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STOCHASTIC CONTROL PROBLEMS, VISCOSITY SOLUTIONS, AND APPLICATION TO FINANCE

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Contents

Introduction

These notes have been prepared for the *Special Research Semester on Financial Markets*, which was held in Pisa, Italy, from April 29 to July 15, 2002.

The lectures were organized into six sessions of two hours each. Unfortunately, I was not able to provide all the information contained in these notes. In particular, I had no time to even start the last chapter on gamma constraints, which contains many open problems. I hope that these notes will motivate some people to make some progress on this problem.

I would like to thank all participants to these lectures. It was a pleasure for me to share my experience on this subject with the excellent audience that was offered by this special research semester. Special thanks go to Maurizion Prattelli for his excellent organization, his permanent availability, and his warm hospitality. I am also grateful to Patrick Cheridito for a careful reading of a preliminary version of these notes.

The general topic of these lectures is the Hamilton-Jacobi-Bellman approach to stochastic control problems, with applications to finance. In the first lecture, I introduced the classical *standard* class of stochastic control problems, the associated *dynamic programming principle*, and the resulting HJB equation describing the local behavior of the value function of the control problem. Throughout this first introduction to HJB equation, the value function is assumed to be as smooth as required.

The second lecture was dedicated to the verification theorem with two applications. First, the classical Merton portfolio selection problem, which was the starting point of the use of stochastic control techniques in the financial literature. As a second application, we present a recent result on the law of iterated logarithm for double stochastic integrals, which is needed in the problem of hedging under gamma constraints of Section 4.

The regularity issue was discussed in the third lecture. I first established the continuity of the value function when the controls take values in a bounded domain, then I provided some examples proving that, in general, one should not expect more regularity (in the classical sense). This motivated the need for a weak notion of solution of the HJB equation : the theory of viscosity solutions.

In the next lecture, I showed how the HJB equation can now be written rigorously in the viscosity sense, without any regularity assumption on the value function. I put a special emphasis on the fact that these proofs are only slight modifications of the proofs in the smooth case.

The remaining part of the lecture focuses on the problem of super-replicating some given European contingent claim in a Markov diffusion model, under portfolio constraints. This is a very popular problem in finance which, unfortunately, does not fit in the class of standard control problems treated in the first part of these notes. However, one can derive a dual formulation of this problem, which turns out to be a standard control problem with unbounded controls set. Control problems with controls taking values in an unbounded set are said to be *singular*. This is the contain of the fifth lecture. The last lecture uses the results of the first sections to derive the HJB equation satisfied by the super-replication value (in the viscosity sense), and studies precisely the terminal condition. The main results exhibit the so-call *face-lifting* phenomenon in the context of the Black and Scholes model.

1 Stochastic control problems and the associated Hamilton-Jacobi-Bellman equation

1.1 Stochastic control problems in standard form

Throughout these notes, $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a filtered probability space with filtration $\mathbb{F} = \{\mathcal{F}_t,\ t \geq 0\}$ satisfying the usual conditions. Let $W = \{W_t,\ t \geq 0\}$ be a Brownian motion valued in $\mathbb{R}^d$, defined on $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

Control processes. Given a subset U of $\mathbb{R}^k$, we denote by $\mathcal{U}_0$ the set of all progressively measurable processes $\nu = \{\nu_t, t \geq 0\}$ valued in U. The elements of $\mathcal{U}_0$ are called control processes.

Controlled Process. Let

$$b \ : \ (t,x,u) \in \mathbb{R}_+ \times \mathbb{R}^n \times U \quad \longrightarrow \quad b(t,x,u) \in \mathbb{R}^n$$

and

$$\sigma \ : \ (t,x,u) \in \mathbb{R}_+ \times \mathbb{R}^n \times U \quad \longrightarrow \quad \sigma(t,x,u) \in \mathcal{M}_{\mathbb{R}}(n,d)$$

be two given functions satisfying the uniform Lipschitz condition

$$|b(t,x,u) - b(t,y,u)| + |\sigma(t,x,u) - \sigma(t,y,u)| \quad \leq \quad K\,|x-y|\ , \qquad (1.1)$$

for some constant K independent of (t,x,y,u). For each control process $\nu \in \mathcal{U}$, we consider the state stochastic differential equation :

$$dX_t \quad = \quad b(t,X_t,\nu_t)dt + \sigma(t,X_t,\nu_t)dW_t \qquad (1.2)$$

If the above equation has a unique solution X, for a given initial data, then the process X is called the controlled process, as his dynamics is driven by the action of the control process ν.

Admissible control processes. Let $T > 0$ be some given time horizon. We shall denote by $\mathcal{U}$ the subset of all control processes $\nu \in \mathcal{U}_0$ which satisfy the additional requirement :

$$E\int_0^T \Big(|b(t,x,\nu_t)| + |\sigma(t,x,\nu_t)|^2\Big)\,dt \;<\; \infty \quad \text{for} \quad x \in I\!R^n\;. \qquad (1.3)$$

This condition guarantees the existence of a controlled process for each given initial condition and control, under the above uniform Lipschitz condition on the coefficients b and σ. This is a consequence of a more general existence theorem for stochastic differential equations with random coefficients, see e.g. Protter [21].

Theorem 1.1 *Let Condition* (1.1) *hold. Then, for each $\mathcal{F}_0$ random variable $\xi \in L^2(\Omega)$, there exists a unique $I\!F-$adapted process X satisfying* (1.2) *together with the initial condition $X_0 = \xi$. Moreover, we have*

$$E\left[\sup_{0\le s\le t} |X_s|^2\right] \;<\; \infty\;. \qquad (1.4)$$

<u>*Cost functional*</u>. Let

$$f, k \;:\; [0,T]\times I\!R^n \times U \;\longrightarrow\; I\!R \quad \text{and} \quad g \;:\; I\!R^n \;\longrightarrow\; I\!R$$

be given functions. We assume that $\|k^-\|_\infty < \infty$ (i.e. $\max(-k, O)$ is uniformly bounded), and f and g satisfy the quadratic growth condition :

$$|f(t,x,u)| + |g(x)| \;\le\; C(1+|x|^2) \quad \text{for some constant} \quad C \text{ independent of } (t,u)\;,$$

We define the cost function J on $[0,T]\times I\!R^n \times \mathcal{U}$ by :

$$J(t,x,\nu) \;:=\; E_{t,x}\left[\int_t^T \beta(t,s) f(s,X_s,\nu_s)ds + \beta(t,T)g(X_T)\right]$$

with

$$\beta(t,s) \;:=\; e^{-\int_t^s k(r,X_r,\nu_r)dr}\;.$$

Here $E_{t,x}$ is the expectation operator conditional on $X_t = x$, and X is the solution of the SDE 1.2 with control ν and initial condition $X_t = x$.

