

Maximum Maximum of Martingales given Marginals

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Abstract

We consider the problem of superhedging under volatility uncertainty for an investor allowed to dynamically trade the underlying asset and statically trade European call options for all possible strikes and finitely-many maturities. We present a general duality result which converts this problem into a min-max calculus of variations problem where the Lagrange multipliers correspond to the static part of the hedge. Following Galichon, Henry-Labordère and Touzi [19], we apply stochastic control methods to solve it explicitly for Lookback options with a non-decreasing payoff function. The first step of our solution recovers the extended optimal properties of the Azéma-Yor solution of the Skorokhod embedding problem obtained by Hobson and Klimmek [22] (under slightly different conditions). The two marginal case corresponds to the work of Brown, Hobson and Rogers [9].

The robust superhedging cost is complemented by (simple) dynamic trading and leads to a class of semi-static trading strategies. The superhedging property then reduces to a functional inequality which we verify independently. The optimality follows from existence of a model which achieves equality which is obtained in Oblój and Spoida [33].

Key words: Optimal control, robust pricing and hedging, volatility uncertainty, optimal transportation, pathwise inequalities, lookback option.

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1 Introduction

The classical framework underpinning much of the quantitative finance starts by postulating a probabilistic model for future prices of risky assets. The models, from their origins in Samuleson [36], Merton [28] and Black and Scholes [6] to the present day, have seen a remarkable evolution and ever increasing sophistication. Nevertheless, the essence remained the same: no arbitrage ensures that under an equivalent probability measure (discounted) asset prices are martingales and that the fair price of a future payoff is given by the capital needed to replicate that payoff. That capital is then computed as the (risk-neutral) expectation of the payoff.

The classical framework has been very influential both in terms of its impact on academic research as well as on the financial industry. However, as every modelling framework, it has its important limitations. The fundamental criticism is related to the distinction between *risk* and *uncertainty* dating back to Knight [25]. The classical approach starts by postulating a stochastic universe $(\Omega, \mathbb{F}, \mathbb{P})$ which is meant to model a financial environment and capture its riskiness. What it fails to capture however is the uncertainty in the choice of \mathbb{P} , i.e. the possibility that the model itself is wrong, also called *the Knightian uncertainty*. To account for model uncertainty it is natural to consider simultaneously a whole family $\{\mathbb{P}_\alpha : \alpha \in \mathcal{A}\}$ of probability measures. When all \mathbb{P}_α are absolutely continuous w.r.t. one reference measure \mathbb{P} we speak of *drift uncertainty* or *dominated* setting. This has important implications for portfolio choice problems, see Föllmer, Schied and Weber [18], but is not different from an incomplete market setup in terms of option pricing. However the *non-dominated* setup when \mathbb{P}_α may be mutually singular posed new challenges and was investigated starting with Avellaneda et al. [1] and Lyons [26], through Denis and Martini [16] to several recent works e.g. Peng [34], Soner, Touzi and Zhang [37], see also [19] and the references therein.

Naturally as one relaxes the classical setup one has to abandon its precision: under model uncertainty we do not try to have a unique price but rather to obtain an interval of no-arbitrage prices. Its bounds are given by seller's and buyer's "safe" prices, the superreplication and the subreplication prices, which can be enforced by trading strategies which work in all considered models. These bounds can be made more efficient by enlarging the set of hedging instruments. Indeed, in the financial markets certain derivatives on the underlying we try to model are liquid and have well defined market prices. Without one fixed model, these options can be included in traded asset without creating an arbitrage opportunity. By allowing to trade dynamically in the underlying and statically (today) in a range of options one hopes to have a more efficient approach with smaller intervals of possible no-arbitrage prices. This constitutes the basis of the so-called *robust approach to pricing and hedging*.

We contribute to this literature. Our objective here is to derive in an explicit form the superhedging cost of a Lookback option given that the underlying asset is available for

frictionless continuous-time trading, and that European options for all strikes are available for trading for a finite set of maturities. In a zero interest rate financial market, it essentially follows from the no-arbitrage condition, as observed by Breeden and Litzenberger [8], that these trading possibilities restrict the underlying asset price process to be a martingale with given marginals. Since a martingale can be written as a time changed Brownian motion, and the maximum of the processes is not altered by a time change, the one-marginal constraint version of this problem can be converted into the framework of the Skorokhod embedding problem (SEP). This observation is the starting point of the seminal paper by Hobson [20] who exploited the already known optimality result of the Azéma-Yor solution to the SEP and, more importantly, provided an explicit static superhedging strategy. This methodology was subsequently used to derive robust prices and super/sub-hedging strategies for barrier options in Brown, Hobson and Rogers [10], for options on local time in Cox, Hobson and Oblój [12], for double barrier options in Cox and Oblój [13, 14] and for options on variance in Cox and Wang [15], see Oblój [32] and Hobson [21] for more details.

The above works focused on finding explicitly robust prices and hedges for an option maturing at T given market prices of call/put options co-maturing at T . For lookback options, an extension to the case where prices at a further intermediate maturity are given can be deduced from Brown, Hobson and Rogers [9]. More recently, Hobson and Neuberger [23] treated forward starting straddle also using option prices at two maturities. Otherwise, and excluding the trivial cases when intermediate laws have no constraining effect (see e.g. the iterated Azéma-Yor setting in Madan and Yor [27]), we are not aware of any explicit robust pricing/hedging results when prices of call options for several maturities are given. The most likely reason for this is that the SEP-based methodology pioneered in Hobson [20] starts with a good guess for the superhedge/embedding and these become much more difficult when more marginals are involved.

Our approach is to exploit a duality transformation which converts our problem into a martingale transportation problem: maximize the expected coupling defined by the payoff so as to transport the Dirac measure along the given distributions μ_1, \dots, μ_n by means of a continuous-time process restricted to be a martingale. This approach was simultaneously suggested by [4] in the discrete-time case, and [19] in continuous-time. We refer to Bonnans and Tan [7] for a numerical approximation in the context of variance options, and Tan and Touzi [39] for a general version of the optimal transportation problem under controlled dynamics.

Our general duality result converts the original problem into a min-max calculus of variations problem where the Lagrange multipliers encode the intermediate marginal constraints. An important financial interpretation is that the multiplier represent the optimal static position in Vanilla options so as to reduce the risk induced by the derivative security. Following Galichon, Henry-Labordère and Touzi [19], we apply stochastic control methods to solve the

new problem explicitly. The first step of our solution recovers the extended optimal properties of the Azéma-Yor solution of the Skorokhod embedding problem obtained by Hobson and Klimmek [22] (under slightly different conditions). The two marginal case corresponds to the work of Brown, Hobson and Rogers [9]. However the stochastic control only allows us to prove an upper bound on the superreplication price. To show that the bound is optimal we need to construct a model which fits the given marginals and attains the bound. To do this we revert to the SEP methodology.

Equipped with a candidate for the static position in the optimal hedge we are able to guess the corresponding dynamic counterpart and obtain a class of semi-static trading strategies. The superhedging property then reduces to a functional inequality which we verify independently. The optimality then follows from existence of a model which achieves equality and which is derived from SEP results obtained in Obłój and Spoida [33].

The paper is organized as follows. Section 2, provides the precise mathematical formulation of the problem and states the main pricing/hedging duality result for arbitrary measurable claims under n-marginal constraints. It also discusses the link with martingale optimal transport. Our main result is given in Section 3. Pathwise arguments, including the superreplicating strategy, provide a first self-contained proof of the main theorem and are reported in Section 4. The stochastic control approach which allowed us to guess the correct quantities for the pathwise arguments, is reported in Section 5. Additional arguments for the one marginal case are given in Section 6. A proof of one technical lemma is relegated to the Appendix.

2 Robust superhedging of Lookback options

2.1 Modeling the volatility uncertainty

The probabilistic setting is the same as in [19] and we introduce it briefly. Further, we limit ourselves to a one-dimensional setting which is mostly relevant here.

Let $\Omega_x := \{\omega \in C([0, T], \mathbb{R}^1) : \omega_0 = x\}$ and write $\Omega := \Omega_0$. Consider Ω as the canonical space equipped with the uniform norm $\|\omega\|_\infty := \sup_{0 \leq t \leq T} |\omega_t|$, B the canonical process, \mathbb{P}_0 the Wiener measure, $\mathbb{F} := \{\mathcal{F}_t\}_{0 \leq t \leq T}$ the filtration generated by B . Throughout the paper, X_0 is some given initial value in \mathbb{R} , and we denote

$$X_t := X_0 + B_t \quad \text{for } t \in [0, T].$$

For all \mathbb{F} -progressively measurable processes σ with values in \mathbb{R}^+ and satisfying $\int_0^T \sigma_s^2 ds < \infty$, \mathbb{P}_0 -a.s., we define the probability measures on (Ω, \mathcal{F}) :

$$\mathbb{P}^\sigma := \mathbb{P}_0 \circ (X^\sigma)^{-1} \quad \text{where } X_t^\sigma := X_0 + \int_0^t \sigma_r dB_r, \quad t \in [0, T], \quad \mathbb{P}_0 - \text{a.s.}$$

so that X is a \mathbb{P}^σ -local martingale. \mathcal{P}_S denotes the collection of all such probability measures on (Ω, \mathcal{F}) . The quadratic variation process $\langle X \rangle = \langle B \rangle$ is universally defined and takes values in the set of all non-decreasing continuous functions with $\langle B \rangle_0 = 0$. Moreover, for any $\mathbb{P}^\sigma \in \mathcal{P}_S$, $\langle B \rangle_t$ is absolutely continuous with respect to the Lebesgue measure.

In this section, we shall consider a convenient subset $\mathcal{P} \subset \mathcal{P}_S$, satisfying some technical conditions. For all $\mathbb{P} \in \mathcal{P}$, we think of $(\Omega, \mathcal{F}_T, \mathbb{F}, \mathbb{P})$ as a possible model for our financial market, where (X_t) denotes the forward price of the underlying, i.e. we use the discounted units and the money market account is just constant.

The coordinate process stands for the price process of an underlying security and we focus on the situation when prices of liquidly traded options allow to back out the (risk-neutral) distribution of the underlying security, as observed by Breeden and Litzenberger [8]. To this end we shall focus on the probability measures in $\mathbb{P} \in \mathcal{P}$ which satisfy the following requirement¹

$$\{X_t : t \leq T\} \text{ is a } \mathbb{P} \text{ - uniformly integrable martingale.} \quad (2.1)$$

For all $\mathbb{P} \in \mathcal{P}$, we denote by $\mathbb{H}^0(\mathbb{P})$ the collection of all (\mathbb{P}, \mathbb{F}) -progressively measurable processes, and

$$\mathbb{H}_{\text{loc}}^2(\mathbb{P}) := \left\{ H \in \mathbb{H}^0(\mathbb{P}) : \int_0^T |H_t|^2 d\langle B \rangle_t < \infty, \mathbb{P} \text{ - a.s.} \right\} .$$

Finally, throughout the paper, all functions are implicitly taken to be Borel measurable.

2.2 Robust super-hedging by trading the underlying

We consider the robust superhedging problem of some derivative security defined by the payoff $\xi : \Omega_{X_0} \rightarrow \mathbb{R}$ at some given maturity $T > 0$. We assume that ξ is \mathcal{F}_T -measurable.

Under the self-financing condition, for any portfolio process H , the portfolio value process

$$Y_t^H := Y_0 + \int_0^t H_s \cdot dB_s, \quad t \in [0, T], \quad (2.2)$$

is well-defined \mathbb{P} -a.s., whenever $H \in \mathbb{H}_{\text{loc}}^2(\mathbb{P})$, for every $\mathbb{P} \in \mathcal{P}$. This stochastic integral should be rather denoted $Y_t^{H^\mathbb{P}}$ to emphasize its dependence on \mathbb{P} , see however Nutz [29].

Let ξ be an \mathcal{F}_T -measurable random variable. We introduce the subset of martingale measures:

$$\mathcal{P}(\xi) := \{ \mathbb{P} \in \mathcal{P} : \mathbb{E}^\mathbb{P}[\xi^-] < \infty \}.$$

¹We thus rule out cases when the coordinate process is a strict local martingale, which may be of interest in modelling financial bubbles, see e.g. Cox and Hobson [11], Jarrow, Protter and Shimbo [24].

The reason for restricting to this class of models is that, under the condition that $\mathbb{E}^{\mathbb{P}}[\xi^+] < \infty$, the hedging cost of ξ under \mathbb{P} is expected to be $-\infty$ whenever $\mathbb{E}^{\mathbb{P}}[\xi^-] = \infty$. As usual, in order to avoid doubling strategies, we introduce the set of admissible portfolios:

$$\mathcal{H}(\xi) := \{H : H \in \mathbb{H}_{\text{loc}}^2 \text{ and } Y^H \text{ is a } \mathbb{P} - \text{supermartingale for all } \mathbb{P} \in \mathcal{P}(\xi)\}.$$

The robust superhedging problem is defined by:

$$U^0(\xi) := \inf \{Y_0 : \exists H \in \mathcal{H}(\xi), Y_1^H \geq \xi, \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}(\xi)\}. \quad (2.3)$$

Theorem 2.1 in [19] gives a dual representation of $U^0(\xi)$ for an arbitrary payoff ξ satisfying some uniform continuity assumptions. More recently, Neufeld and Nutz [30] relaxed the uniform continuity condition, allowing for a larger class of random variables including measurable ones. The following extension of [30], reported in [35], is better suited to our context:

Theorem 2.1 *Assume $\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi^+] < \infty$. Then $U^0(\xi) = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi]$. Moreover, existence holds for the robust superhedging problem $U^0(\xi)$, whenever $U^0(\xi) < \infty$.*

2.3 Robust superhedging with additional trading of Vanillas

Let n be some positive integer, $0 = t_0 < \dots < t_n = T$ be some partition of the interval $[0, T]$. In addition to the continuous-time trading of the primitive securities, we assume that the investor can take static positions in European call or put options with all possible strikes and maturities $t_1 < \dots < t_n$. The market price of the European call option with strike $K \in \mathbb{R}$ and maturity t_i is denoted

$$c_i(K), \quad i = 1, \dots, n, \quad \text{and we denote } c_0(K) := (X_0 - K)^+.$$

Consider a model $\mathbb{P} \in \mathcal{P}$ which is calibrated to the market, i.e. $\mathbb{E}^{\mathbb{P}}[(X_{t_i} - K)^+] = c_i(K)$ for all $1 \leq i \leq n$ and $K \in \mathbb{R}$. Differentiating in K , as observed by Breeden and Litzenberger [8], we see that

$$\mathbb{P}(X_{t_i} > K) = -c'_i(K+) =: \mu_i((K, \infty))$$

is uniquely specified by the market prices and is independent of \mathbb{P} . Let $\mu = (\mu_1, \dots, \mu_n)$ and

$$\mathcal{P}(\mu) := \{\mathbb{P} \in \mathcal{P} : X_{t_i} \sim \mu_i, \quad 1 \leq i \leq n\}$$

be the set of calibrated market models. As X is a \mathbb{P} -martingale, the necessary and sufficient condition for $\mathcal{P}(\mu) \neq \emptyset$ is that the μ_i 's are nondecreasing in convex order or, equivalently,

$$\int |x| d\mu_i(x) < \infty, \quad \int x d\mu_i(x) = X_0, \quad \text{and } c_{i-1} \leq c_i \quad \text{for all } 1 \leq i \leq n, \quad (2.4)$$

where now $c_i(K) = \int_K^\infty (x - K) d\mu_i(x)$. This is a direct extension of the Strassen Theorem [38]. The necessity follows from Jensen's inequality. For sufficiency, an explicit model can be constructed using techniques of Skorokhod embeddings, see Obłój [31]. In consequence, the t_j -maturity European derivative defined by the payoff $\lambda_i(X_{t_i})$ has an un-ambiguous market price

$$\mu_i(\lambda_i) := \int \lambda_i d\mu_i = \mathbb{E}^\mathbb{P}[\lambda(X_{t_i})], \quad \text{for all } \mathbb{P} \in \mathcal{P}(\mu).$$

The condition $\mathcal{P}(\mu) \neq \emptyset$ embodies the fact that the market prices observed today do not admit arbitrage. By this we mean that there exists a classical model in mathematical finance which admits no arbitrage (no free lunch with vanishing risk) and reprices the call options through risk neutral expectation. For that reason we sometimes refer to (2.4) as the *no-arbitrage condition*.

