi) + |\pi| \sum_{j=i}^{n-1} \mathbf{1}_{\tau > t_j} f(t_j, X_{t_j}^\pi, Y_{t_j}^\pi) \right].$$

By the Lipschitz conditions on  $f$  and  $g$ , we then estimate that :

$$\begin{aligned} |R_{t_i}^\pi - Y_{t_i}^\pi| &\leq C \operatorname{ess\,sup}_{\tau \in \Theta_i} \left\{ E_i^\pi |X_\tau^\pi - X_\tau| + |\pi| \sum_{j=i}^n E_i^\pi \left( |\pi| + |X_{t_j} - X_{t_j}^\pi| + |Y_{t_j} - Y_{t_j}^\pi| \right) \right\} \\ &\leq C \left\{ E_i^\pi \left[ \max_{i \leq j \leq n} |X_{t_j} - X_{t_j}^\pi| \right] \right. \\ &\quad \left. + |\pi| \sum_{j=i}^n E_i^\pi \left( |\pi| + |Y_{t_j} - R_{t_j}^\pi| + |R_{t_j}^\pi - Y_{t_j}^\pi| \right) \right\}. \end{aligned}$$

It follows from the arbitrariness of  $i$ , that for each integer  $i \leq \ell \leq n$  :

$$\begin{aligned} E_i^\pi |R_{t_\ell}^\pi - Y_{t_\ell}^\pi| &\leq C \left\{ E_i^\pi \left[ \max_{i \leq j \leq n} |X_{t_j} - X_{t_j}^\pi| \right] \right. \\ &\quad \left. + |\pi| \sum_{j=i}^n E_i^\pi \left( |\pi| + |Y_{t_j} - R_{t_j}^\pi| + |R_{t_j}^\pi - Y_{t_j}^\pi| \right) \right\}. \end{aligned}$$

Using the discrete time version of Gronwall's Lemma, we therefore obtain :

$$|R_{t_i}^\pi - Y_{t_i}^\pi| \leq C \left\{ E_i^\pi \left[ \max_{i \leq j \leq n} |X_{t_j} - X_{t_j}^\pi| \right] + |\pi| \sum_{j=i}^n E_i^\pi \left( |\pi| + |Y_{t_j} - R_{t_j}^\pi| \right) \right\},$$

for some constant  $C$  independent of  $\pi$ . (7.7) is then obtained by using (3.2)-(7.5)-(7.6).  $\square$

## 7.2 A priori bounds on the discrete-time approximation

Consider the sequence of maps

$$\bar{\varphi}_i^\pi(x) = \alpha_i^\pi + \beta_i^\pi |x|, \quad 0 \leq i \leq n,$$

where the  $(\alpha_i^\pi, \beta_i^\pi)$  are defined as in Lemma 3.3. Observe that  $\bar{\varphi}_i^\pi \geq \bar{\varphi}_{i+1}^\pi \geq |g|$ ,  $0 \leq i \leq n-1$ .

Let  $\varphi^\pi = \{(g, \bar{\varphi}_i^\pi)\}_{0 \leq i \leq n}$ . Then, it follows from the same arguments as in Lemma 3.3, that :

$$\mathbf{T}_i^{\varphi^\pi}(Y_{t_i}^\pi) = Y_{t_i}^\pi, \quad 0 \leq i \leq n.$$

In particular, this induces similar bounds on  $E_{t_{i-1}}^\pi [Y_{t_i}^\pi]$  by direct computations.

## 7.3 Simulation

As in the non-reflected case, we define the approximation  $\hat{Y}^\pi$  of  $Y^\pi$  by :

$$\begin{aligned} \hat{Y}_1^\pi &= g(X_1^\pi), \\ \check{Y}_{t_{i-1}}^\pi &= \hat{E}_{t_{i-1}}^\pi [Y_{t_i}^\pi] + f\left(t_{i-1}, X_{t_{i-1}}^\pi, \check{Y}_{t_{i-1}}^\pi\right) \Delta_i^\pi \\ \hat{Y}_{t_{i-1}}^\pi &= \mathbf{T}_{t_{i-1}}^{\varphi^\pi}(\check{Y}_{t_{i-1}}^\pi) = \left(g(X_{t_{i-1}}^\pi) \vee \check{Y}_{t_{i-1}}^\pi\right) \wedge \bar{\varphi}_{i-1}^\pi(X_{t_{i-1}}^\pi), \quad 1 \leq i \leq n, \end{aligned} \quad (7.8)$$

where  $\hat{E}^\pi$  is some approximation of  $E^\pi$ .

With this construction, the estimation of the regression error of Theorem 4.1 immediately extends to the context of reflected backward SDE's approximation. In particular, we obtain the same  $L^p$  error estimate of the regression approximation as in the non-reflected case :

**Theorem 7.2** *Let  $p \geq 1$  be given. Then, there is a constant  $C > 0$  which only depends on  $(K, p)$  such that :*

$$\left\| \hat{Y}_{t_i}^\pi - Y_{t_i}^\pi \right\|_{L^p} \leq \frac{C}{|\pi|} \max_{0 \leq j \leq n-1} \left\| (\hat{E}_j^\pi - E_j^\pi) \left[ \hat{Y}_{t_{j+1}}^\pi \right] \right\|_{L^p}$$

for all  $0 \leq i \leq n$ .

From this theorem, we can now deduce an estimate of the  $L^p$  error  $\hat{Y}^\pi - Y^\pi$  in the case where  $\hat{E}^\pi$  is defined as in Section 6.2. Let  $\varphi \in \mathcal{L}$  satisfying

$$\sum_{k=0}^d \sum_{I \in \mathcal{J}_k} \int_{\mathbb{R}^d} |u|^{4p+2} \partial_I \varphi(u)^2 du < \infty,$$

for some  $p \geq 1$ . Consider the approximation  $\hat{Y}^\pi$  obtained by the above simulation scheme, where  $\hat{E}^\pi$  is defined as in Section 6.2 with normalized localizing function  $\varphi^\pi(x) = \varphi(|\pi|^{-1/2}x)$ . Then, we have the following  $L^p$  estimate of the error due to the regression approximation :

$$\limsup_{|\pi| \rightarrow 0} \max_{0 \leq i \leq n} |\pi|^{p+d/4} N^{1/2} \left\| \hat{Y}_{t_i}^\pi - Y_{t_i}^\pi \right\|_{L^p}^p < \infty.$$

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