Observe that the quadratic growth condition on f and g together with the bound on k^- ensure that $J(t,x,\nu)$ is well-defined for all admissible controls $\nu \in \mathcal{U}$, as a consequence of Theorem 1.1.

The stochastic control problem. The purpose of this section is to study the minimization problem

$$V(t,x) \;:=\; \inf_{\nu\in\mathcal{U}} J(t,x,\nu) \quad \text{for} \quad (t,x) \in [0,T) \times I\!R^n \;.$$

The main concern of this section is to describe the local behavior of the value function V by means of the so-called *Hamilton-Jacobi-Bellman* equation, and to see under which circumstances V is characterized by its local behavior.

We conclude this section by some remarks.

1. Although the cost function $J(t,x,\nu)$ may depend on the information preceding time t, the value function $V(t,x)$ depends only on the present information (t,x) at time t. We refer to Hausmann (1983) or ElKaroui, Jeanblanc and N'guyen (1983) for the proof of this deep result.

2. If $V(t,x) = J(t,x,\hat{\nu}_{t,x})$, we call $\hat{\nu}_{t,x}$ an *optimal control* for the problem $V(t,x)$.

3. The following are some interesting subsets of controls :

- a process $\nu \in \mathcal{U}$ which is adapted to the natural filtration $I\!F^X$ of the associated state process is called *feedback control*,

- a process $\nu \in \mathcal{U}$ which can be written in the form $\nu_s = \tilde{u}(s,X_s)$ for some measurable map $\tilde{u}$ from $[0,T]\times I\!R^n$ into U, is called *Markovian control*; notice that any Markovian control is a feedback control,

- the deterministic processes of $\mathcal{U}$ are called *open loop controls.*

4. Let (Y,Z) be the controlled processes defined by

$$dY_s \;=\; Z_s f(s,X_s,\nu_s)ds \quad \text{and} \quad dZ_s \;=\; -Z_s k(s,X_s,\nu_s)ds \;,$$

and define the augmented state process $\bar{X} := (X,Y,Z)$. Then, the above value function V can be written in the form :

$$V(t,x) \;=\; \bar{V}(t,x,0,1)\ ,$$

where $\bar{x} = (x,y,z)$ is some initial data for the augmented state process $\bar{X}$,

$$\bar{V}(t,\bar{x}) \;:=\; E_{t,\bar{x}}\left[\bar{g}(\bar{X}_T)\right] \quad \text{and} \quad \bar{g}(x,y,z) \;:=\; y + g(x)z\ .$$

Hence the stochastic control problem V can be reduced without loss of generality to the case where $f = k \equiv 0$. We shall appeal to this reduced form whenever convenient for the exposition.

1.2 The dynamic programming principle

The dynamic programming principle is the main tool in the theory of stochastic control. A rigorous proof of this result is beyond the scope of these notes, as it appeals to delicate measurable selection arguments.

Theorem 1.2 *Let $(t,x) \in [0,T) \times I\!R^n$ be given. Then, for every stopping time θ valued in $[t,T]$, we have*

$$V(t,x) \;=\; \inf_{\nu\in\mathcal{U}} E_{t,x}\left[\int_t^\theta \beta(t,s)f(s,X_s,\nu_s)ds + \beta(t,\theta)V(\theta,X_\theta)\right]\ . \tag{1.5}$$

Before sketching the proof of this result, let us make some comments.
1. In the discrete-time framework, the dynamic programming principle can be stated as follows :

$$V(t,x) \;=\; \inf_{u\in U} E_{t,x}\left[f(t,X_t,u) + e^{-k(t+1,X_{t+1},\nu_{t+1})}V(t+1,X_{t+1})\right]\ .$$

Observe that the infimum is now taken over the subset U of the finite dimensional space R^k. Hence, the dynamic programming principle allows to reduce the initial minimization problem, over the subset $\mathcal{U}$ of the infinite dimensional set of $I\!R^k-$valued processes, into a finite dimensional minimization problem. However, we are still facing an infinite dimensional problem since the dynamic programming principle relates the value function at time t to the value function at time $t+1$.

2. In the context of the above discrete-time framework, notice that the dynamic programming principle suggests the following backward algorithm to compute V as well as the associated optimal strategy (when it exists). Since $V(T,\cdot)=g$ is known, the above dynamic programming principle can be applied recursively in order to deduce the value function $V(t,x)$ for every t.

3. Back to the continuous time setting. There is no counterpart to the above backward algorithm. But, as the stopping time θ approaches t, the above dynamic programming principle implies a special local behavior for the value function V. When V is known to be smooth, this will be obtained by means of Itô's lemma.

4. It is usually very difficult to determine *a priori* the regularity of V. The situation is even worse since there are many counter-examples showing that the value function V can not be expected to be smooth in general; see Section 1.5. This problem is solved by appealing to the notion of viscosity solutions, which provides a weak local characterization of the value function V.

5. Once the local behavior of the value function is characterized, we are faced to the important uniqueness issue, which implies that V is completely characterized by its local behavior together with some convenient boundary condition.

Sketch of the proof of Theorem 1.2. Let $\tilde{V}(t,x)$ denote the right hand-side of (1.5).