Remark 2.1 For the purpose of the present financial application, we could restrict the measures μ_i to have support in \mathbb{R}_+ and $\mathbb{P} \in \mathcal{P}$ to be such that $X_t \geq 0$ \mathbb{P} -a.s. Note however that this is easily achieved: it suffices to assume that $X_0 > 0$ and $c_n(K) = X_0 - K$ for $K \leq 0$. Then $\mu_n((K, \infty)) = 1$, $K < 0$, and hence $\mu_n([0, \infty)) = 1$. Then for any $\mathbb{P} \in \mathcal{P}(\mu)$ we have $X_t = \mathbb{E}^\mathbb{P}[X_T | \mathcal{F}_t] \geq 0$ \mathbb{P} -a.s. for $t \in [0, T]$. In particular, $\mu_i([0, \infty)) = \mathbb{P}(X_{t_i} \geq 0) = 1$.

As it will be made clear in our subsequent Proposition 2.1, the function λ_i will play the role of a Lagrange multiplier for the constraint $X_{t_i} \sim \mu_i$, $i = 1, \dots, n$.

We denote $\mathbf{t} := (t_1, \dots, t_n)$, $\lambda = (\lambda_1, \dots, \lambda_n)$,

$$\mu(\lambda) := \sum_{i=1}^n \mu_i(\lambda_i), \quad \lambda(\mathbf{x}_\mathbf{t}) := \sum_{i=1}^n \lambda_i(\mathbf{x}_{t_i}), \quad \text{and } \xi^\lambda(\mathbf{x}, \mathbf{t}) := \xi(\mathbf{x}) - \lambda(\mathbf{x}_\mathbf{t}) \quad (2.5)$$

for $\mathbf{x} \in C([0, T])$. The set of Vanilla payoffs which may be used by the hedger are naturally taken in the set

$$\Lambda_n^\mu(\xi) := \left\{ \lambda \in \Lambda_n^\mu : \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^\mathbb{P}[(\xi^\lambda)^+] < \infty \right\}, \quad \text{where } \Lambda_n^\mu := \left\{ \lambda : \lambda_i \in \mathbb{L}^1(\mu_i), 1 \leq i \leq n \right\}. \quad (2.6)$$

The superreplication upper bound is defined by:

$$U_n^\mu(\xi) := \inf \left\{ Y_0 : \exists \lambda \in \Lambda_n^\mu(\xi) \text{ and } H \in \mathcal{H}(\xi^\lambda), \bar{Y}_T^{H, \lambda} \geq \xi, \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}(\xi^\lambda) \right\}, \quad (2.7)$$

where $\bar{Y}^{H, \lambda}$ denotes the portfolio value of a self-financing strategy with continuous trading H in the primitive securities, and static trading λ_i in the t_i -maturity European calls with all strikes:

$$\bar{Y}_T^{H, \lambda} := Y_T^H - \mu(\lambda) + \lambda(X_\mathbf{t}), \quad (2.8)$$

indicating that the investor has the possibility of buying at time 0 any derivative security with payoff $\lambda_i(X_{t_i})$ for the price $\mu_i(\lambda_i)$. $U_n^\mu(\xi)$ is an upper bound on the price of ξ necessary for absence of strong (model-independent) arbitrage opportunities: selling ξ at a higher price, the hedger could set up a portfolio with a negative initial cost and a non-negative payoff under any market scenario.

Similar to [19], the next result is a direct consequence of the robust superhedging dual formulation of Theorem 2.1.

Proposition 2.1 *Assume that $\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}[\xi^+] < \infty$, and let the family of probability measures μ_i , $i = 1, \dots, n$ be as in (2.4). Then:*

$$U_n^\mu(\xi) = \inf_{\lambda \in \Lambda_n^\mu(\xi)} \sup_{\mathbb{P} \in \mathcal{P}} \{ \mu(\lambda) + \mathbb{E}^\mathbb{P}[\xi - \lambda(X_t)] \}.$$

Our objective in the subsequent sections is to use the last dual formulation in order to obtain a closed form expression for the above upper bound in the following special cases:

- Lookback option $\xi := g(X_T, X_T^*)$, with $X_T^* := \max_{t \leq T} X_t$, under one-marginal constraint $n = 1$, and some “monotonicity” condition of $m \mapsto g(x, m)$;
- Lookback option $\xi := \phi(X_T^*)$, for some nondecreasing function ϕ , under multiple marginal constraints.

The one-marginal result is reported in Section 6, and has been recently established by Hobson and Klimmek [22] under slightly different assumptions; therefore it must be viewed as an alternative approach to that of [22]. In contrast, the multiple-marginal result of Sections 4 and 5 is new to the literature, and generalizes the earlier contribution of Brown, Hobson and Rogers [9] in the two-marginal case. It also encompasses the trivial case where one can simply iterate the one-dimensional Azéma-Yor [2] construction, see also Madan and Yor [27].

2.4 Optimal transportation and Skorokhod embedding problem

In this short section we discuss the connection of our problem to optimal transportation theory, on one the hand, and to the Skorokhod embedding problem, on the other hand.

First, by formally inverting the inf-sup in the dual formulation of Proposition 2.1, we see that $U_n^\mu(\xi)$ is related to the optimization problem:

$$\sup_{\mathbb{P} \in \mathcal{P}(\mu)} \mathbb{E}^\mathbb{P}[\xi] \tag{2.9}$$

which falls in the recently introduced class of optimal transportation problems under controlled stochastic dynamics, see [4, 19, 39]. In words, the above problem consists in maximizing the expected transportation cost of the Dirac measure $\delta_{\{X_0\}}$ along the given marginals

μ_1, \dots, μ_n with transportation scheme constrained to a specific subclass of martingales. The cost of transportation in our context is defined by the path-dependent payoff $\xi(\mathbf{x})$.

The validity of the equality between the value function in (2.9) and our problem $U_n^\mu(\xi)$ was established recently by Dolinsky and Soner [17] for Lipschitz payoff function $\omega \mapsto \xi(\omega)$ and $n = 1$. The corresponding duality result in the discrete time framework was obtained in [4].

Note that if we can find a trading strategy $\bar{Y}_T^{H,\lambda}$ as in (2.8) which superreplicates ξ : $\bar{Y}_T^{H,\lambda} \geq \xi$ \mathbb{P} -a.s. for all $\mathbb{P} \in \mathcal{P}(\xi^\lambda)$ and a $\mathbb{P}^{\max} \in \mathcal{P}(\mu) \cap \mathcal{P}(\xi^\lambda)$ such that $\mathbb{E}^{\mathbb{P}^{\max}}[\xi] = Y_0$ then trivially

$$Y_0 \leq \sup_{\mathbb{P} \in \mathcal{P}(\mu)} \mathbb{E}^{\mathbb{P}}[\xi] \leq U_n^\mu(\xi) \leq Y_0$$

and it follows that we have equalities throughout. This line of attack has been at the heart of the approach to robust pricing and hedging based on the Skorokhod embedding problem, as in [20, 10, 13, 14, 15]. It relies crucially on the ability to make a correct guess for the cheapest superhedge $\bar{Y}_T^{H,\lambda}$. This becomes increasingly difficult when one considers information about prices at several maturities, $n > 1$. In this paper, we follow the above methodology in Section 4 to provide a first proof of our main result, Theorem 3.1. Sections 5–6 then provide a second proof based on stochastic control methods. The latter is longer and more involved than the former however it was in fact necessary in order to guess the right quantities for the former.

We now specialize the discussion to the case of a Lookback option $\xi = G(X_{\mathbf{t}}, X_T^*)$, for some payoff function G . By the Dambis-Dubins-Schwartz time change theorem, we may re-write the problem (2.9) as a multiple stopping problem (see Proposition 3.1 in [19]):

$$\sup_{(\tau_1, \dots, \tau_n) \in \mathcal{T}(\mu)} \mathbb{E}^{\mathbb{P}_0} [G(X_{\tau_1}, \dots, X_{\tau_n}, X_{\tau_n}^*)], \quad (2.10)$$

where the $\mathcal{T}(\mu)$ is the set of ordered stopping times $\tau_1 \leq \dots \leq \tau_n < \infty$ \mathbb{P}_0 -a.s. with $X_{\tau_i} \sim_{\mathbb{P}_0} \mu_i$ for all $i = 1, \dots, n$ and $(X_{t \wedge \tau_n})$ being a uniformly integrable martingale. Elements of $\mathcal{T}(\mu)$ are solutions to the iterated (multi-marginal) version of the so-called Skorokhod embedding problem (SEP), cf. [31]. Here, the formulation (2.10) is directly searching for a solution to the SEP which maximizes the criterion defined by the coupling $G(x, m)$. Previous works have focused mainly on single marginal constraint ($n = 1$). The case $G(x, m) = \phi(m)$ for some non-decreasing function ϕ is solved by the so-called Azéma-Yor embedding [2, 3, 20], see also [19] which recovered this result by stochastic-control approach of Section 5. The case $G(x, m)$ was considered recently by Hobson and Klimmek [22], where the optimality of the Azéma-Yor solution of the SEP is shown to be valid under convenient conditions on the function G . This case is also solved in Section 6 of the present paper with our approach, leading to the same results than [22] but under slightly different conditions.

The case $G(x_1, \dots, x_n, m) = \phi(m)$ for some nonincreasing function ϕ is also trivially solved by $\tau^{AY}(\mu_n)$ in the special case when the single marginal solutions are naturally ordered: $\tau^{AY}(\mu_i) \leq \tau^{AY}(\mu_{i+1})$. This is called the *increasing mean residual value property* by Madan and Yor [27] who established in particular strong Markov property of the resulting time-changed process. The case of arbitrary measures which satisfy (2.4) for $n = 2$ was solved in Brown, Hobson and Rogers [9]. In this paper we consider $n \in \mathbb{N}$.

3 Main result

Consider the Lookback option defined by the payoff

$$\xi = \phi(X_T^*),$$

for some nondecreasing function ϕ .

The key ingredient for the solution of the present n -marginals Skorokhod embedding problem turns out to be the following minimization problem:

$$C(m) := \min_{\zeta_1 \leq \dots \leq \zeta_n \leq m} \sum_{i=1}^n \left(\frac{c_i(\zeta_i)}{m - \zeta_i} - \frac{c_i(\zeta_{i+1})}{m - \zeta_{i+1}} \mathbf{1}_{\{i < n\}} \right) \quad \text{for all } m \geq X_0, \quad (3.1)$$

where we understand the value in (3.1) for $\zeta_k < \zeta_{k+1} = \dots = \zeta_n = m$ as limit of the value $\zeta_{k+1} = \dots = \zeta_n = \zeta \rightarrow m$ which is clearly either $+\infty$ or is well defined in terms of the derivative of the call function at m .

For a fixed m , the minimum above is attained by some $-\infty < \zeta_1^* \leq \dots \leq \zeta_n^* \leq m$. To see this, we first observe that, by taking $\zeta_1 = \dots = \zeta_n$ it follows that (3.1) simplifies to $\min_{\zeta_n \leq m} \frac{c_n(\zeta_n)}{m - \zeta_n}$ which is the slope of the tangent to c_n intersecting the x -axis in m and is strictly smaller than 1. Then $C(m) < 1$. On the other hand, let $(\zeta_1^k, \dots, \zeta_n^k)$ be a sequence which attains minimum. Notice that $\frac{c_1(\zeta_1)}{m - \zeta_1} \rightarrow 1$ as $\zeta_1 \rightarrow -\infty$ and the remaining terms in the sum in (3.1) are non-negative. Then, if we can extract a subsequence of $(\zeta_1^k)_k$ converging to $-\infty$, we obtain by sending $k \rightarrow \infty$ along such a subsequence that $C(m) \geq 1$, a contradiction. Hence ζ_1^k is bounded from below, implying that $-\infty < \inf_j \zeta_1^j \leq \zeta_1^k \leq \dots \leq \zeta_n^k \leq m$, thus reducing (3.1) to a minimisation problem of a continuous function in a compact subset of \mathbb{R}^n .

Theorem 3.1 *Let ϕ be a non-decreasing function and assume that the no-arbitrage condition (2.4) holds. Let $\zeta_1^*(m), \dots, \zeta_n^*(m)$ be a solution to (3.1) for a fixed m . Then,*

$$U_n^\mu(\xi) \leq U := \phi(X_0) + \sum_{i=1}^n \int_{X_0}^{\infty} \left(\frac{c_i(\zeta_i^*(m))}{m - \zeta_i^*(m)} - \frac{c_i(\zeta_{i+1}^*(m))}{m - \zeta_{i+1}^*(m)} \mathbf{1}_{\{i < n\}} \right) \phi'(m) dm. \quad (3.2)$$

Moreover, there exist $\lambda \in \Lambda_n^\mu$, explicitly given in (4.4), and trading strategies $H = H^{\text{stock}} + H^{\text{fwd}} \in \mathcal{H}(\xi^\lambda)$, explicitly given in (4.5)-(4.6), such that $U = \phi(X_0) + \mu(\lambda)$ and

$$U + \lambda(X_t) - \mu(\lambda) + \sum_{i=1}^n H_{t_{i-1}}(X_{t_i} - X_{t_{i-1}}) \geq \phi(X_T^*) \quad \text{for all } \omega \in \Omega_{X_0}. \quad (3.3)$$

Assume further that μ_1, \dots, μ_n satisfy Assumption A in [33]. Then, equality holds in (3.2).

As explained before, we shall provide two alternative proofs of this result. The first one, reported in Section 4, consists in a short pathwise argument, based on a guess of the form of the optimal superhedging strategy for a simple one-touch barrier option, combined with the Skorokhod embedding results of [33].

The second proof, reported in Section 5, is more involved, and builds on the stochastic control approach of [19] which develops a systematic way of solving superhedging problems under marginals constraints. As in [19], the stochastic control tools provide the upper bound, and the optimality of the bound follows from the Skorokhod embedding results of [33]. It is our intention to demonstrate that the arguments we give in these sections prove the upper bound in Theorem 3.1 and hence emphasis will be put on rigour.

A further reason for reporting the stochastic control proof in detail is that in fact it was a starting point for both the pathwise arguments in Section 4 as well as for the construction of the embedding in [33]. More precisely, it allowed us to identify the static part λ of the optimal hedge. It was then possible to guess the dynamic part of the super-hedging strategy following the intuition of two-marginal ($n = 2$) case in [9] to trade only at the intermediate maturities and when the barrier is hit. The embedding construction was tailored as to provide a model in which the optimal super-hedge is in fact a perfect hedge, see Section 4 for details.

Remark 3.1 It follows from [33, Section 4] that if their Assumption A fails then the bound (3.1) is not necessarily optimal.

4 The pathwise approach

4.1 A trajectorial inequality

The following trajectorial inequality is the building block for robust superhedging of the Lookback option in the n -marginal case.

Proposition 4.1 *Let ω be a càdlàg path and denote $\omega_t^* := \sup_{0 \leq s \leq t} \omega_s$. Then, for $m > \omega_0$ and $\zeta_1 \leq \dots \leq \zeta_n < m$:*

$$\begin{aligned} \mathbf{1}_{\{\omega_{t_n}^* \geq m\}} &\leq \sum_{i=1}^n \left(\frac{(\omega_{t_i} - \zeta_i)^+}{m - \zeta_i} + \mathbf{1}_{\{\omega_{t_{i-1}}^* < m \leq \omega_{t_i}^*\}} \frac{m - \omega_{t_i}}{m - \zeta_i} \right) \\ &\quad - \sum_{i=1}^{n-1} \left(\frac{(\omega_{t_i} - \zeta_{i+1})^+}{m - \zeta_{i+1}} + \mathbf{1}_{\{m \leq \omega_{t_i}^*, \zeta_{i+1} \leq \omega_{t_i}\}} \frac{\omega_{t_{i+1}} - \omega_{t_i}}{m - \zeta_{i+1}} \right). \end{aligned} \quad (4.1)$$

Proof Denote by A_n the right hand-side of (4.1), and let us prove the claim by induction. First, in the case $n = 1$, the required inequality is the same as that of [9, Lemma 2.1]:

$$A_1 = \frac{(\omega_{t_1} - \zeta_1)^+ + \mathbf{1}_{\{\omega_{t_0}^* < m \leq \omega_{t_1}^*\}} (m - \omega_{t_1})}{m - \zeta_1} \geq \frac{\omega_{t_1} - \zeta_1 + m - \omega_{t_1}}{m - \zeta_1} \mathbf{1}_{\{m \leq \omega_{t_1}^*\}} \geq \mathbf{1}_{\{m \leq \omega_{t_1}^*\}}.$$

We next assume that $A_{n-1} \geq \mathbf{1}_{\{\omega_{t_{n-1}}^* \geq m\}}$ for some $n \geq 2$, and show that $A_n \geq \mathbf{1}_{\{\omega_{t_n}^* \geq m\}}$. We consider two cases.

Case 1: $\omega_{t_{n-1}}^* \geq m$. Then $\omega_{t_n}^* \geq m$, and it follows from the induction hypothesis that $1 = \mathbf{1}_{\{\omega_{t_n}^* \geq m\}} = \mathbf{1}_{\{\omega_{t_{n-1}}^* \geq m\}} \leq A_{n-1}$. In order to see that $A_{n-1} \leq A_n$, we compute directly that, in the present case,

$$A_n - A_{n-1} = \frac{\omega_{t_n} - \zeta_n}{m - \zeta_n} \left(\mathbf{1}_{\{\omega_{t_n} \geq \zeta_n\}} - \mathbf{1}_{\{\omega_{t_{n-1}} \geq \zeta_n\}} \right) \geq 0. \quad (4.2)$$

Case 2: $\omega_{t_{n-1}}^* < m$. As (ω_t^*) is non-decreasing, it follows that $\omega_{t_i}^* < m$ for all $i \leq n-1$. With a direct computation we obtain:

$$A_n = A_n^0 + \frac{(\omega_{t_n} - \zeta_n)^+}{m - \zeta_n} + \mathbf{1}_{\{m \leq \omega_{t_n}^*\}} \frac{m - \omega_{t_n}}{m - \zeta_n}, \text{ where } A_n^0 := \sum_{i=1}^{n-1} \left(\frac{(\omega_{t_i} - \zeta_i)^+}{m - \zeta_i} - \frac{(\omega_{t_i} - \zeta_{i+1})^+}{m - \zeta_{i+1}} \right).$$

Since $m > \omega_{t_i}^* \geq \omega_{t_i}$ for $i \leq n-1$, the functions $\zeta \mapsto (\omega_{t_i} - \zeta)^+ / (m - \zeta)$ are non-increasing. This implies that $A_n^0 \geq 0$ by the fact that $\zeta_i \leq \zeta_{i+1}$ for all $i \leq n$. Then:

$$\begin{aligned} A_n &\geq \frac{(\omega_{t_n} - \zeta_n)^+ + \mathbf{1}_{\{m \leq \omega_{t_n}^*\}} (m - \omega_{t_n})}{m - \zeta_n} \geq \frac{(\omega_{t_n} - \zeta_n)^+ + m - \omega_{t_n}}{m - \zeta_n} \mathbf{1}_{\{m \leq \omega_{t_n}^*\}} \\ &\geq \frac{\omega_{t_n} - \zeta_n + m - \omega_{t_n}}{m - \zeta_n} \mathbf{1}_{\{m \leq \omega_{t_n}^*\}} = \mathbf{1}_{\{m \leq \omega_{t_n}^*\}}. \end{aligned} \quad (4.3)$$

□

4.2 Financial interpretation

We develop now a financial interpretation of the right hand side of (4.1) as a (pathwise) superhedging strategy for a simple knock-in digital barrier option with payoff $\xi = \mathbf{1}_{\{X_T^* \geq m\}}$.

It consists of three elements: a static position in call options, a forward transaction (with the shortest available maturity) when the barrier m is hit and rebalancing thereafter at times t_i . More precisely:

(i) *Static position in calls:*

$$\lambda_\zeta(X_{\mathbf{t}}) := \sum_{i=1}^n \left(\frac{(X_{t_i} - \zeta_i)^+}{m - \zeta_i} - \frac{(X_{t_i} - \zeta_{i+1})^+}{m - \zeta_{i+1}} \right).$$

For $1 \leq i < n$, we hold a portfolio long and short calls with maturity t_i and strikes ζ_i and ζ_{i+1} respectively. This yields a “tent like” payoff which becomes negative only if the underlying exceeds level m . Note that by setting $\zeta_i = \zeta_{i+1}$ we may avoid trading the t_i -maturity calls. For maturity t_n we are only long in a call with strike ζ_n .

(ii) *Forward transaction if the barrier m is hit:* $\mathbf{1}_{\{X_{t_{i-1}}^* < m \leq X_{t_i}^*\}} \frac{m - X_{t_i}}{m - \zeta_i}$

At the moment when the barrier m is hit, say between maturities t_{i^*-1} and t_{i^*} , we enter into forward contracts with maturity t_{i^*} .

Note that the long call position with maturity t_{i^*} together with the forward then superhedge the knock-in digital barrier option, cf. (4.3). This resembles the robust semi-static hedge in the one-marginal case, cf. [9, Lemma 2.4]. All the “tent like” payoffs up to maturity t_{i^*-1} are non-negative.

(iii) *Rebalancing of portfolio to hedge calendar spreads:* $-\sum_{i=1}^{n-1} \mathbf{1}_{\{m \leq X_{t_i}^*, \zeta_{i+1} \leq X_{t_i}\}} \frac{X_{t_{i+1}} - X_{t_i}}{m - \zeta_{i+1}}$

After the barrier m was hit, we start trading at times t_i in such a way that a potential negative payoff of the calendar spreads $\frac{(X_{t_{i+1}} - \zeta_{i+1})^+}{m - \zeta_{i+1}} - \frac{(X_{t_i} - \zeta_{i+1})^+}{m - \zeta_{i+1}}$, $i^* \leq i \leq n$, is offset, cf. (4.2).

In the above (ii) and (iii) are instances of dynamic trading which is done in a self-financing way. Their combined payoff may be written as $\int_0^T H_s^\zeta dX_s$ for a suitable choice of (simple) integrand H^ζ . Note that here $\xi \geq 0$ so $\mathcal{P}(\xi) = \mathcal{P}$ and $H^\zeta \in \mathcal{H}(\xi^\zeta)$. Let $Y_0 = \mu(\lambda_\zeta) = \sum_{i=1}^n \left(\frac{c_i(\zeta_i)}{m - \zeta_i} - \frac{c_i(\zeta_{i+1})}{m - \zeta_{i+1}} \mathbf{1}_{\{i < n\}} \right)$, which is the initial cost entering into the static position in (i), see (2.5). Then

$$Y_T^\zeta = Y_0 + \int_0^T H_s^\zeta dX_s + \lambda_\zeta(X_{\mathbf{t}}) - \mu(\lambda_\zeta)$$

is an example of a semi-static trading strategy as in (2.8) and the inequality (4.1) now simply says that for any choice of $\zeta_1 \leq \dots \leq \zeta_n < m$, our strategy Y^ζ superreplicates ξ . Our candidate superhedge for ξ is the cheapest among all Y^ζ . Its cost is given by (3.1) and corresponds to minimizers $\zeta_1^*(m), \dots, \zeta_n^*(m)$ of the optimization problem (3.1). To prove that indeed $U_n^\mu(\xi) = \mu(\lambda_{\zeta^*})$ it suffices, as observed in Section 2.4, to find one $\mathbb{P} \in \mathcal{P}(\mu)$ such that $\mathbb{E}^\mathbb{P}[\xi] = \mu(\lambda_{\zeta^*})$. This is done below in Section 4.3 where, under Assumption A in [33] and using the results therein, we actually exhibit \mathbb{P} such that $\xi = Y_T^{\zeta^*}$ \mathbb{P} -a.s. Moreover, \mathbb{P} is independent of m .

We finally extend the above hedging strategy to the context of a general Lookback payoff $\phi(X_T^*)$ for some nondecreasing ϕ . The superhedging property (3.3) is again obtained by integrating both sides of inequality $\mathbf{1}_{\{X_T^* \geq m\}} \leq Y_T^{\zeta^*(m)}$ against ϕ' . The resulting optimal hedging strategy is then characterised by:

$$\lambda_i(x) := \int_{X_0}^{\infty} \left(\frac{(x - \zeta_i^*(m))^+}{m - \zeta_i^*(m)} - \frac{(x - \zeta_{i+1}^*(m))^+}{m - \zeta_{i+1}^*(m)} \mathbf{1}_{\{i < n\}} \right) \phi'(m) dm \quad (4.4)$$

$$H_t^{\text{stock}}(\omega) := - \int_{X_0}^{\infty} \frac{\mathbf{1}_{\{\omega_{[t]}^* \geq m, \omega_{[t]} \geq \zeta_{\text{ind}(t)}^*(m)\}}}{m - \zeta_{\text{ind}(t)}^*(m)} \phi'(m) dm \quad (4.5)$$

$$H_t^{\text{fwd}}(\omega) := - \int_{X_0}^{\infty} \frac{\mathbf{1}_{\{\omega_{[t]}^* < m, \omega_t^* \geq m\}}}{m - \zeta_{\text{ind}(t)}^*(m)} \phi'(m) dm. \quad (4.6)$$

with $[t] := \max\{t_i : t_i < t, i < n\}$, $\text{ind}(t) := \min\{i \leq n : t_i > t\}$.

The above integrals are well defined² and from (4.4), (4.5) and (4.6) it is more transparent what kind of integrability conditions one has to impose on ϕ in order to ensure admissible trading strategies and a finite superhedging cost. Note however that as long as $\zeta_i^*(m) \neq m$ we have $H^{\text{stock}} < \infty$, $H^{\text{fwd}} < \infty$ and the stochastic integral $\int (H^{\text{stock}} + H^{\text{fwd}}) dX$ is a simple sum and hence is well defined pathwise.

4.3 First proof of Theorem 3.1

As argued above, to finish the proof of Theorem 3.1 it suffices to exhibit $\mathbb{P}^{\max} \in \mathcal{P}(\mu)$ such that $\mathbf{1}_{\{X_T^* \geq m\}} = Y_T^{\zeta^*}$ \mathbb{P}^{\max} -a.s. for all $m \geq X_0$. The case of general lookback payoff then follows since ϕ is non-decreasing.

Under Assumption A, Oblój and Spoida [33] construct an iterated extension of the Azéma-Yor embedding for μ_1, \dots, μ_n in a Brownian motion. They define functions η_i and stopping times τ_i , $\tau_0 = 0$, $\tau_i := \inf\{t \geq \tau_{i-1} : X_t \leq \eta_i(X_t^*)\}$, $1 \leq i \leq n$, such that $X_{\tau_i} \sim_{\mathbb{P}_0} \mu_i$ and $(X_{t \wedge \tau_n})$ is uniformly integrable. Further, they compute explicitly the distribution of $X_{\tau_i}^*$. We note that in fact $\max_{j \leq i} \eta_j(m) = \zeta_i^*(m)$ for all $m \geq X_0$ and all $1 \leq i \leq n$. Consider a time change of X :

$$Z_t := X_{\tau_i \wedge (\tau_{i-1} \vee \frac{t - t_{i-1}}{t_i - t})}, \quad \text{for } t_{i-1} < t \leq t_i, \quad i = 1, \dots, n$$

²Indeed, the integrands are non-negative. Further, note that under Assumption A in [33] the functions $\zeta_i^*(m)$ are continuous and in particular measurable but the latter can be assumed in all generality. Indeed, for a fixed m the set $F(m)$ of minimisers in (3.1) is closed and for any closed $K \subset \mathbb{R}^n$, $\{m : F(m) \cap K \neq \emptyset\}$ is equal to $\{m : C(m) = C_K(m)\}$ where C_K is given as C in (3.1) but with a further requirement that $(\zeta_1, \dots, \zeta_n) \in K$. Both C and C_K can be obtained through countable pointwise minimisation of continuous functions and hence are measurable and so is $\{m : C(m) = C_K(m)\}$. Existence of a measurable selector for F now follows from Kuratowski and Ryll-Nardzewski measurable selection theorem, see e.g. [40, Thm. 4.1].

with $Z_0 = X_0$ and observe that (Z_t) is a continuous, uniformly integrable, martingale on $[0, t_n]$ with $Z_{t_i} = X_{\tau_i} \sim_{\mathbb{P}_0} \mu_i$. In consequence, the distribution of Z , $\mathbb{P}^{\max} := \mathbb{P}_0 \circ (Z)^{-1}$, is an element of $\mathcal{P}(\mu)$. By going back to the proof of Proposition 4.1 and inspecting the cases where a strict inequality occurs one shows using the definition of τ_i , see [33, Definition 2.2, Lemma 3.2], that for Z these are of measure zero.