By the tower Property of the conditional expectation operator, it is easily checked that

$$J(t,x,\nu) \;=\; E_{t,x}\left[\int_t^\theta \beta(t,s)f(s,X_s,\nu_s)ds+\beta(t,\theta)J(\theta,X_\theta,\nu)\right] .$$

Since $J(\theta, X_\theta, \nu) \geq V(\theta, X_\theta)$, this proves that $V \geq \tilde{V}$. To prove the reverse inequality, let $\mu \in \mathcal{U}$ and $\varepsilon > 0$ be fixed, and consider an $\varepsilon-$optimal control ν^ε for the problem $V(\theta, X_\theta)$, i.e.

$$J(\theta, X_\theta, \nu^\varepsilon) \quad \leq \quad V(\theta, X_\theta) + \varepsilon \; .$$

Clearly, one can choose $\nu^\varepsilon = \mu$ on the stochastic interval $[t, \theta]$. Then

$$\begin{aligned} V(t,x) \quad &\leq \quad J(t,x,\nu^\varepsilon) \;=\; E_{t,x}\left[\int_t^\theta \beta(t,s) f(s, X_s, \mu_s) ds + \beta(t,\theta) J(\theta, X_\theta, \nu^\varepsilon)\right] \\ &\leq \quad E_{t,x}\left[\int_t^\theta \beta(t,s) f(s, X_s, \mu_s) ds + \beta(t,\theta) V(\theta, X_\theta)\right] + \varepsilon E_{t,x}[\beta(t,\theta)] \; . \end{aligned}$$

This provides the required inequality by the arbitrariness of $\mu \in \mathcal{U}$ and $\varepsilon > 0$.

□

Exercise. Where is the gap in the above sketch of the proof ?

1.3 The Hamilton-Jacobi-Bellman equation

In this paragraph, we introduce the Hamilton-Jacobi-Bellman equation by deriving it from the dynamic programming principle under smoothness assumptions on the value function. Let $H : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ ($\mathcal{S}^n$ is the set of all $n \times n$ symmetric matrices with real coefficients) be defined by :

$$\begin{aligned} &H(t,x,r,p,A) \\ :=\; &\inf_{u \in U} \left\{ -k(t,x,u) r + b(t,x,u)' p + \frac{1}{2} \mathrm{Tr}[\sigma\sigma'(t,x,u) A] + f(t,x,u) \right\} \; , \end{aligned}$$

where prime denotes transposition. We also need to introduce the linear second order operator $\mathcal{L}^u$ associated to the controlled process $\{\beta(0,t) X_t, t \geq 0\}$ controlled by the constant control process u :

$$\begin{aligned} \mathcal{L}^u \varphi(t,x) \quad := \quad & -k(t,x,u)\varphi(t,x) + b(t,x,u)' D\varphi(t,x) \\ & + \frac{1}{2} \mathrm{Tr}[\sigma\sigma'(t,x,u) D^2 \varphi(t,x)] \; , \end{aligned}$$

where D and D^2 denote the gradient and the Hessian operator with respect to the x variable. With this notation, we have by Itô's lemma

$$\begin{aligned}\beta^\nu(0,s)\varphi(s,X_s^\nu)-\beta^\nu(0,t)\varphi(t,X_t^\nu) &= \int_t^s \beta^\nu(0,r)\left(\frac{\partial}{\partial t}+\mathcal{L}^{\nu_r}\right)\varphi(r,X_r^\nu)dr \\ &\quad + \int_t^s \beta^\nu(0,r)D\varphi(r,X_r^\nu)'\sigma(r,X_r^\nu,\nu_r)dW_r\end{aligned}$$

for every smooth function $\varphi \in C^{1,2}([0,T],\mathbb{R}^n)$ and each admissible control process $\nu \in \mathcal{U}$.

Proposition 1.1 *Assume the value function $V \in C^{1,2}([0,T),\mathbb{R}^n)$, and let the coefficients $k(\cdot,\cdot,u)$ and $f(\cdot,\cdot,u)$ be continuous in (t,x) for all fixed $u \in U$. Then, for all $(t,x) \in [0,T) \times \mathbb{R}^n$:*

$$\frac{\partial V}{\partial t}(t,x)+H\left(t,x,V(t,x),DV(t,x),D^2V(t,x)\right) \;\geq\; 0 \tag{1.6}$$

Proof. Let $(t,x) \in [0,T)\times\mathbb{R}^n$ and $u \in U$ be fixed and consider the constant control process $\nu = u$, together with the associated state process X with initial data $X_t = x$. For all $h > 0$, Define the stopping time :

$$\theta_h \;:=\; \inf\{s > t \;:\; (s-t, X_s - x) \notin [0,h) \times \alpha B\} \;,$$

where $\alpha > 0$ is some given constant, and B denotes the unit ball of $\mathbb{R}^n$. Notice that $\theta_h \longrightarrow t$ as $h \searrow 0$ and $\theta_h = h$ for $h \leq \bar{h}(\omega)$ sufficiently small.
1. From the dynamic programming principle, it follows that :

$$\begin{aligned} 0 &\geq E_{t,x}\left[\beta(0,t)V(t,x)-\beta(0,\theta_h)V(\theta_h,X_{\theta_h})-\int_t^{\theta_h}\beta(0,r)f(r,X_r,\nu_r)dr\right] \\ &= -E_{t,x}\left[\int_t^{\theta_h}\beta(0,r)(V_t+\mathcal{L}^{\cdot}V+f)(r,X_r,u)dr\right] \\ &\quad - E_{t,x}\left[\int_t^s \beta(0,r)DV(r,X_r)'\sigma(r,X_r,u)dW_r\right] \;, \end{aligned}$$

where V_t denotes the partial derivative with respect to t; the last equality follows from Itô's lemma and uses the crucial smoothness assumption on V.

2. Observe that $\beta(0,r)DV(r,X_r)'\sigma(r,X_r,u)$ is bounded on the stochastic interval $[t,\theta_h]$. Therefore, the second expectation on the right hand-side of the last inequality vanishes, and we obtain :

$$-E_{t,x}\left[\frac{1}{h}\int_t^{\theta_h}\beta(0,r)(V_t+\mathcal{L}^{\cdot}V+f)(r,X_r,u)dr\right] \;\leq\; 0$$

We now send h to zero. The a.s. convergence of the random value inside the expectation is easily obtained by the mean value Theorem; recall that $\theta_h = h$ for sufficiently small $h > 0$. Since the random variable $h^{-1}\int_t^{\theta_h}\beta(0,r)(\mathcal{L}^{\cdot}V+f)(r,X_r,u)dr$ is essentially bounded, uniformly in h, on the stochastic interval $[t,\theta_h]$, it follows from the dominated convergence theorem that :

$$-\frac{\partial V}{\partial t}(t,x) - \mathcal{L}^u V(t,x) - f(t,x,u) \;\leq\; 0\;,$$

which is the required result, since $u \in U$ is arbitrary. $\Box$

We next wish to show that V satisfies the nonlinear partial differential equation (1.6) with equality. This is a more technical result which can be proved by different methods. We shall report a proof, based on a contradiction argument, which provides more intuition on this result, although it might be slightly longer than the usual proof reported in standard textbooks.

Proposition 1.2 *Assume the value function $V \in C^{1,2}([0,T),I\!R^n)$, and let the function H be continuous, and $\|k^+\|_\infty < \infty$. Then, for all $(t,x) \in [0,T)\times I\!R^n$:*

$$\frac{\partial V}{\partial t}(t,x) + H\Big(t,x,V(t,x),DV(t,x),D^2V(t,x)\Big) \;\leq\; 0 \tag{1.7}$$

Proof. Let $(t_0,x_0) \in [0,T)\times I\!R^n$ be fixed, assume to the contrary that

$$\frac{\partial V}{\partial t}(t_0,x_0) + H\Big(t_0,x_0,V(t_0,x_0),DV(t_0,x_0),D^2V(t_0,x_0)\Big) \;>\; 0\;, \tag{1.8}$$

and let us work towards a contradiction.

1. For a given parameter $\varepsilon > 0$, define the smooth function $\varphi \leq V$ by

$$\varphi(t,x) \ := \ V(t,x) - \varepsilon\frac{1}{2}|x-x_0|^2 .$$

Then

$$(V-\varphi)(t_0,x_0) = 0, \quad (DV - D\varphi)(t_0,x_0) = 0, \quad (V_t - \varphi_t)(t_0,x_0) = 0, \\ \text{and} \ \ (D^2V - D^2\varphi)(t_0,x_0) = \varepsilon I_n ,$$

where I_n is the $n \times n$ identity matrix. By continuity of H, it follows from (1.8) that

$$h(t_0,x_0) \ := \ \frac{\partial\varphi}{\partial t}(t_0,x_0) + H\left(t_0,x_0,\varphi(t_0,x_0),D\varphi(t_0,x_0),D^2\varphi(t_0,x_0)\right) \ > \ 0$$

for a sufficiently small $\varepsilon > 0$.

2. For $\eta > 0$, define the open neighborhood of (t_0,x_0) :

$$\mathcal{N}_\eta \ := \ \{(t,x) \ : \ (t-t_0,x-x_0) \in (-\eta,\eta) \times \eta B \text{ and } h(t,x) > 0\} ,$$

and observe that

$$2\gamma e^{\eta\|k^+\|_\infty} \ := \ \min_{\partial\mathcal{N}_\eta}(V-\varphi) \ = \ \frac{\varepsilon}{2}\min_{\partial\mathcal{N}_\eta}|x-x_0|^2 \ > \ 0 . \qquad (1.9)$$

Next, let $\tilde{\nu}$ be a $\gamma-$optimal control for the problem $V(t_0,x_0)$, i.e.

$$J(t_0,x_0,\tilde{\nu}) \ \leq \ V(t_0,x_0) + \gamma . \qquad (1.10)$$

We shall denote by $\tilde{X}$ and $\tilde{\beta}$ the controlled process and the discount factor defined by $\tilde{\nu}$ and the initial data $\tilde{X}_{t_0} = x_0$.

3. Consider the stopping time

$$\theta \ := \ \inf\left\{s > t \ : \ (s,\tilde{X}_s) \notin \mathcal{N}_\eta\right\} ,$$

and observe that, by continuity of the state process, $(\theta,\tilde{X}_\theta) \in \partial\mathcal{N}_\eta$, so that :

$$(V-\varphi)(\theta,\tilde{X}_\theta) \ \geq \ 2\gamma e^{\eta\|k^+\|_\infty}$$

by (1.9). We now compute that :

$$\begin{aligned} & \tilde\beta(t_0,\theta)V(\theta,\tilde X_\theta) - \tilde\beta(t_0,t_0)V(t_0,x_0) \\ \geq & \int_{t_0}^{\theta} d[\tilde\beta(t_0,r)\varphi(r,\tilde X_r)] + 2\gamma e^{h\|k^+\|_\infty}\tilde\beta(t_0,\theta) \\ \geq & \int_{t_0}^{\theta} d[\tilde\beta(t_0,r)\varphi(r,\tilde X_r)] + 2\gamma \ . \end{aligned}$$

By Itô's lemma, this provides :

$$V(t_0,x_0) \ \leq \ E_{t_0,x_0}\left[\tilde\beta(t_0,\theta)V(\theta,\tilde X_\theta) - \int_{t_0}^{\theta}(\varphi_t + \mathcal{L}^{\tilde\nu_r}\varphi)(r,\tilde X_r)dr\right] - 2\gamma \ ,$$

where the "dW" integral term has zero mean, as its integrand is bounded on the stochastic interval $[t_0,\theta]$. Observe also that $(\varphi_t + \mathcal{L}^{\tilde\nu_r}\varphi)(r,\tilde X_r) + f(r,\tilde X_r,\tilde\nu_r) \geq h(r,\tilde X_r) \geq 0$ on the stochastic interval $[t_0,\theta]$. We therefore deduce that :

$$\begin{aligned} V(t_0,x_0) \ &\leq \ -2\gamma + E_{t_0,x_0}\left[\int_{t_0}^{\theta}\tilde\beta(t_0,r)f(r,\tilde X_r,\tilde\nu_r) + \tilde\beta(t_0,\theta)V(\theta,\tilde X_\theta)\right] \\ &\leq \ -2\gamma + J(t_0,x_0,\tilde\nu) \\ &\leq \ V(t_0,x_0) - \gamma \ , \end{aligned}$$

where the last inequality follows by (1.10). This completes the proof. □

As a consequence of Propositions 1.1 and 1.2, we have the main result of this section :

Theorem 1.3 *Let the conditions of Propositions 1.1 and 1.2 hold. Then, the value function V solves the Hamilton-Jacobi-Bellman equation*

$$\frac{\partial V}{\partial t}(t,x) + H\left(t,x,V(t,x),DV(t,x),D^2V(t,x)\right) \ = \ 0 \qquad (1.11)$$

on $[0,T)\times I\!R^n$.