5 The stochastic control approach

We now present the methodology which led us to conjecture (3.1) as the solution to the superhedging cost. Our objective in this section is to derive the upper bound of Theorem 3.1 from the dual formulation of Proposition 2.1. Our first observation is that, from the nondecrease of the payoff function ϕ , it follows from the monotone convergence theorem that:

$$U_n^\mu(\xi) = \inf_{\lambda \in \Lambda_n^\mu(\xi)} \sup_{\mathbb{P} \in \mathcal{P}^*} \{ \mu(\lambda) + \mathbb{E}^\mathbb{P}[\xi - \lambda(X_t)] \}, \quad (5.1)$$

where

$$\mathcal{P}^* := \{ \mathbb{P} \in \mathcal{P} : \mathbb{E}^\mathbb{P}[X_T^*] < \infty \}. \quad (5.2)$$

In the present approach, we assume in addition that

$$\begin{aligned} & \phi \in C^1 \text{ Lipschitz, bounded, } \text{Supp}(\phi') \text{ bounded from above,} \\ \text{and } & \int_{X_0}^\infty \left(\frac{c_i(\zeta_i^*(m))}{m - \zeta_i^*(m)} + \frac{c_i(\zeta_{i+1}^*(m))}{m - \zeta_{i+1}^*(m)} \mathbf{1}_{\{i < n\}} \right) \phi'(m) dm < \infty. \end{aligned} \quad (5.3)$$

We start with an essential ingredient, namely a general one-marginal construction which allows to move from $(n - 1)$ to n marginals.

5.1 The one marginal problem

For an inherited maximum $M_0 \geq X_0$, we introduce the process:

$$M_t := M_0 \vee X_t^* \quad \text{for } t \geq 0.$$

The process (X, M) takes values in the state space $\mathbf{\Delta} := \{(x, m) \in \mathbb{R}^2 : x \leq m\}$. Our interest in this section is on the upper bound on the price of the one-marginal ($n = 1$) Lookback option defined by the payoff

$$\xi = g(X_T, X_T^*) \quad \text{for some } g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}. \quad (5.4)$$

Assumption A Function $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is C^1 in (x, m) , Lipschitz in m uniformly in x , and g_{xx} exists as a measure.

Assumption B The function $x \mapsto \frac{g_m(x, m)}{m-x}$ is non-decreasing.

For a function $\lambda : \mathbb{R} \rightarrow \mathbb{R}$, we denote $g^\lambda := g - \lambda$, and we write simply

$$\Lambda^\mu = \left\{ \lambda \in \mathbb{L}^1(\mu) : \sup_{\mathbb{P} \in \mathcal{P}^*} \mathbb{E}[g^\lambda(X_T, M_T)^+] < \infty \right\} \quad (5.5)$$

for all probability measure $\mu \in M(\mathbb{R})$. Similar to Proposition 3.1 in [19], it follows from the Dambis-Dubins-Schwartz time change theorem that the model-free upper bound can be converted into:

$$U^\mu(\xi) := \inf_{\lambda \in \Lambda^\mu} \sup_{\tau \in \mathcal{T}^*} \left\{ \mu(\lambda) + J(\lambda, \tau) \right\} \quad \text{where} \quad J(\lambda, \tau) := \mathbb{E}^{\mathbb{P}^0} [g^\lambda(X_\tau, X_\tau^*)], \quad (5.6)$$

and \mathcal{T}^* is the collection of all stopping times τ such that

$$\{X_{t \wedge \tau}, t \geq 0\} \text{ is a } \mathbb{P}_0\text{-uniformly integrable martingale with } \mathbb{E}^{\mathbb{P}^0} [X_\tau^*] < \infty. \quad (5.7)$$

Then for every fixed multiplier $\lambda \in \Lambda^\mu$, we are facing the infinite horizon optimal stopping problem

$$u^\lambda(x, m) := \sup_{\tau \in \mathcal{T}^*} \mathbb{E}_{x, m}^{\mathbb{P}^0} [g^\lambda(X_\tau, M_\tau)], \quad (x, m) \in \mathbf{\Delta}, \quad (5.8)$$

where $\mathbb{E}_{x, m}^{\mathbb{P}^0}$ denotes the conditional expectation operator $\mathbb{E}^{\mathbb{P}^0}[\cdot | (X_0, M_0) = (x, m)]$.

Finally, the set Λ^μ of (5.5) translates in the present context to:

$$\Lambda^\mu = \left\{ \lambda \in \mathbb{L}^1(\mu) : \sup_{\tau \in \mathcal{T}^*} \mathbb{E}^{\mathbb{P}^0} [g^\lambda(X_\tau, M_\tau)^+] < \infty \right\}. \quad (5.9)$$

Remark 5.1 The condition $\mathbb{E}^{\mathbb{P}^0} [X_\tau^*] < \infty$ is equivalent to $\mathbb{E}^{\mathbb{P}^0} [X_\tau (\ln X_\tau)^+ \mathbf{1}_{X_\tau > 0}] < \infty$, by Doob's \mathbb{L}^1 -inequality.

The dynamic programming equation corresponding to the optimal stopping problem u^λ defined in (5.8) is:

$$\begin{aligned} \min \{u - g^\lambda, -u_{xx}\} &= 0 \quad \text{for } (x, m) \in \mathbf{\Delta} \\ u_m(m, m) &= 0 \quad \text{for } m \geq 0. \end{aligned} \quad (5.10)$$

It is then natural to introduce a candidate solution for the dynamic programming equation defined by a free boundary $\{x = \psi(m)\}$, for some convenient function ψ :

$$v^\psi(x, m) = g^\lambda(x \wedge \psi(m), m) + (x - \psi(m))^+ g_x^\lambda(\psi(m), m) \quad (5.11)$$

$$= g^\lambda(x, m) - \int_{\psi(m)}^{x \vee \psi(m)} (x - \xi) g_{xx}^\lambda(\xi, m) d\xi, \quad 0 \leq x \leq m, \quad (5.12)$$

i.e. $v^\psi(\cdot, m)$ coincides with the obstacle g^λ before the exercise boundary $\psi(m)$, and satisfies $v_{xx}^\psi(\cdot, m) = 0$ in the continuation region $[\psi(m), m]$. However, the candidate solution needs to satisfy more conditions. Namely $v^\psi(\cdot, m)$ must be above the obstacle, concave in x on $(-\infty, m]$, and it needs to satisfy the Neumann condition in (5.10).

For this reason, our strategy of proof consists in first restricting the minimization in (5.6) to those multipliers λ in the set:

$$\hat{\Lambda}^\mu := \left\{ \lambda \in \Lambda^\mu : v^\psi \text{ concave in } x \text{ and } v^\psi \geq g^\lambda \text{ for some } \psi \in \Psi^\lambda \right\}, \quad (5.13)$$

where the set Ψ^λ is defined in (5.16) below so that our candidate solution v^ψ satisfies the Neumann condition in (5.10). Namely, by formal differentiation of v^ψ , the Neumann condition reduces to the ordinary differential equation (ODE):

$$-\psi' g_{xx}^\lambda(\psi, m) = \gamma(\psi, m) \quad \text{where} \quad \gamma(x, m) := (m - x) \frac{\partial}{\partial x} \left\{ \frac{g_m(x, m)}{m - x} \right\} \quad (5.14)$$

exists a.e. in view of Assumption B. Similar to [19], we need for technical reasons to consider this ODE in the relaxed sense. Since g^λ is concave in x on $(-\infty, \psi(m)]$, the partial second derivative g_{xx}^λ is well-defined as a measure on \mathbb{R} . We then introduce the weak formulation of the ODE (5.14):

$$\begin{aligned} & \psi(m) < m \text{ for all } m \in \mathbb{R}, \\ \text{and } & - \int_{\psi(E)} g_{xx}^\lambda(\cdot, \psi^{-1})(d\xi) = \int_E \gamma(\psi, \cdot)(dm) \text{ for all } E \in \mathcal{B}(\mathbb{R}), \end{aligned} \quad (5.15)$$

where ψ is chosen in its right-continuous version, and is non-decreasing by the concavity of g^λ and the non-negativity of γ implied by Assumption B. We introduce the collection of all relaxed solutions of (5.14):

$$\Psi^\lambda := \left\{ \psi : \mathbb{R} \rightarrow \mathbb{R} \text{ right-continuous and satisfies (5.15)} \right\}. \quad (5.16)$$

Notice that the ODE (5.14), which motivates the relaxation (5.15), does not characterize the free boundary ψ as it is not complemented by any boundary condition.

Remark 5.2 For later use, we observe that (5.15) implies by direct integration that

$$\text{the function } x \mapsto \lambda(x) - \int_{\psi(X_0)}^x \int_{X_0}^{\psi^{-1}(y)} \frac{g_m(\psi(\xi), \xi)}{\xi - \psi(\xi)} d\xi dy - \int_{\psi(X_0)}^x g_x(\xi, \psi^{-1}(\xi)) d\xi \text{ is affine.}$$

Proposition 5.1 *Let Assumptions A and B hold true. Then:*

$$u^\lambda \leq v^\psi \text{ for any } \lambda \in \hat{\Lambda}^\mu \text{ and } \psi \in \Psi^\lambda.$$

Proof Assumption B guarantees that the function ψ that we will be manipulating is non-decreasing and has a well defined right-continuous inverse. We proceed in three steps.

1. We first prove that v^ψ is differentiable in m on the diagonal with

$$v_m^\psi(m, m) = 0 \quad \text{for all } m \in \mathbb{R}. \quad (5.17)$$

Indeed, since $\psi \in \Psi_\lambda$, it follows from Remark 5.2 that

$$\lambda(x) = \alpha_0 + \alpha_1 x + \int_{\psi(X_0)}^x \int_{X_0}^{\psi^{-1}(y)} \frac{g_m(\psi(\xi), \xi)}{\xi - \psi(\xi)} d\xi dy + \int_{\psi(X_0)}^x g_x(\xi, \psi^{-1}(\xi)) d\xi$$

for some constants α_0, α_1 . Plugging this expression into (5.11), we see that for $\psi(m) \leq x \leq m$:

$$\begin{aligned} v^\psi(x, m) &= g(\psi(m), m) - \left(\alpha_1 + \int_{X_0}^m \frac{g_m(\psi(\xi), \xi)}{\xi - \psi(\xi)} d\xi \right) (x - \psi(m)) \\ &\quad - \left(\alpha_0 + \alpha_1 \psi(m) + \int_{\psi(X_0)}^{\psi(m)} \int_{X_0}^{\psi^{-1}(y)} \frac{g_m(\psi(\xi), \xi)}{\xi - \psi(\xi)} d\xi dy + \int_{\psi(X_0)}^{\psi(m)} g_x(\xi, \psi^{-1}(\xi)) d\xi \right) \\ &= g(\psi(m), m) - \alpha_0 - \alpha_1 x - \int_{X_0}^m g_m(\psi(\xi), \xi) \frac{x - \psi(\xi)}{\xi - \psi(\xi)} d\xi - \int_{\psi(X_0)}^{\psi(m)} g_x(\xi, \psi^{-1}(\xi)) d\xi. \end{aligned}$$

Since g is C^1 , (5.17) follows by direct differentiation with respect to m .

2. Let $\tau \in \mathcal{T}^*$ be arbitrary. Clearly, it is sufficient to restrict attention to those $\tau \in \mathcal{T}^*$ such that $g^\lambda(X_\tau, M_\tau) \in \mathbb{L}^1(\mathbb{P}_0)$.

Define the sequence of stopping times $\tau_n := \tau \wedge \inf\{t > 0 : |X_t - x| > n\}$. Since v^ψ is concave, it follows from the Itô-Tanaka formula that:

$$v^\psi(x, m) \geq v^\psi(X_{\tau_n}, M_{\tau_n}) - \int_0^{\tau_n} v_x^\psi(X_t, M_t) dB_t - \int_0^{\tau_n} v_m^\psi(X_t, M_t) dM_t$$

Notice that $(M_t - X_t) dM_t = 0$. Then since $v_m^\psi(m, m) = 0$, it follows that $v_m^\psi(X_t, M_t) dM_t = v_m^\psi(M_t, M_t) dM_t = 0$, and therefore:

$$v^\psi(x, m) \geq v^\psi(X_{\tau_n}, M_{\tau_n}) - \int_0^{\tau_n} v_x^\psi(X_t, M_t) dX_t.$$

Taking expectations in the last inequality, we see that:

$$v^\psi(x, m) \geq \mathbb{E}_{x, m}^{\mathbb{P}_0} [v^\psi(X_{\tau_n}, M_{\tau_n})]. \quad (5.18)$$

3. By the Lipschitz property of g in m uniformly in x (Assumption A):

$$|g^\lambda(X_\tau, M_{\tau_n})| \leq |g^\lambda(X_\tau, M_\tau)| + \kappa M_\tau$$

for some constant κ . Since $g^\lambda(X_\tau, M_\tau) \in \mathbb{L}^1(\mathbb{P}_0)$ and $M_\tau \in \mathbb{L}^1(\mathbb{P}_0)$, by the definition of \mathcal{T}^* , this shows that $g^\lambda(X_\tau, M_{\tau_n}) \in \mathbb{L}^1(\mathbb{P}_0)$. We next deduce from the concavity of v^ψ in x that:

$$v^\psi(X_{\tau_n}, M_{\tau_n}) + v_x^\psi(X_{\tau_n}, M_{\tau_n})(X_\tau - X_{\tau_n}) \geq v^\psi(X_\tau, M_{\tau_n}).$$

Since $(X_{t \wedge \tau})_{t \geq 0}$ is a uniformly integrable martingale, this provides:

$$v^\psi(X_{\tau_n}, M_{\tau_n}) \geq \mathbb{E}^{\mathbb{P}^0} \left[v^\psi(X_\tau, M_{\tau_n}) | \mathcal{F}_{\tau_n} \right] \geq \mathbb{E}^{\mathbb{P}^0} \left[g^\lambda(X_\tau, M_{\tau_n}) | \mathcal{F}_{\tau_n} \right], \quad (5.19)$$

where the last inequality follows from the fact that v^ψ is above the obstacle g^λ . Then it follows from (5.18) together with the tower property of conditional expectations that $v(x, m) \geq \mathbb{E}_{x,m}^{\mathbb{P}^0} [g^\lambda(X_\tau, M_{\tau_n})]$. Using again Assumption A, we then see that:

$$v^\psi(x, m) \geq \mathbb{E}_{x,m}^{\mathbb{P}^0} [g^\lambda(X_\tau, M_\tau) - \kappa(M_\tau - M_{\tau_n})] \nearrow \mathbb{E}_{x,m}^{\mathbb{P}^0} [g^\lambda(X_\tau, M_\tau)],$$

by the monotone convergence theorem. By the arbitrariness of $\tau \in \mathcal{T}^*$, this implies that $v^\psi \geq u^\lambda$. \square

Remark 5.3 In the special case $g(x, m) = \phi(m)$ for some C^1 non-decreasing function ϕ , the Lipschitz property in Assumption A can be dropped by using the monotone convergence theorem in the passage to the limit after equation (5.19), see [19].