1.4 Solving a control problem by verification

In this paragraph, we provide a first answer towards the uniqueness problem. Namely, given a smooth solution v of the Hamilton-Jacobi-Bellman equation, we give sufficient conditions which allow to conclude that v coincides with the value function V. This is the so-called *verification result*. The statement of this result is heavy, but its proof is simple and relies essentially on Itô's lemma. We conclude this section by two examples of application of the verification theorem.

1.4.1 The verification theorem

Theorem 1.4 *Let v be a $C^{1,2}([0,T),\mathbb{R}^n) \cap C([0,T] \times \mathbb{R}^n)$ function. Assume that $\|k^-\|_\infty < \infty$ and v and f have quadratic growth, i.e. there is a constant C such that*

$$|f(t,x,u)| + |v(t,x)| \;\le\; C(1+|x|^2) \quad \textit{for all} \quad (t,x,u) \in [0,T) \times \mathbb{R}^n \times U \;.$$

(i) *Suppose that $v(T,\cdot) \le g$ and*

$$\frac{\partial v}{\partial t}(t,x) + H\Big(t,x,v(t,x),Dv(t,x),D^2v(t,x)\Big) \;\ge\; 0$$

on $[0,T) \times \mathbb{R}^n$. Then $v \le V$ on $[0,T] \times \mathbb{R}^n$.
(ii) *Assume further that $v(T,\cdot) = g$, and there exists a minimizer $\hat{u}(t,x)$ of $u \longmapsto \mathcal{L}^u v(t,x) + f(t,x,u)$ such that*

$$\begin{aligned} 0 &= \frac{\partial v}{\partial t}(t,x) + H\Big(t,x,v(t,x),Dv(t,x),D^2v(t,x)\Big) \\ &= \frac{\partial v}{\partial t}(t,x) + \mathcal{L}^{\hat{u}(t,x)}v(t,x) + f(t,x,u) \;, \end{aligned}$$

the stochastic differential equation

$$dX_s \;=\; b\left(s,X_s,\hat{u}(s,X_s)\right)ds + \sigma\left(s,X_s,\hat{u}(s,X_s)\right)dW_s$$

defines a unique solution X for each given initial date $X_t = x$, and the process $\hat{\nu}_s := \hat{u}(s,X_s)$ is a well-defined control process in $\mathcal{U}$.

Then $v = V$, and $\hat{\nu}$ is an optimal Markov control process.

Proof. Let $\nu \in \mathcal{U}$ be an arbitrary control process, X the associated state process with initial date $X_t = x$, and define the stopping time

$$\theta_n \;:=\; T \wedge \inf\{s > t \;:\; |X_s - x| \geq n\}\ .$$

By Itô's lemma, we have

$$\begin{aligned} v(t,x) &= \beta(t,\theta_n) v\left(\theta_n, X_{\theta_n}\right) - \int_t^{\theta_n} \beta(t,r)(v_t + \mathcal{L}^{\nu(r)} v)(r, X_r)dr \\ &\quad - \int_t^{\theta_n} \beta(t,r) Dv(r,X_r)\sigma(r,X_r,\nu_r)dW_r \end{aligned}$$

Observe that $v_t + \mathcal{L}^u v + f(\cdot,\cdot,u) \geq v_t + H(\cdot,\cdot,v,Dv,D^2v) \geq 0$, and that the integrand in the stochastic integral is bounded on $[t,\theta_n]$, a consequence of the continuity of Dv, σ and the condition $\|k^-\|_\infty < \infty$. Then :

$$v(t,x) \;\leq\; E\left[\beta(t,\theta_n) v\left(\theta_n, X_{\theta_n}\right) + \int_t^{\theta_n} \beta(t,r) f(r,X_r,\nu_r)dr\right]\ . \qquad (1.12)$$

We now take the limit as n increases to infinity. Since $\theta_n \longrightarrow T$ a.s. and

$$\begin{aligned} &\left|\beta(t,\theta_n) v\left(\theta_n, X_{\theta_n}\right) + \int_t^{\theta_n} \beta(t,r) f(r,X_r,\nu_r)dr\right| \\ &\quad\leq\; Ce^{T\|k^-\|_\infty}(1 + |X_{\theta_n}|^2 + T + \textstyle\int_t^T |X_s|^2 ds) \\ &\quad\leq\; Ce^{T\|k^-\|_\infty}(1+T)(1 + \sup_{t\leq s\leq T}\ |X_s|^2) \;\in\; L^1\ , \end{aligned}$$

by the estimate (1.4) of Theorem 1.1, it follows from the dominated convergence that

$$\begin{aligned} v(t,x) &\leq\; E\left[\beta(t,T) v(T,X_T) + \int_t^T \beta(t,r) f(r,X_r,\nu_r)dr\right] \\ &\leq\; E\left[\beta(t,T) g(X_T) + \int_t^T \beta(t,r) f(r,X_r,\nu_r)dr\right]\ , \end{aligned}$$

where the last inequality uses the condition $v(T,\cdot) \leq g$. Since the control $\nu \in \mathcal{U}$ is arbitrary, this completes the proof of (i).

Statement (ii) is proved by repeating the above argument and observing that the control $\hat{\nu}$ achieves equality at the crucial step (1.12). $\square$

Remark 1.1 When U is reduced to a singleton, the optimization problem V is degenerate. In this case, the HJB equation is linear, and the verification theorem reduces to the so-called *Feynman-Kac formula*.

We conclude this section by a discussion of the existence of a classical solution to the HJB equation. The verification theorem assumes the existence of such a solution, and is by no means an existence result. However, it provides uniqueness in the class of function with quadratic growth.

We now state without proof an existence result for the HJB equation together with the terminal condition $V(T, \cdot) = g$ (see [18] for the detailed proof). The main assumption is the so-called *uniform parabolicity* condition :

$$\begin{array}{c} \text{there is a constant } c > 0 \text{ such that} \\ \xi' \ \sigma\sigma'(t, x, u) \ \xi \ \geq \ c|\xi|^2 \text{ for all } (t, x, u) \in [0, T] \times I\!R^n \times U \ . \end{array} \tag{1.13}$$

In the following statement, we denote by $C_b^k(I\!R^n)$ the space of bounded functions whose partial derivatives of orders $\leq k$ exist and are bounded continuous. We similarly denote by $C_b^{p,k}([0, T], I\!R^n)$ the space of bounded functions whose partial derivatives with respect to t, of orders $\leq p$, and with respect to x, of order $\leq k$, exist and are bounded continuous.

Theorem 1.5 *Let Condition 1.13 hold, and assume further that :*
- *U is compact;*
- *b, σ and f are in $C_b^{1,2}([0, T], I\!R^n)$;*
- *$g \in C_b^3(I\!R^n)$.*

Then the HJB equation (1.11) *with the terminal data $V(T, \cdot) = g$ has a unique solution $V \in C_b^{1,2}([0, T] \times I\!R^n)$.*

1.4.2 Application 1 : optimal portfolio allocation

We now apply the verification theorem to a classical example in finance, which was introduced by Merton [19], and generated a huge literature since then.

Consider a financial market consisting of a non-risky asset S^0 and a risky one S. The dynamics of the price processes are given by

$$dS_t^0 \;=\; S_t^0 r dt \quad \text{and} \quad dS_t \;=\; S_t[\mu dt + \sigma dW_t] \; .$$

Here, r, μ and σ are some given positive constants, and W is a one-dimensional Brownian motion.

The investment policy is defined by an $\mathbb{F}-$adapted process $\pi = \{\pi_t, t \in [0,T]\}$, where π_t represents the proportion of wealth invested in the risky asset at time t; The remaining $(1-\pi_t)$ proportion of wealth is invested in the risky asset. Therefore, the wealth process satisfies

$$\begin{aligned} dX_t &= \pi_t X_t \frac{dS_t}{S_t} + (1-\pi_t) X_t \frac{dS_t^0}{S_t^0} \\ &= X_t \left[(r + (\mu - r)\pi_t)\, dt + \sigma \pi_t dW_t \right] \; . \end{aligned} \tag{1.14}$$

Such a process π is said to be admissible if

$$E\left[\int_0^T |\pi_t|^2 dt\right] \;<\; \infty \; .$$

We denote by $\mathcal{U}$ the set of all admissible portfolios. Observe that, in view of the particular form of our controlled process X, this definition agrees with (1.3).

Let γ be an arbitrary parameter in $(0,1)$ and define the *power utility function* :

$$U(x) \;:=\; x^\gamma \quad \text{for} \quad x \geq 0 \; .$$

The parameter γ is called the relative risk premium coefficient.

The objective of the investor is to choose an allocation of his wealth so as to maximize the expected utility of his terminal wealth, i.e.

$$V(t,x) \;:=\; \sup_{\pi \in \mathcal{U}} E_{t,x}\left[U(X_T)\right] \; .$$

The HJB equation associated with this problem is :

$$\frac{\partial w}{\partial t}(t,x) + \sup_{u \in \mathbb{R}} \mathcal{L}^u w(t,x) \;=\; 0 \; , \tag{1.15}$$

where $\mathcal{L}^u$ is the second order linear operator :

$$\mathcal{L}^u w(t,x) \ := \ (r+(\mu-r)u)x \,\frac{\partial w}{\partial x}(t,x) + \frac{1}{2}\sigma^2 u^2 x^2 \,\frac{\partial^2 w}{\partial x^2}(t,x) \ .$$

From the definition of X in (1.14), we see that :

$$X_s \ = \ X_t \exp\left(r(s-t) + (\mu-r)\int_t^s \pi_\tau d\tau - \frac{1}{2}\sigma^2 \int_t^s \pi_\tau^2 d\tau + \sigma \int_t^s \pi_\tau dW_\tau \right)$$

so that

$$E_{t,x}\left[U(X_T)\right] \ = \ x^\gamma E_{t,1}\left[U(X_T)\right] \quad \text{and} \quad V(t,x) \ = \ x^\gamma V(t,1) \ .$$

Set $h(t) := V(t,1)$, and plug the above separability property of V in (1.15). The result is the following ordinary differential equation on h :

$$0 \ = \ h' + \gamma h \sup_{u\in I\!R} \left\{ r + (\mu-r)u + \frac{1}{2}(\gamma-1)\sigma^2 u^2 \right\} \tag{1.16}$$

$$= \ h' + \gamma h \left[r + \frac{1}{2}\frac{(\mu-r)^2}{(1-\gamma)\sigma^2} \right] \ , \tag{1.17}$$

where the maximizer is :

$$\hat{u} \ := \ \frac{\mu - r}{(1-\gamma)\sigma^2} \ .$$

Since $V(T,\cdot) = U(x)$, we seek for a function h satisfying the above ordinary differential equation together with the boundary condition $h(T) = 1$. This allows to select a unique candidate for the function h :

$$h(t) \ := \ e^{a(T-t)} \quad \text{with} \quad a \ := \ \gamma \left[r + \frac{1}{2}\frac{(\mu-r)^2}{(1-\gamma)\sigma^2} \right] \ .$$

Hence, the function $(t,x) \longmapsto x^\gamma h(t)$ is a classical solution of the HJB equation (1.15). It is easily checked that the conditions of Theorem 1.4 are all satisfied in this context. Then $V(t,x) = x^\gamma h(t)$, and the optimal portfolio allocation policy is given by the constant process :

$$\hat{\pi}_t \ := \ \hat{u} \ = \ \frac{\mu - r}{(1-\gamma)\sigma^2} \ .$$

1.4.3 Application 2 : the law of iterated logarithm for double stochastic integrals

The main object of this paragraph is Theorem 1.6 below, reported from [6], which describes the local behavior of double stochastic integrals near the starting point zero. This result will be needed in the problem of hedging under gamma constraints which will be discussed later in these notes. An interesting feature of the proof of Theorem 1.6 is that it relies on a verification argument. However, the problem does not fit exactly in the setting of Theorem 1.4. Therefore, this is an iteresting exercise on the verification concept.

Given a bounded predictable process b, we define the processes

$$Y_t^b \ := \ Y_0 + \int_0^t b_r dW_r \quad \text{and} \quad Z_t^b \ := \ Z_0 + \int_0^t Y_r^b dW_r \,, \quad t \geq 0 \,,$$

where Y_0 and Z_0 are some given initial data in $\mathbb{R}$.