Remark 5.4 The analysis of the present section can be developed further to prove that, under the present conditions, the Azéma-Yor solution of the Skorokhod embedding problem defines the optimal upper bound for the one-marginal constraint problem. Since this result is not needed for the proof of Theorem 3.1, we report it for completeness in Section 6.

5.2 Multiple-marginals penalized value function

We now continue our general methodology and return to the multiple-marginal problem of Section 2.3. Our aim is to prove Theorem 3.1 and derive the robust superhedging bounds for a Lookback derivative security

$$\phi(X_T^*) \quad \text{given the marginals } X_{t_i} \sim \mu_i \text{ for all } i = 1, \dots, n. \quad (5.20)$$

We recall that the probability measures μ_i are defined from market prices which do not admit arbitrage, i.e. (2.4) holds.

Using the notation introduced in Section 2.3, we recall that the robust superhedging bound can be expressed in the dual formulation of Proposition 2.1 as:

$$U_n^\mu(\xi) := \inf_{\lambda \in \Lambda_n^\mu(\xi)} \left\{ \mu(\lambda) + u^\lambda(X_0, X_0) \right\}, \quad \text{where } u^\lambda(x, m) := \sup_{\mathbb{P} \in \mathcal{P}^*} \mathbb{E}^{\mathbb{P}} \left[\phi^\lambda(X_t, M_{t_n}) \right] \quad (5.21)$$

with $\phi^\lambda := \phi - \sum_{i=1}^n \lambda_i$ as in (2.5), and the set of Lagrange multipliers is:

$$\Lambda_n^\mu(\xi) = \left\{ \lambda = (\lambda_1, \dots, \lambda_n) : \lambda_i \in \mathbb{L}^1(\mu_i) \text{ and } \sup_{\mathbb{P} \in \mathcal{P}^*} \mathbb{E}^{\mathbb{P}} [\phi^\lambda(X_t, X_{t_n}^*)^+] < \infty \right\}. \quad (5.22)$$

Our approach to solve the present n -marginals Skorokhod embedding problem is to introduce the sequence of intermediate optimization problems:

$$u_n(x, m) = \phi(m) \quad \text{and} \quad u_{k-1}(x, m) = \sup_{\mathbb{P} \in \mathcal{P}^*} \mathbb{E}_{t_{k-1}, x, m}^{\mathbb{P}} \left[u_k^\lambda(X_{t_k}, M_{t_k}) \right], \quad k \leq n, \quad (5.23)$$

where $\mathbb{E}_{t_{k-1}, x, m}^{\mathbb{P}} = \mathbb{E}^{\mathbb{P}}[\cdot | (X, M)_{t_{k-1}} = (x, m)]$, and:

$$u_k^\lambda(x, m) := u_k(x, m) - \lambda_k(x) \quad \text{for} \quad (x, m) \in \mathbf{\Delta}. \quad (5.24)$$

Given this iterative sequence of value functions, it follows from the dynamic programming principle that our problem of interest is given by:

$$u^\lambda = u_0^\lambda \quad \text{for all} \quad \lambda \in \Lambda_n^\mu(\xi).$$

From the Dambis-Dubins-Schwartz theorem (see Proposition 3.1 in [19]), we may convert the stochastic control problem in (5.23) into a sequence of optimal stopping problems:

$$u_{k-1}(x, m) = \sup_{\tau \in \mathcal{T}^*} \mathbb{E}_{x, m}^{\mathbb{P}_0} \left[u_k^\lambda(X_\tau, M_\tau) \right]. \quad (5.25)$$

Then, denoting by $\mathcal{S}_n^* := \{\tau = (\tau_1, \dots, \tau_n) \in \mathcal{T}^* : \tau_1 \leq \dots \leq \tau_n\}$, we see that

$$U_n^\mu(\xi) = \inf_{\lambda \in \Lambda_n^\mu(\xi)} \left\{ \mu(\lambda) + u_0^\lambda(X_0, X_0) \right\} \quad \text{where} \quad u_0^\lambda(x, m) := \sup_{\tau \in \mathcal{S}_n^*} \mathbb{E}_{x, m}^{\mathbb{P}_0} \left[\phi^\lambda(X_\tau, M_{\tau_n}) \right], \quad (5.26)$$

and the set $\Lambda_n^\mu(\xi)$ of (5.22) translates in the present context to:

$$\Lambda_n^\mu(\xi) = \left\{ \lambda = (\lambda_1, \dots, \lambda_n) : \lambda_i \in \mathbb{L}^1(\mu) \text{ and } \sup_{\tau \in \mathcal{S}_n^*} \mathbb{E}^{\mathbb{P}_0} \left[\phi^\lambda(X_\tau, X_{\tau_n}^*)^+ \right] < \infty \right\}. \quad (5.27)$$

5.3 Preparation for the upper bound

The function u_{k-1} corresponds to the optimization problem considered in Section 5.1 with a payoff $g(x, m) = u_k(x, m)$ depending on the spot and the running maximum. This was our original motivation for isolating the one-marginal problem.

To solve the multiple marginals problem, we introduce the iterative sequence of candidate value functions:

$$\begin{aligned} v_n(x, m) &:= \phi(m), \quad v_k^\lambda(x, m) := v_k(x, m) - \lambda_k(x), \quad \text{and} \\ v_{k-1}(x, m) &:= v_k^\lambda(x \wedge \psi_k(m), m) + (x - \psi_k(m))^+ \partial_x v_k^\lambda(\psi_k(m), m) \\ &= v_k^\lambda(x, m) - \int_{\psi_k(m)}^{x \vee \psi_k(m)} (x - \xi) \partial_{xx} v_k^\lambda(d\xi, m), \end{aligned} \quad (5.28)$$

where $\psi = (\psi_1, \dots, \psi_n)$ with ψ_i defined as an arbitrary solution of the ordinary differential equation

$$-\psi_k' \partial_{xx} v_k^\lambda(\psi_k, m) = \gamma_k(\psi_k, m), \quad \text{with} \quad \gamma_k(x, m) := (m - x) \partial_x \left\{ \frac{\partial_m v_k(x, m)}{m - x} \right\}, \quad (5.29)$$

which stays strictly below the diagonal. Notice that, in contrast to the one-marginal case, we have dropped here the dependence of v_k in ψ by simply denoting $v_k := v_k^\psi$ and $v_k^\lambda := v_k^{\psi, \lambda}$.

Similar to the one-marginal case, we introduce the weak formulation of this ODE:

$$\begin{aligned} \psi_k(m) &< m \text{ for all } m \geq 0, \text{ and} \\ - \int_{\psi(E)} \partial_{xx} v_k^\lambda(\cdot, \psi_k^{-1})(d\xi) &= \int_E \gamma_k(\psi_k, \cdot)(dm) \text{ for all } E \in \mathcal{B}(\mathbb{R}), \end{aligned} \quad (5.30)$$

and we introduce the set

$$\Psi_n^\lambda := \left\{ \psi : \mathbb{R} \rightarrow \mathbb{R}^n \text{ with right-continuous entries } \psi_k \text{ satisfying (5.30), } k \leq n \right\}. \quad (5.31)$$

We also follow the one-marginal case by restricting the minimization in (5.26) to those multipliers λ in the set:

$$\hat{\Lambda}_n^\mu(\xi) := \left\{ \lambda \in \Lambda_n^\mu(\xi) : v_{k-1} \text{ concave in } x \text{ and } v_{k-1} \geq v_k^\lambda \text{ for all } k \leq n \right\}. \quad (5.32)$$

Lemma 5.1 *Let ϕ be a $C^1(\mathbb{R})$ non-decreasing Lipschitz function. Then:*

- (i) *for all $i = 1, \dots, n$, the function v_i satisfies Assumptions A and B, i.e. v_i is C^1 in (x, m) , Lipschitz in m uniformly in x , $\partial_{xx} v_i$ exists a.e. and $x \mapsto \partial_m v_i(x, m)/(m - x)$ is non-decreasing,*
- (ii) *for all $i = 1, \dots, n$, the function $\partial_m v_i$ is concave in x ,*
- (iii) *$u^\lambda(X_0, X_0) \leq v_0(X_0, X_0)$ for all $\lambda \in \hat{\Lambda}_n^\mu$ and $\psi \in \Psi_n^\lambda$.*

Proof We first prove (i). First $v_n = \phi$ satisfies Assumptions A and B as it is independent of the x -variable, non-decreasing and C^1 Lipschitz. For the remaining cases $i \leq n - 1$, we proceed by induction, assuming that v_{i+1} satisfies Assumptions A and B, and we intend to show that v_i does as well. We first observe that either one of the following condition is also satisfied by v_{i+1} :

$$v_i(x, m) = \phi(m) \text{ non-decreasing, or } \partial_m v_i(m, m) = 0, \quad (5.33)$$

where the first alternative holds for $i = n$. v_{i-1} is clearly C^1 , and by using the ODE (5.29) satisfied by v_i , we directly compute that

$$\partial_m v_{i-1}(x, m) = \begin{cases} \partial_m v_i(x, m) & \text{for } x \in (-\infty, \psi_i(m)] \\ \partial_m v_i(\psi_i(m), m) \frac{m-x}{m-\psi_i(m)} & \text{for } x \in [\psi_i(m), m]. \end{cases} \quad (5.34)$$

Then v_{i-1} inherits the Lipschitz property of g in m , uniformly in x . Moreover, $x \mapsto \partial_m v_{i-1}(x, m)/(m - x)$ is non-decreasing whenever $x \mapsto \partial_m v_i(x, m)/(m - x)$ is.

We next prove (iii). By the previous step, v_i satisfies Assumptions A and B for all $i = 1, \dots, n$. Then it follows from Proposition 5.1 that $u_{n-1} \leq v_{n-1}$ for all $\psi \in \Psi^\lambda$. Therefore

$$u_{n-2}(x, m) \leq \sup_{\tau_{n-1} \in \mathcal{T}^*} \mathbb{E}_{x, m}^{\mathbb{P}} \left[v_{n-1}^\lambda(X_{\tau_{n-1}}, X_{\tau_{n-1}}^*) \right],$$

and we deduce from a second application of Proposition 5.1 that $u_{n-2} \leq v_{n-2}$. The required inequality follows by a backward iteration of this argument.

We finally prove (ii). From (5.34), we see that $\partial_m v_{i-1}$ is concave in x on $(-\infty, \psi_i(m))$ and on $(\psi_i(m), m]$. It remains to verify that $\partial_m v_{i-1}$ is concave at the point $x = \psi_i(m)$. We directly calculate that

$$\partial_{xm} v_{i-1}(\psi_i(m)-, m) = \partial_{xm} v_i(\psi_i(m)-, m) \quad \text{and} \quad \partial_{xm} v_{i-1}(\psi_i(m)+, m) = \frac{-\partial_m v_i(\psi_i(m), m)}{m - \psi_i(m)}.$$

Then, by the concavity of $\partial_m v_i$ in x , together with (5.33), we have

$$\partial_m v_i(\psi_i(m), m) + \partial_{xm} v_i(\psi_i(m)+, m)(m - \psi_i(m)) \geq \partial_m v_i(m, m) \geq 0,$$

which implies that $\partial_{xm} v_{i-1}(\psi_i(m)-, m) \geq \partial_{xm} v_{i-1}(\psi_i(m)+, m)$. \square

Our next result uses the notation:

$$\delta_i(x, m) := c_i(x) - c_0(x) \mathbf{1}_{\{m < X_0\}} \quad (x, m) \in \mathbf{\Delta}. \quad (5.35)$$

Lemma 5.2 *Let ϕ be a C^1 non-decreasing Lipschitz function. Then, for all $\lambda \in \hat{\Lambda}_n^\mu$ and $\psi \in \Psi_n^\lambda$, we have:*

$$\begin{aligned} \mu(\lambda) + u^\lambda(X_0, X_0) &\leq \mu(\lambda) + v_0(X_0, X_0) \\ &= \phi(X_0) + \sum_{i=1}^n \int \delta_i(\xi, \psi_i^{-1}(\xi)) \lambda_i''(d\xi) - \int_{\psi_i(X_0)}^{X_0} c_0(\xi) \partial_{xx} v_i(\xi, X_0) d\xi. \end{aligned}$$

Proof This is a direct consequence of Lemma 5.1 obtained by substituting the expression of the v_i 's, and using the fact that $\mu_i(\lambda_i) - \lambda_i(X_0) = \int \lambda_i'' d(\mu - \delta_{X_0})$. \square

The following result provides the necessary calculations for the terms which appear in Lemma 5.2. We denote:

$$\bar{\psi}_i := \psi_i \wedge \dots \wedge \psi_n \quad \text{for all } i = 1, \dots, n, \quad (5.36)$$

and we set $\bar{\psi}_{n+1}(m) := m$, $m \geq 0$.

Lemma 5.3 *For a C^1 non-decreasing Lipschitz function ϕ , $\lambda \in \Lambda_n^\mu(\xi)$, $\psi \in \Psi_n^\lambda$, and $i \leq n$, we have:*

$$\begin{aligned} \text{(i)} \quad &\int \delta_i(\cdot, \psi_i^{-1}) d\lambda_i' = \int \left(\frac{\delta_i(\bar{\psi}_i, \cdot)(m)}{m - \bar{\psi}_i(m)} - \frac{\delta_i(\bar{\psi}_{i+1}, \psi_i^{-1} \circ \bar{\psi}_{i+1})(m)}{m - \bar{\psi}_{i+1}(m)} \mathbf{1}_{\{i < n\}} \right) \left(\mathbf{1}_{\{\bar{\psi}_i < \bar{\psi}_{i+1}\}} \phi' \right)(m) dm, \\ \text{(ii)} \quad &\int_{\psi_i(X_0)}^{X_0} c_0(\xi) \partial_{xx} v_i(\xi, X_0) d\xi = -\mathbf{1}_{\{i < n\}} \int_0^{X_0} \frac{c_0(\bar{\psi}_{i+1}(m))}{m - \bar{\psi}_{i+1}(m)} \mathbf{1}_{\{\bar{\psi}_{i+1}(m) > \psi_i(X_0)\}} \phi'(m) dm. \end{aligned}$$

Proof See Section 7. □

Plugging these calculations into the estimate of Lemma 5.2 provides:

Lemma 5.4 *Let ϕ be a C^1 non-decreasing Lipschitz function. Then, for all $\lambda \in \hat{\Lambda}_n^\mu$ and $\psi \in \Psi_n^\lambda$, we have:*

$$\begin{aligned} & \mu(\lambda) + u^\lambda(X_0, X_0) \\ & \leq \mu(\lambda) + v_0(X_0, X_0) \\ & = \phi(X_0) + \int \sum_{i=1}^n \left(\frac{\delta_i(\bar{\psi}_i(m), m)}{m - \bar{\psi}_i(m)} - \frac{\delta_i(\bar{\psi}_{i+1}(m), m)}{m - \bar{\psi}_{i+1}(m)} \mathbf{1}_{\{i < n\}} \right) \left(\phi' \mathbf{1}_{\{\bar{\psi}_i < \bar{\psi}_{i+1}\}} \right) (m) dm. \end{aligned}$$

with $\bar{\psi}_{n+1}(m) := m$, $m \geq 0$.

Proof By Lemmas 5.2 and 5.3 (i), we have

$$\mu(\lambda) + u^\lambda(X_0, X_0) \leq \phi(X_0) + \int \phi'(m) \sum_{i=1}^n \mathbf{1}_{\{\bar{\psi}_i(m) < \bar{\psi}_{i+1}(m)\}} A_i(m) dm,$$

where

$$\begin{aligned} A_i(m) & = \frac{c_i(\bar{\psi}_i(m)) - c_0(\bar{\psi}_i(m)) \mathbf{1}_{\{m < X_0\}}}{m - \bar{\psi}_i(m)} \\ & \quad - \frac{c_i(\bar{\psi}_{i+1}(m)) - c_0(\bar{\psi}_{i+1}(m)) (\mathbf{1}_{\{\bar{\psi}_{i+1}(m) < \psi_i(X_0)\}} + \mathbf{1}_{\{m < X_0\}} \mathbf{1}_{\{\bar{\psi}_{i+1}(m) > \psi_i(X_0)\}})}{m - \bar{\psi}_{i+1}(m)}. \end{aligned}$$

Notice that $m < X_0$ on $\{\bar{\psi}_i(m) < \bar{\psi}_{i+1}(m)\}$. Then

$$\begin{aligned} A_i(m) & = \frac{c_i(\bar{\psi}_i(m)) - c_0(\bar{\psi}_i(m)) \mathbf{1}_{\{m < X_0\}}}{m - \bar{\psi}_i(m)} - \frac{c_i(\bar{\psi}_{i+1}(m)) - c_0(\bar{\psi}_{i+1}(m)) \mathbf{1}_{\{m < X_0\}}}{m - \bar{\psi}_{i+1}(m)} \\ & \text{on } \{\bar{\psi}_i(m) < \bar{\psi}_{i+1}(m)\}. \end{aligned}$$

□

We now consider the problem of minimisation inside the integral in the expression obtained in Lemma 5.4, forgetting about the constraints on the ψ_i 's.

Lemma 5.5 *Under the no-arbitrage condition (2.4), we have*

$$\min_{\zeta_1 \leq \dots \leq \zeta_n < m} \sum_{i=1}^n \left\{ \frac{\delta_i(\zeta_i, m)}{m - \zeta_i} - \frac{\delta_i(\zeta_{i+1}, m)}{m - \zeta_{i+1}} \right\} = 0 \quad \text{for } m < X_0,$$

and the minimum is attained at $\zeta_i^* = 0$, $i = 1, \dots, n$.

Proof Since $m < X_0$ and $\zeta_i < m$ for all $i \leq n$, it follows that $c_0(\zeta_i)\mathbf{1}_{\{m < X_0\}} = c_0(\zeta_i)$. We proceed by induction.

1. Notice that ζ_1 only appears in the first term of the sum. The partial minimization with respect to ζ_1 reduces to

$$\min_{\zeta_1 \leq \zeta_2} \frac{c_1(\zeta_1) - c_0(\zeta_1)}{m - \zeta_1}.$$

By the no-arbitrage condition the function to be minimized is nonnegative, and is zero for $\zeta_1^* = 0$.

2. For $2 \leq i \leq n$, assume that $\zeta_{i-1}^* = 0$ realizes the minimum over ζ_{i-1} . Then, the partial minimization with respect to ζ_i reduces to

$$\min_{0 \leq \zeta_i < m} \frac{c_i(\zeta_i) - c_0(\zeta_i)}{m - \zeta_i} \mathbf{1}_{\{\zeta_i < \zeta_{i+1}\}} - \frac{c_{i-1}(\zeta_i) - c_0(\zeta_i)}{m - \zeta_i}.$$

Since $c_i \geq c_{i-1}$ by the no-arbitrage condition, it is clear that the latter minimum is zero and attained at $\zeta_i^* = 0$. \square

5.4 Second Proof of Theorem 3.1 under (5.3)

1. Given the results of Lemma 5.4, we prove in this first step that the pointwise minimisation of Lemma 5.5 and (3.1) can be achieved by some vector of Lagrange multipliers $\lambda^* = (\lambda_1^*, \dots, \lambda_n^*) \in \Lambda_n^\mu(\xi)$, thus implying that our required upper bound satisfies:

$$U_n^\mu(\xi) \leq \phi(X_0) + \int_{X_0}^{\infty} \sum_{i=1}^n \left(\frac{c_i(\zeta_i^*(m))}{m - \zeta_i^*(m)} - \frac{c_i(\zeta_{i+1}^*(m), m)}{m - \zeta_{i+1}^*(m)} \mathbf{1}_{\{i < n\}} \right) \phi'(m) dm. \quad (5.37)$$

In order to define λ^* , we take a family of functions ψ_i^* satisfying:

$$\psi_1^* := b_1^{-1}, \quad \psi_n^* := \zeta_n^*, \quad \text{and} \quad \bar{\psi}_i^* := \psi_i^* \wedge \dots \wedge \psi_n^* = \zeta_i^*, \quad 1 < i < n,$$

i.e. ψ_i^* is an extension of ζ_i^* for all $i = 1, \dots, n$. Here b_1 is the barycentre function of μ_1 , see (6.1) below, and the fact that $\psi_1^* := b_1^{-1}$ is a minimizer in (3.1) is easily verified as in [9].

For our purpose here the precise form of ψ_i^* does not matter. Indeed, direct verification reveals that the functions λ_i introduced in (4.4) solve system of ODEs (5.29). Under our assumption that the support of ϕ' is bounded from above, it also follows that, up to a linear function, λ_i is bounded from below. Then $\sup_{\mathbb{P} \in \mathcal{P}^*} \mathbb{E}^\mathbb{P} [(\phi(X_T^*) - \lambda(X_t))^+] \leq \sup_{\mathbb{P} \in \mathcal{P}^*} \mathbb{E}^\mathbb{P} [\phi(X_T^*)^+] < \infty$.

The final ingredient to verify, in order for $\lambda^* \in \Lambda_n^\mu(\xi)$ which implies that inequality (5.37) holds, is that $\lambda_i^* \in \mathbb{L}^1(\mu_i)$. To see this, we follow the same calculations as in the proof of Lemma 5.4 to see that

$$\mu_i(\lambda_i^*) \leq \text{Const} + \int \left(\frac{c_i(\bar{\psi}_i^*(m))}{m - \bar{\psi}_i^*(m)} - \frac{c_i(\bar{\psi}_{i+1}^*(m))}{m - \bar{\psi}_{i+1}^*(m)} \right) \phi'(m) \mathbf{1}_{\{\bar{\psi}_i^*(m) < \bar{\psi}_{i+1}^*(m)\}} dm$$

proving the required integrability by our condition (5.3).

2. Now we prove that equality holds in (3.2) if [33, Assumption A] is in place.

Recall that [33], under their Assumption A, construct an embedding $\hat{\tau}_1, \dots, \hat{\tau}_n$ of μ_1, \dots, μ_n . In addition they obtain the law of $X_{\hat{\tau}_n}^*$ which we plug in at the last equality sign of the following display. By the expression of $U_n^\mu(\xi)$ in (5.26), it follows that:

$$\begin{aligned} U_n^\mu(\xi) &\geq \inf_{\lambda \in \Lambda_n^\mu(\xi)} \left\{ \mu(\lambda) + \mathbb{E}^{\mathbb{P}^0} \left[\phi(X_{\hat{\tau}_n}^*) - \sum_{i=1}^n \lambda_i(X_{\hat{\tau}_i}) \right] \right\} = \mathbb{E}^{\mathbb{P}^0} [\phi(X_{\hat{\tau}_n}^*)] \\ &\stackrel{[33]}{=} \phi(X_0) + \sum_{i=1}^n \int_{X_0}^{\infty} \left(\frac{c_i(\tilde{\zeta}_i(m))}{m - \tilde{\zeta}_i(m)} - \frac{c_i(\tilde{\zeta}_{i+1}(m))}{m - \tilde{\zeta}_{i+1}(m)} \mathbf{1}_{\{i < n\}} \right) \phi'(m) dm, \end{aligned} \quad (5.38)$$

for some $\tilde{\zeta}_1(y) \leq \dots \leq \tilde{\zeta}_n(y)$. The proof is complete by comparing the expression inside the integral in (5.38) with (3.1) and recalling that $\zeta_1^*, \dots, \zeta_n^*$ was chosen as the minimizer of the latter.

6 The Azéma-Yor embedding solves the one-marginal problem

In this subsection, we return to the one-marginal context of Subsection 5.1. The endpoints of the support of the distribution μ are denoted by:

$$\ell^\mu := \sup \{x : \mu([x, \infty)) = 1\} \quad \text{and} \quad r^\mu := \inf \{x : \mu((x, \infty)) = 0\}$$

We introduce the so-called barycentre function:

$$b(x) := \frac{\int_{[x, \infty)} y \mu(dy)}{\mu([x, \infty))} \mathbf{1}_{\{x < r^\mu\}} + x \mathbf{1}_{\{x \geq r^\mu\}}, \quad x \geq 0. \quad (6.1)$$

The Azéma-Yor [2, 3] solution of the *Skorokhod Embedding Problem* is:

$$\tau^* := \inf \{t > 0 : X_t^* \geq b(X_t)\}. \quad (6.2)$$

Plugging $\psi^* := b^{-1}$ in the ODE (5.15), we obtain the function

$$\lambda^*(x) := \int_{\ell^\mu}^x \int_{\ell^\mu}^y g_m(\xi, b(\xi)) \frac{\mu(d\xi)}{\mu([\xi, \infty))} dy + \int_{\ell^\mu}^x g_x(\xi, b(\xi)) d\xi; \quad x \in (-\infty, r^\mu), \quad (6.3)$$

whose well-posedness will be guaranteed by the following condition.

In this subsection, we need the following additional condition on the payoff function $g(x, m)$:

Assumption C g is Lipschitz in x , uniformly in m , $(g - \lambda^*)^+$ is bounded, and

$$g_{xx}(dx, m) - g_{xx}(dx, b(x)) \leq \gamma(x, b(x))b(dx) \quad \text{whenever } b(x) \leq m.$$

Under this additional condition, we now verify that $\lambda^* \in \mathbb{L}^1(\mu)$. Indeed, following Step 1 of the proof of Lemma 3.2 in [19], this is equivalent to the integrability of $c(\cdot)$ with respect to the measure $(\lambda^*)''$, and it follows from the ODE (5.15) that

$$\begin{aligned} \int c(x)(\lambda^*)''(dx) &= \int_{X_0}^{\infty} c(\psi^*(m))(\gamma(\psi^*(m), m)dm + g_{xx}(\psi^*(m), m)d\psi^*(m)) \\ &= \int_{X_0}^{\infty} c(\psi^*(m))\left(\frac{g_m(\psi^*(m), m)}{m - \psi^*(m)}dm + dg_x(\psi^*(m), m)\right) \\ &= \int_{X_0}^{\infty} \frac{c(\psi^*(m))}{m - \psi^*(m)}g_m(\psi^*(m), m)dm + \int_{X_0}^{\infty} c(\psi^*(m))dg_x(\psi^*(m), m) \end{aligned}$$

Since $c(\psi^*(\infty)) = 0$ and $c(\ell^\mu) = X_0$, the second integral is well-defined and finite either by the boundedness of g_x in Assumption C.

As for the first integral, it follows from the boundedness of g_m in Assumption A that $\int_{X_0}^{\infty} \frac{c(\psi^*(m))}{m - \psi^*(m)}|g_m(\psi^*(m), m)|dm \leq |g_m|_{\infty} \int_{X_0}^{\infty} \frac{c(\psi^*(m))}{m - \psi^*(m)}dm < \infty$. Hence $\lambda^* \in \mathbb{L}^1(\mu)$.

The following result has been by Hobson and Klimmek [22] under slightly different conditions than those in Assumption 6. Our objective is to derive it directly from our stochastic control approach.

Theorem 6.1 *Let $\xi = g(X_T, X_T^*)$ for some payoff function g satisfying Assumptions A, B, and C. Then, for any $\mu \in M(\mathbb{R})$, the pair (λ^*, τ^*) is a solution of the problem $U^\mu(\xi)$, and:*

$$U^\mu(\xi) = J(\lambda^*, \tau^*) = \mathbb{E}^{\mathbb{P}^0} [g(X_{\tau^*}, X_{\tau^*}^*)].$$

The remaining part of this section is dedicated to the proof of this result.

We first observe that, under the present assumptions, we may also restrict the maximization to the subset \mathcal{P}^* . This is due to the fact that Proposition 5.1 is only needed to be applied with λ^* , so that the boundedness assumption on $(g - \lambda^*)^+$ justifies the restriction to \mathcal{P}^* .

Our starting point is the result of Proposition 5.1 which provides an upper bound for the value function $U^\mu(\xi)$ for every choice of a multiplier $\lambda \in \hat{\Lambda}^\mu$ and a corresponding solution $\psi \in \Psi^\lambda$ of the ODE (5.15):

$$U^\mu(\xi) \leq \mu(\lambda) + v^\psi(X_0, X_0) \quad \text{for all } \lambda \in \hat{\Lambda}^\mu \text{ and } \psi \in \Psi^\lambda. \quad (6.4)$$

Alternatively, for any choice of a non-decreasing function ψ with $\psi(m) < m$ for all $m \in \mathbb{R}$, we may define a corresponding multiplier function λ by (5.15), or equivalently by (5.14),

in the distribution sense. Then $\psi \in \Psi^\lambda$. If in addition v^ψ is concave in x and above the corresponding obstacle g^λ , then $\lambda \in \hat{\Lambda}^\mu$ and we may conclude by Proposition 5.1 that $U^\mu(\xi) \leq \mu(\lambda)v^\psi$. The next result exhibits this bound for the choice $\psi = b^{-1}$, the right-continuous inverse of the barycenter function.