Lemma 1.1 *Let λ and T be two positive parameters with $2\lambda T < 1$. Then :*

$$E\left[e^{2\lambda Z_T^b}\right] \ \leq \ E\left[e^{2\lambda Z_T^1}\right] \quad \textit{for each predictable process } b \textit{ with } \ \|b\|_\infty \ \leq \ 1 \,.$$

Proof. We split the argument into three steps.
1. We first directly compute that

$$E\left[e^{2\lambda Z_T^1}\Big|\,\mathcal{F}_t\right] \quad = \quad v(t, Y_t^1, Z_t^1) \ ,$$

where, for $t \in [0, T]$, and $y, z \in \mathbb{R}$, the function v is given by :

$$\begin{aligned} v(t,y,z) \quad &:= \quad E\left[\exp\left(2\lambda\left\{z + \int_t^T (y + W_u - W_t)\, dW_u\right\}\right)\right] \\ &= \quad e^{2\lambda z} E\left[\exp\left(\lambda\{2yW_{T-t} + W_{T-t}^2 - (T-t)\}\right)\right] \\ &= \quad \mu \exp\left[2\lambda z - \lambda(T-t) + 2\mu^2\lambda^2(T-t)y^2\right] \,, \end{aligned}$$

where $\mu := [1 - 2\lambda(T-t)]^{-1/2}$. Observe that

$$\text{the function } v \text{ is strictly convex in } y, \tag{1.18}$$

and

$$yD^2_{yz}v(t,y,z) \;=\; 8\mu^2\lambda^3(T-t)\; v(t,y,z)\; y^2 \;\geq\; 0 \;. \tag{1.19}$$

2. For an arbitrary real parameter β, we denote by $\mathcal{L}^\beta$ the Dynkin operator associated to the process $\left(Y^b, Z^b\right)$:

$$\mathcal{L}^\beta \;:=\; D_t + \frac{1}{2}\beta^2 D^2_{yy} + \frac{1}{2}y^2 D^2_{zz} + \beta y D^2_{yz} \;.$$

In this step, we intend to prove that for all $t \in [0,T]$ and $y,z \in I\!R$:

$$\max_{|\beta|\leq 1} \mathcal{L}^\beta v(t,y,z) \;=\; \mathcal{L}^1 v(t,y,z) \;=\; 0 \;. \tag{1.20}$$

The second equality follows from the fact that $\{v(t, Y^1_t, Z^1_t),\ t \leq T\}$ is a martingale . As for the first equality, we see from (1.18) and (1.19) that 1 is a maximizer of both functions $\beta \longmapsto \beta^2 D^2_{yy}v(t,y,z)$ and $\beta \longmapsto \beta y D^2_{yz}v(t,y,z)$ on $[-1,1]$.

3. Let b be some given predictable process valued in $[-1,1]$, and define the sequence of stopping times

$$\tau_k \;:=\; T \wedge \inf\left\{t \geq 0 \;:\; (|Y^b_t| + |Z^b_t| \geq k\right\} \;,\quad k \in I\!N \;.$$

By Itô's lemma and (1.20), it follows that :

$$\begin{aligned}
v(0,Y_0,Z_0) \;&=\; v\left(\tau_k, Y^b_{\tau_k}, Z^b_{\tau_k}\right) - \int_0^{\tau_k} [bD_y v + yD_z v]\left(t, Y^b_t, Z^b_t\right) dW_t \\
&\quad - \int_0^{\tau_k} \mathcal{L}^{b_t} v\left(t, Y^b_t, Z^b_t\right) dt \\
&\geq\; v\left(\tau_k, Y^b_{\tau_k}, Z^b_{\tau_k}\right) - \int_0^{\tau_k} [bD_y v + yD_z v]\left(t, Y^b_t, Z^b_t\right) dW_t \;.
\end{aligned}$$

Taking expected values and sending k to infinity, we get by Fatou's lemma :

$$\begin{aligned}
v(0,Y_0,Z_0) \;&\geq\; \liminf_{k\to\infty}\; E\left[v\left(\tau_k, Y^b_{\tau_k}, Z^b_{\tau_k}\right)\right] \\
&\geq\; E\left[v\left(T, Y^b_T, Z^b_T\right)\right] \;=\; E\left[e^{2\lambda Z^b_T}\right] \;,
\end{aligned}$$

which proves the lemma. □

We are now able to prove the law of the iterated logarithm for double stochastic integrals by a direct adaptation of the case of the Brownian motion. Set

$$h(t) \;:=\; 2t \log\log \frac{1}{t} \quad \text{for} \quad t > 0 \; .$$

Theorem 1.6 *Let b be a predictable process valued in a bounded interval $[\beta_0, \beta_1]$ for some real parameters $0 \le \beta_0 < \beta_1$, and $X^b_t := \int_0^t \int_0^u b_v dW_v dW_u$. Then :*

$$\beta_0 \;\le\; \limsup_{t \searrow 0} \frac{2X^b_t}{h(t)} \;\le\; \beta_1 \qquad a.s.$$

Proof. We first show that the first inequality is an easy consequence of the second one. Set $\bar\beta := (\beta_0 + \beta_1)/2 \ge 0$, and set $\delta := (\beta_1 - \beta_0)/2$. By the law of the iterated logarithm for the Brownian motion, we have

$$\bar\beta \;=\; \limsup_{t \searrow 0} \frac{2X^{\bar\beta}_t}{h(t)} \;\le\; \delta \; \limsup_{t \searrow 0} \frac{2X^{\tilde b}_t}{h(t)} + \limsup_{t \searrow 0} \frac{2X^b_t}{h(t)} \; ,$$

where $\tilde b := \delta^{-1}(\bar\beta - b)$ is valued in $[-1, 1]$. It then follows from the second inequality that :

$$\limsup_{t \searrow 0} \frac{2X^b_t}{h(t)} \;\ge\; \bar\beta - \delta \;=\; \beta_0 \; .$$