Proposition 6.1 *Let ξ be given by (5.4). Then, under Assumptions A, B and C, we have:*

$$U^\mu(\xi) \leq \mu(\lambda^*) + J(\lambda^*, \tau^*) = \mathbb{E}^{\mathbb{P}^0}[g(X_{\tau^*}, X_{\tau^*}^*)].$$

Proof It is immediately checked that $\psi^* := b^{-1} \in \Psi^{\lambda^*}$. Moreover, by Assumption C and the subsequent discussion, $\lambda^* \in \mathbb{L}^1(\mu)$, and $\sup_{\tau \in \mathcal{T}^*} \mathbb{E}^{\mathbb{P}^0}[g^{\lambda^*}(X_\tau, X_\tau^*)] < \infty$. Then, in view of the previous discussion, the required inequality follows from Proposition 5.1 once we prove that v^{ψ^*} is concave, and that $v^{\psi^*} \geq g^{\lambda^*}$.

1. We first verify that v^{ψ^*} is concave. By direct computation using the expression of λ^* in (6.3) together with the identity

$$\frac{b(dx)}{b(x) - x} = \frac{\mu(dx)}{\mu([x, \infty))},$$

we see that

$$g_{xx}^{\lambda^*}(x, m) = g_{xx}(x, m) - g_{xx}(x, b(x)) - \gamma(x, b(x))b'(x) \quad (6.5)$$

in the distribution sense. By Assumption C, it follows that $x \mapsto g^{\lambda^*}(x, m)$ is concave on $(-\infty, \psi^*(m)]$. Since $v^\psi(\cdot, m)$ is linear on $[\psi^*(m), m]$ and C^1 across the boundary $\psi^*(m)$, this proves that v^ψ is concave.

2. We next check that $v^{\psi^*} \geq g^{\lambda^*}$. Since equality holds on $(-\infty, \psi^*(m)]$, we only compute for $x \in [\psi^*(m), m]$ that:

$$\begin{aligned} (v^{\psi^*} - g^{\lambda^*})(x, m) &= \int_{\psi^*(m)}^x (g_x^{\lambda^*}(\psi^*(m), m) - g_x^{\lambda^*}(\xi, m))d\xi \\ &= - \int_{\psi^*(m)}^x \int_{\psi^*(m)}^\xi g_{xx}^{\lambda^*}(y, m)dyd\xi. \end{aligned}$$

By (6.5), this provides:

$$\begin{aligned} (v^{\psi^*} - g^{\lambda^*})(x, m) &= - \int_{\psi^*(m)}^x \left(g_x(\xi, m) - g_x(\xi, b(\xi)) - \int_{\psi^*(m)}^\xi \frac{g_m(y, b(y))}{b(y) - y} b(dy) \right) d\xi \\ &= \int_{\psi^*(m)}^x \int_{\psi^*(m)}^\xi \left(g_{xm}(\xi, b(y)) + \frac{g_m(y, b(y))}{b(y) - y} \right) b(dy) d\xi \\ &= \int_{\psi^*(m)}^x \left(\int_y^x g_{xm}(\xi, b(y)) d\xi + \frac{g_m(y, b(y))}{b(y) - y} \right) b(dy) \\ &= \int_{\psi^*(m)}^x (b(y) - x) \left(\frac{g_m(x, b(y))}{b(y) - x} - \frac{g_m(y, b(y))}{b(y) - y} \right) b(dy) \geq 0, \end{aligned}$$

where the last inequality follows from the nondecrease of b and $x \mapsto g_m(x, m)/(m - x)$ (Assumption B), together with the fact that $b(y) \geq x$ for $\psi(m) \leq y \leq x \leq m$. \square

Proof of Theorem 6.1 To complete the proof of the theorem, it remains to prove that

$$\inf_{\lambda \in \Lambda^\mu} \{ \mu(\lambda) + u^\lambda(X_0, X_0) \} \geq \mathbb{E}_{X_0, X_0}^{\mathbb{P}_0} [g(X_{\tau^*}, X_{\tau^*})].$$

To see this, we use the fact that the stopping time τ^* defined in (6.2) is a solution of the Skorokhod embedding problem, i.e. $X_{\tau^*} \sim \mu$ and $(X_{t \wedge \tau^*})_{t \geq 0}$ is a uniformly integrable martingale, see Azéma and Yor [2, 3]. Moreover X_{τ^*} is integrable. Then, for all $\lambda \in \Lambda^\mu$, it follows from the definition of u^λ that $u^\lambda(X_0, X_0) \geq J(\lambda, \tau^*)$, and therefore:

$$\mu(\lambda) + u^\lambda(X_0, X_0) \geq \mu(\lambda) + \mathbb{E}_{X_0, X_0}^{\mathbb{P}_0} [g(X_{\tau^*}, X_{\tau^*}) - \lambda(X_{\tau^*})] = \mathbb{E}_{X_0, X_0}^{\mathbb{P}_0} [g(X_{\tau^*}, X_{\tau^*})].$$

\square

We conclude this section by a formal justification that the function b^{-1} appears naturally if one searches for the best upper bound in (6.4).

Step 1: using the expression (5.12) of v^ψ , we directly compute that

$$\begin{aligned} \mu(\lambda) + u^\lambda(X_0, X_0) &= \mu(g(\cdot, X_0)) + \mu(g^\lambda(\cdot, X_0)) - \int_{\psi(X_0)}^{X_0} g_{xx}^\lambda(\xi, X_0)(X_0 - \xi) d\xi \\ &= \mu(g(\cdot, X_0)) + \int g_{xx}^\lambda(\xi, X_0)(c(\xi) - c_0(\xi)\mathbf{1}_{\{\xi \leq \psi(X_0)\}}) d\xi \\ &= \mu(g(\cdot, X_0)) + \int g_{xx}^\lambda(\xi, \psi^{-1}(\xi))(c(\xi) - c_0(\xi)\mathbf{1}_{\{\xi \leq \psi(X_0)\}}) d\xi \\ &\quad + \int (g_{xx}(\xi, X_0) - g_{xx}(\xi, \psi^{-1}(\xi)))(c(\xi) - c_0(\xi)\mathbf{1}_{\{\xi \leq \psi(X_0)\}}) d\xi, \end{aligned}$$

where the second equality follows from two integrations by parts together with the fact that $\int x \mu(dx) = X_0$, see Step 1 of the proof of Lemma 3.2 in [19]. Then, by using the ODE (5.15) satisfied by ψ to change variables in the last integral, we see that:

$$\mu(\lambda) + u^\lambda(X_0, X_0) = \mu(g(\cdot, X_0)) + \int \{ -\gamma(\psi(m), m) + G(\psi(m), m)\psi'(m) \} \delta(\psi(m), m) dm,$$

where we denoted:

$$\begin{aligned} \delta(x, m) &:= c(x) - c_0(x)\mathbf{1}_{\{m \leq X_0\}}, \quad c_0(x) := (X_0 - x)^+, \\ \text{and } G(x, m) &:= g_{xx}(x, X_0) - g_{xx}(x, m). \end{aligned}$$

Step 2: The expression of $\mu(\lambda) + v^\psi$ derived in the previous step only involves the function $\psi \in \Psi^\lambda$. Forgetting about all constraints on ψ , we treat our minimization problem as

a standard problem of calculus of variations. The local Euler-Lagrange equation for this problem is:

$$\frac{d}{dx}(G\delta)(\psi, m) = -(\gamma\delta)_x(\psi, m) + (G\delta)_x(\psi, m)\psi'.$$

Since $(G\delta_m)(x, m) = 0$, this reduces to

$$\begin{aligned} 0 &= (G_m\delta + \gamma\delta_x + \gamma_x\delta)(\psi, m) \\ &= (m - \psi)\gamma(\psi, m)\frac{\partial}{\partial x}\left\{\frac{\delta(x, m)}{m - x}\right\}_{x=\psi}. \end{aligned}$$

This shows formally that the solution of the minimization problem:

$$\min_{\xi < m} \frac{\delta(x, m)}{m - x}$$

provides a solution to the local Euler-Lagrange equation. Finally, the solution of the above minimization problem is known to be given by the right inverse barycenter function b^{-1} , see the proof of Lemma 3.3 in [19].

7 Appendix

This section contains the proof of Lemma 5.3. We start with the computation of $\gamma_i(\psi_i, \cdot)$, as defined in (5.29), in terms of g and the ψ_i 's.

Lemma 7.1 *For all $i < n$, we have $\gamma_i(\psi_i(m), m) = \frac{\phi'(m)}{m - \psi_i(m)} \mathbf{1}_{\{\psi_i < \bar{\psi}_{i+1}\}}$.*

Proof By direct differentiation of (5.28), we see that:

$$\begin{aligned} \partial_m v_{i-1}(x, m) &= \partial_m v_i(x \wedge \psi_i(m), m) \\ &\quad + (x - \psi_i(m))^+ [\partial_{xx} v_i(\psi_i(m), m)\psi_i'(m) + \partial_{xm} v_i(\psi_i(m), m)]. \end{aligned}$$

Using the ODE satisfied by ψ_i , this provides:

$$\begin{aligned} \partial_m v_{i-1}(x, m) &= \partial_m v_i(x \wedge \psi_i(m), m) + \frac{(x - \psi_i(m))^+}{m - \psi_i(m)} \partial_m v_i(x \wedge \psi_i(m), m) \\ &= \frac{m - x \vee \psi_i(m)}{m - \psi_i(m)} \partial_m v_i(x \wedge \psi_i(m), m). \end{aligned} \tag{7.1}$$

Differentiating this expression with respect to x , we also compute that:

$$\begin{aligned} \partial_{mx} v_{i-1}(x, m) &= \mathbf{1}_{\{x < \psi_i(m)\}} \partial_{mx} v_i(x \wedge \psi_i(m), m) \\ &\quad + \mathbf{1}_{\{x > \psi_i(m)\}} \frac{-1}{m - \psi_i(m)} \partial_m v_i(x \wedge \psi_i(m), m). \end{aligned} \tag{7.2}$$

From the expression of γ_i , it follows from (7.1) and (7.2) that:

$$\gamma_{i-1}(x, m) = \mathbf{1}_{\{x < \psi_i(m)\}} \gamma_i(x, m) = \cdots = \mathbf{1}_{\{x < \bar{\psi}_i(m)\}} \gamma_n(x, m) = \mathbf{1}_{\{x < \bar{\psi}_i(m)\}} \frac{\phi'(m)}{m - x}.$$

□

Proof of Lemma 5.3 (i) In this proof, we ignore the possible discontinuities of ψ_i and ψ_i^{-1} for simplicity. For any integrable function φ , the following claim:

$$\begin{aligned} \int \varphi(\xi) \lambda_i''(\xi) d\xi &= \int \left(\frac{\varphi(\psi_i(m))}{m - \psi_i(m)} \mathbf{1}_{\{\psi_i(m) < \bar{\psi}_{i+1}(m)\}} - \sum_{j=i+1}^k \frac{\varphi(\psi_j(m))}{m - \psi_j(m)} \mathbf{1}_{\{\psi_i(m) < \psi_j(m) = \bar{\psi}_j(m)\}} \right) \phi'(m) dm \\ &\quad + \int \varphi(\xi) \left[\partial_{xx} v_k(\xi, \psi_i^{-1}(\xi)) - \partial_{xx} v_k(\xi, (\psi_{i+1}^{-1} \vee \cdots \vee \psi_k^{-1})(\xi)) \right] \\ &\quad \mathbf{1}_{\{\psi_i^{-1}(\xi) > (\psi_{i+1}^{-1} \vee \cdots \vee \psi_k^{-1})(\xi)\}} d\xi, \end{aligned} \quad (7.3)$$

which will be proved below by induction, implies the required result by applying it to the function $\varphi(\xi) = \delta_i(\xi, \psi_i^{-1}(\xi))$, with $k = n - 1$, and using the fact that $v_n = \phi$ is independent of x .

We next start verifying (7.3) for $k = i + 1$. The first ingredient for the verification of (7.3) is the fact that

$$\partial_{xx} v_j(x, m) = \partial_{xx} v_{j+1}^\lambda(x, m) \mathbf{1}_{\{x < \psi_{j+1}(m)\}}, \quad \text{where } v_j^\lambda = v_j - \lambda_j. \quad (7.4)$$

which can be immediately checked from the expression of v_i in (5.28).

1. To see that (7.3) holds true with $k = i + 1$, we first decompose the integral so as to use the ODE satisfied by ψ_i :

$$\begin{aligned} \int \varphi \lambda_i'' &= - \int \varphi(\xi) \partial_{xx} v_i^\lambda(\xi, \psi_i^{-1}(\xi)) d\xi + \int \varphi(\xi) \partial_{xx} v_i(\xi, \psi_i^{-1}(\xi)) d\xi \\ &= \int \varphi(\psi_i(m)) \gamma_i(\psi_i(m), m) dm + \int \varphi(\xi) \partial_{xx} v_i(\xi, \psi_i^{-1}(\xi)) d\xi. \end{aligned}$$

We next substitute the expression of $\gamma_i(\psi_i, \cdot)$ from Lemma 7.1, and use (7.4) for the second integral:

$$\begin{aligned} \int \varphi \lambda_i'' &= \int \frac{\varphi(\psi_i(m))}{m - \psi_i(m)} \mathbf{1}_{\{\psi_i(m) < \bar{\psi}_{i+1}(m)\}} dm + \int \varphi(\xi) \partial_{xx} v_{i+1}^\lambda(\xi, \psi_i^{-1}(\xi)) \mathbf{1}_{\{\psi_{i+1}^{-1}(\xi) < \psi_i^{-1}(\xi)\}} d\xi \\ &= \int \frac{\varphi(\psi_i(m))}{m - \psi_i(m)} \mathbf{1}_{\{\psi_i(m) < \bar{\psi}_{i+1}(m)\}} dm + \int \varphi(\xi) \partial_{xx} v_{i+1}^\lambda(\xi, \psi_{i+1}^{-1}(\xi)) \mathbf{1}_{\{\psi_{i+1}^{-1}(\xi) < \psi_i^{-1}(\xi)\}} d\xi \\ &\quad + \int \varphi(\xi) \left[\partial_{xx} v_{i+1}^\lambda(\xi, \psi_i^{-1}(\xi)) - \partial_{xx} v_{i+1}^\lambda(\xi, \psi_{i+1}^{-1}(\xi)) \right] \mathbf{1}_{\{\psi_{i+1}^{-1}(\xi) < \psi_i^{-1}(\xi)\}} d\xi. \end{aligned}$$

Then, by using again the ODE (5.15) satisfied by ψ_{i+1} together with the expression of $\gamma_{i+1}(\psi_{i+1}, \cdot)$ from Lemma 7.1, we get:

$$\begin{aligned} \int \varphi \lambda_i'' &= \int \frac{\varphi(\psi_i(m))}{m - \psi_i(m)} \mathbf{1}_{\{\psi_i(m) < \bar{\psi}_{i+1}(m)\}} dm - \int \frac{\varphi(\psi_{i+1}(m))}{m - \psi_{i+1}(m)} \mathbf{1}_{\{\psi_i(m) < \psi_{i+1}(m) = \bar{\psi}_{i+1}(m)\}} d\xi \\ &\quad + \int \varphi(\xi) [\partial_{xx} v_{i+1}^\lambda(\xi, \psi_i^{-1}(\xi)) - \partial_{xx} v_{i+1}^\lambda(\xi, \psi_{i+1}^{-1}(\xi))] \mathbf{1}_{\{\psi_{i+1}^{-1}(\xi) < \psi_i^{-1}(\xi)\}} d\xi, \end{aligned}$$

which we recognize to be the required equality (7.3) for $k = i + 1$.

2. We next assume that (7.3) holds for some $k < n - 1$, and verify it for $k + 1$. For simplicity, we denote $\psi_{i+1,j}^{-1} := \psi_{i+1}^{-1} \vee \dots \vee \psi_j^{-1}$. By (7.4), we compute that:

$$\begin{aligned} A &:= \int \varphi(\xi) \left[\partial_{xx} v_k(\xi, \psi_i^{-1}(\xi)) - \partial_{xx} v_k(\xi, \psi_{i+1,k}^{-1}(\xi)) \right] \mathbf{1}_{\{\psi_i^{-1}(\xi) > \psi_{i+1,k}^{-1}(\xi)\}} d\xi \\ &= \int \varphi(\xi) \mathbf{1}_{\{\psi_i^{-1}(\xi) > \psi_{i+1,k}^{-1}(\xi)\}} \left[\left\{ \partial_{xx} v_{k+1}(\xi, \psi_i^{-1}(\xi)) - \lambda_{k+1}''(\xi) \right\} \mathbf{1}_{\{\psi_{k+1}^{-1}(\xi) < \psi_i^{-1}(\xi)\}} \right. \\ &\quad \left. - \left\{ \partial_{xx} v_{k+1}(\xi, \psi_{i+1,k}^{-1}(\xi)) - \lambda_{k+1}''(\xi) \right\} \mathbf{1}_{\{\psi_{k+1}^{-1}(\xi) < \psi_{i+1,k}^{-1}(\xi)\}} \right] d\xi \\ &= \int \varphi(\xi) \mathbf{1}_{\{\psi_i^{-1}(\xi) > \psi_{i+1,k}^{-1}(\xi)\}} \left[\mathbf{1}_{\{\psi_{i+1,k}^{-1}(\xi) < \psi_{k+1}^{-1}(\xi) < \psi_i^{-1}(\xi)\}} \partial_{xx} v_{k+1}^\lambda(\xi, \psi_i^{-1}(\xi)) \right. \\ &\quad \left. + \mathbf{1}_{\{\psi_{k+1}^{-1}(\xi) < \psi_{i+1,k}^{-1}(\xi)\}} \left\{ \partial_{xx} v_{k+1}(\xi, \psi_i^{-1}(\xi)) - \partial_{xx} v_{k+1}(\xi, \psi_{i+1,k}^{-1}(\xi)) \right\} \right] d\xi \\ &= \int \varphi(\xi) \mathbf{1}_{\{\psi_i^{-1}(\xi) > \psi_{i+1,k}^{-1}(\xi)\}} \left[\mathbf{1}_{\{\psi_{i+1,k}^{-1}(\xi) < \psi_{k+1}^{-1}(\xi) < \psi_i^{-1}(\xi)\}} \partial_{xx} v_{k+1}^\lambda(\xi, \psi_{k+1}^{-1}(\xi)) \right. \\ &\quad \left. + \mathbf{1}_{\{\psi_{i+1,k}^{-1}(\xi) < \psi_{k+1}^{-1}(\xi) < \psi_i^{-1}(\xi)\}} \left\{ \partial_{xx} v_{k+1}^\lambda(\xi, \psi_i^{-1}(\xi)) - \partial_{xx} v_{k+1}^\lambda(\xi, \psi_{k+1}^{-1}(\xi)) \right\} \right. \\ &\quad \left. + \mathbf{1}_{\{\psi_{k+1}^{-1}(\xi) < \psi_{i+1,k}^{-1}(\xi)\}} \left\{ \partial_{xx} v_{k+1}(\xi, \psi_i^{-1}(\xi)) - \partial_{xx} v_{k+1}(\xi, \psi_{i+1,k}^{-1}(\xi)) \right\} \right] d\xi \end{aligned}$$

Putting together the two last terms, we see that:

$$\begin{aligned} A &= \int \varphi(\xi) \mathbf{1}_{\{\psi_i^{-1}(\xi) > \psi_{i+1,k}^{-1}(\xi)\}} \left[\mathbf{1}_{\{\psi_{i+1,k}^{-1}(\xi) < \psi_{k+1}^{-1}(\xi) < \psi_i^{-1}(\xi)\}} \partial_{xx} v_{k+1}^\lambda(\xi, \psi_{k+1}^{-1}(\xi)) \right. \\ &\quad \left. + \mathbf{1}_{\{\psi_{k+1}^{-1}(\xi) < \psi_i^{-1}(\xi)\}} \left\{ \partial_{xx} v_{k+1}^\lambda(\xi, \psi_i^{-1}(\xi)) - \partial_{xx} v_{k+1}^\lambda(\xi, \psi_{i+1,k+1}^{-1}(\xi)) \right\} \right] d\xi \end{aligned}$$

Finally, using the ODE (5.15) satisfied by ψ_{k+1} in the first term, together with the expression of $\gamma_{k+1}(\psi_{k+1}, \cdot)$ from Lemma 7.1, we see that

$$\begin{aligned} A &= - \int \varphi(\psi_{k+1}(m)) \frac{\varphi(\psi_{k+1}(m))}{\psi_{k+1}(m) - m} \mathbf{1}_{\{\psi_i(m) < \psi_{k+1}(m) = \bar{\psi}_{k+1}(m)\}} dm \\ &\quad + \int \varphi(\xi) [\partial_{xx} v_{k+2}^\lambda(\xi, \psi_i^{-1}(\xi)) - \partial_{xx} v_{k+2}^\lambda(\xi, \psi_{i+1,k+2}^{-1}(\xi))] \mathbf{1}_{\{\psi_i^{-1}(\xi) > \psi_{i+1,k+1}^{-1}(\xi)\}} d\xi, \end{aligned}$$

which is precisely the required expression in order to justify that (7.3) holds for $k + 1$. \square

Proof of Lemma 5.3 (ii) By an induction argument on the line of the previous proof of item (i), we see that:

$$\begin{aligned} \int_{\psi_i(X_0)}^{X_0} c_0 \partial_{xx} v_i(\cdot, X_0) = & - \sum_{j=i+1}^k \int_0^{X_0} \frac{c_0(\psi_j(m))}{m - \psi_j(m)} \mathbf{1}_{\{\psi_i(X_0) < \psi_j(m) = \bar{\psi}_{i+1}(m)\}} \phi'(m) dm \\ & + \int c_0(\xi) \mathbf{1}_{\{\psi_{i+1,k}^{-1}(\xi) < X_0 < \psi_i^{-1}(\xi)\}} [\partial_{xx} v_k(\xi, X_0) - \partial_{xx} v_k(\xi, \psi_{i+1,k}^{-1}(\xi))], \end{aligned} \quad (7.5)$$

where we denoted, as in the previous proof, $\psi_{j,k}^{-1} := \psi_j^{-1} \vee \dots \vee \psi_k^{-1}$ for $j \leq k$. The required result follows by taking $k = n$ in (7.5). \square

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References

- [1] Avellaneda, M., Levy, A., and Parás, A. (1995), *Pricing and hedging derivative securities in markets with uncertain volatilities*, *Applied Math. Finance* 2(2), 73–88.
- [2] Azéma, J. and Yor, M. (1979), *Une solution simple au problème de Skorokhod*, *Séminaire de Probabilités XIII*, 90–115, LNM 721.
- [3] Azéma, J. and Yor, M. (1979), *Le problème de Skorokhod: complément à l'exposé précédent*, *Séminaire de Probabilités XIII*, 625–633, LNM 721.
- [4] Beiglböck, M., Henry-Labordère, P. and Penkner, F. (2011), *Model independent Bounds for Option Prices: A Mass Transport Approach*. Preprint, arXiv:1106.5929.
- [5] Benamou, J. D. and Brenier, Y. (2000), *A computational fluid mechanics solution to the Monge Kantorovich mass transfer problem*, *Numer. Math.* 84, 375–393.
- [6] Black, F., and Scholes, M. (1973), *The pricing of options and corporate liabilities*, *Journal of Political Economy*, 81(3), 637–654.
- [7] Bonnans, F. and Tan, X. (2011), *A model-free no-arbitrage price bound for variance options*, to appear in *Applied Mathematics and Optimization*.

- [8] Breeden, D.T. and Litzenberger, R.H. (1978), *Prices of state-contingent claims implicit in option prices*, *J. Business* 51, 621–651.
- [9] Brown, H., Hobson, D. and Rogers, L.C.G. (1998), *The maximum maximum of a martingale constrained by an intermediate law*, *Probability Theory and Related Fields* 119(4), 558–578.
- [10] Brown, H., Hobson, D. and Rogers, L.C.G. (2001), *Robust hedging of barrier options*, *Mathematical Finance* 11(3), 285–314.
- [11] Cox, A. and Hobson, D. (2005), *Local martingales, bubbles and option prices*, *Finance and Stochastics* 9(4), 477–492.
- [12] Cox, A. M. G., Hobson, D. and Obłój, J. (2008), *Pathwise inequalities for local time: applications to Skorokhod embeddings and optimal stopping*, *Ann. Appl. Probab.* 18 (5), 1870–1896.
- [13] Cox, A. and Obłój, J. (2011), *Robust pricing and hedging of double no-touch options*, *Finance and Stochastics* 15(3), 573–605.
- [14] Cox, A. and Obłój, J. (2011), *Robust pricing and hedging of double touch options*, *SIAM Journal on Financial Mathematics* 2, 141–182.
- [15] Cox, A. and Wang, J. (2013), *Root’s barrier: Construction, optimality and applications to variance options*, *Annals of Applied Probability* 23(3), 859–894.
- [16] Denis, L. and Martini, C. (2006), *A Theoretical Framework for the Pricing of Contingent Claims in the Presence of Model Uncertainty*, *Annals of Applied Probability* 16(2), 827–852.
- [17] Dolinsky, J. and Soner, M. (2012), *Martingale optimal transport and robust hedging in continuous time*. Preprint, arXiv:1208.4922.
- [18] Föllmer, H. and Schied, A. and Weber, S. (2009), *Robust Preferences and Robust Portfolio Choice* in *Handbook of Numerical Analysis* (Ciarlet, P.G., Ed.), Elsevier, 29–87.
- [19] Galichon, A., Henry-Labordère, P., Touzi, N. (2011), *A stochastic control approach to no-arbitrage bounds given marginals, with an application to Lookback options*, to appear in *Annals of Applied Probability*.
- [20] Hobson, D. (1998), *Robust hedging of the Lookback option*, *Finance and Stochastics* 2, 329–347.

- [21] Hobson, D. (2010), *The Skorokhod Embedding Problem and Model-Independent Bounds for Option Price* in *Paris-Princeton Lectures on Mathematical Finance*, Springer, LNM 2003.
- [22] Hobson, D. and Klimmek, M. (2010), *Maximising functionals of the joint law of the maximum and terminal value in the Skorokhod embedding problem*, arXiv:1012.3909, to appear in *Annals of Applied Probability*.
- [23] Hobson, D. and Neuberger, A. (2012), *Robust bounds for forward start options*, *Mathematical Finance*, 22, 31–56.
- [24] Jarrow, R., Protter, P. and Shimbo, K. (2010), *Asset price bubbles in incomplete markets*, *Mathematical Finance* 20, 145–185.
- [25] Knight, F. (1921), *Risk, Uncertainty and Profit*, Boston: Houghton Mifflin.
- [26] Lyons, T. J. (1995), *Uncertain volatility and the risk-free synthesis of derivatives*, *Applied Math Finance* 2(2), 117–133.
- [27] Madan, D.B, Yor, M. (2002), *Making Markov martingales meet marginals: with explicit constructions*, *Bernoulli* 8(4), 509–536.
- [28] Merton, R. C. (1973), *Theory of rational option pricing*, *The Bell Journal of Economics and Management Science* 4(1),141–183.
- [29] Nutz, M. (2012). *Pathwise Construction of Stochastic Integrals*. *Electronic Communications in Probability* 17(24), 1–7.
- [30] Neufeld, A. and Nutz, M. (2012), *Superreplication under Volatility Uncertainty for Measurable Claims*. Preprint, arXiv:1208.6486.
- [31] Oblój, J. (2004), *The Skorokhod embedding problem and its offspring*, *Probab. Surv.* 1, 321–390 (electronic).
- [32] Oblój, J. (2010), *Skorokhod Embedding* in *Encyclopedia of Quantitative Finance* (R. Cont, ed.), 1653–1657, Wiley.
- [33] Oblój, J. and Spoida, P. (2013), *An Iterated Azéma-Yor Type Embedding for Finitely Many Marginals*. Preprint, arXiv:1304.0368.
- [34] Peng, S. (2007), *G-Brownian motion and dynamic risk measure under volatility uncertainty*, Lecture notes, arXiv:0711.2834v1.

- [35] Possamai, D., Royer, G. and Touzi, N. (2013). *On the Robust Superhedging of Measurable Claims*. Preprint, arXiv:1302.1850.
- [36] Samuelson, P. (1965), *Rational theory of warrant pricing*, *Industrial Management Rev.* 6, 13–31.
- [37] Soner, H. M., Touzi, N. and Zhang, J. (2009), *Quasi-sure Stochastic Analysis through Aggregation*, arXiv:1003.4431v1. To appear in *Electronic Journal of Probability*.
- [38] Strassen, V. (1965), *The existence of probability measures with given marginals*. *Ann. Math. Statist.* 36, 423–439.
- [39] Tan, X. and Touzi, N. (2011), *Dual Formulation of the optimal transportation problem under controlled stochastic dynamics*. *Annals of Probability*, to appear.
- [40] Wagner, D. (1977), *Survey of measurable selection theorems*, *SIAM J. Control Optim.* 15(5), 859–903.