We now prove the second inequality. Clearly, we can assume with no loss of generality that $\|b\|_\infty \le 1$. Let $T > 0$ and $\lambda > 0$ be such that $2\lambda T < 1$. It follows from Doob's maximal inequality for submartingales that for all $\alpha \ge 0$,

$$\begin{aligned} P\left[\max_{0 \le t \le T} 2X^b_t \;\ge\; \alpha\right] \;&=\; P\left[\max_{0 \le t \le T} \exp(2\lambda X^b_t) \;\ge\; \exp(\lambda\alpha)\right] \\ &\le\; e^{-\lambda\alpha} E\left[e^{2\lambda X^b_T}\right] \; . \end{aligned}$$

In view of Lemma 1.1, this provides :

$$\begin{aligned} P\left[\max_{0 \le t \le T} 2X^b_t \;\ge\; \alpha\right] \;&\le\; e^{-\lambda\alpha} E\left[e^{2\lambda X^1_T}\right] \\ &=\; e^{-\lambda(\alpha + T)}(1 - 2\lambda T)^{-\frac{1}{2}} \; . \qquad (1.21) \end{aligned}$$

We have then reduced the problem to the case of the Brownian motion, and the rest of this proof is identical to the first half of the proof of the law of the iterated logarithm for the Brownian motion. Take θ, $\eta \in (0,1)$, and set for all $k \in I\!\!N$,

$$\alpha_k \;:=\; (1+\eta)^2 h(\theta^k) \quad \text{and} \quad \lambda_k \;:=\; [2\theta^k(1+\eta)]^{-1}\,.$$

Applying (1.21), we see that for all $k \in I\!\!N$,

$$P\left[\max_{0\le t\le\theta^k} 2X_t^b \;\ge\; (1+\eta)^2 h(\theta^k)\right] \;\le\; e^{-1/2(1+\eta)}\left(1+\eta^{-1}\right)^{\frac{1}{2}}(-k\log\theta)^{-(1+\eta)}\,.$$

Since $\sum_{k\ge 0} k^{-(1+\eta)} < \infty$, it follows from the Borel-Cantelli lemma that, for almost all $\omega \in \Omega$, there exists a natural number $K^{\theta,\eta}(\omega)$ such that for all $k \ge K^{\theta,\eta}(\omega)$,

$$\max_{0\le t\le\theta^k} 2X_t^b(\omega) \;<\; (1+\eta)^2 h(\theta^k)\,.$$

In particular, for all $t \in (\theta^{k+1}, \theta^k]$,

$$2X_t^b(\omega) \;<\; (1+\eta)^2 h(\theta^k) \;\le\; (1+\eta)^2\frac{h(t)}{\theta}\,.$$

Hence,

$$\limsup_{t\searrow 0} \frac{2X_t^b}{h(t)} \;<\; \frac{(1+\eta)^2}{\theta} \qquad \text{a.s.}$$

and the required result follows by letting θ tend to 1 and η to 0 along the rationals. □

1.5 On the regularity of the value function

The purpose of this paragraph is to show that the value function should not be expected to be smooth in general. We start by proving the continuity of the value function under strong conditions; in particular, we require the set U in which the controls take values to be bounded. We then give a simple example in the deterministic framework where the value function is not smooth. Since it is well known that stochastic problems are "more regular" than deterministic ones, we also give an example of stochastic control problem whose value function is not smooth.

1.5.1 Continuity of the value function for bounded controls

For notational simplicity, we reduce the stochastic control problem to the case $f = k \equiv 0$, see Remark 4 at the end of Section 1.1. Our main concern, in this section, is to show the standard argument for proving the continuity of the value function. Therefore, the following results assume strong conditions on the coefficients of the model in order to simplify the proofs. We first start by examining the value function $V(t,\cdot)$ for fixed $t \in [0,T]$.

Proposition 1.3 *Let $f = k \equiv 0$, and assume that g is Lipschitz continuous. Then $V(t,\cdot)$ is Lipschitz-continuous for all $t \in [0,T]$.*

Proof. We shall denote here by $X^\nu_{t,x}(\cdot)$ the process controlled by $\nu \in \mathcal{U}$ and starting from the initial date $X^\nu_{t,x}(t) = x$. For $x_1, x_2 \in I\!R^n$ and $\nu \in \mathcal{U}$, we first estimate that :

$$\begin{aligned} |V(t,x_1) - V(t,x_2)| &\leq \sup_{\nu\in\mathcal{U}} E\left|g\left(X^\nu_{t,x_1}(T)\right) - g\left(X^\nu_{t,x_2}(T)\right)\right| \\ &\leq Const \sup_{\nu\in\mathcal{U}} E\left|X^\nu_{t,x_1}(T) - X^\nu_{t,x_2}(T)\right| \qquad (1.22) \end{aligned}$$

by the Lipschitz-continuity of g. Set $h(t) := E\left|X^\nu_{t,x_1}(T) - X^\nu_{t,x_2}(T)\right|^2$. Then, from the dynamics of the state process, we see that :

$$\begin{aligned} h(T) &\leq Const \left\{ |x_1 - x_2|^2 \right. \\ &\qquad + E\int_t^T \left|b\left(r, X^\nu_{t,x_1}(r), \nu_r\right) - b\left(r, X^\nu_{t,x_2}(r), \nu_r\right)\right|^2 dr \\ &\qquad \left. + E\int_t^T \left|\sigma\left(r, X^\nu_{t,x_1}(r), \nu_r\right) - \sigma\left(r, X^\nu_{t,x_2}(r), \nu_r\right)\right|^2 dr \right\} \\ &\leq Const \left\{ |x_1 - x_2|^2 + \int_t^T h(r)dr \right\} , \end{aligned}$$

where the last inequality follows from the Lipschitz-continuity of b and σ in the x variable, uniformly in (t,u). By the Gronwall Lemma, this provides the estimate $h(t)^2 \leq Const\ |x_1 - x_2|^2$, which provides the required result by going back to (1.22). $\Box$

We next turn to the continuity in the t variable. This requires that the set U, in which the controls take values, be bounded.

Proposition 1.4 *Let $f = k \equiv 0$, and assume that*
- *g is Lipschitz-continuous.*
- *U is bounded,*
- *the coefficients b and σ are continuous in (t,x,u), and Lipschitz in $x \in I\!\!R^n$ uniformly in $(t,u) \in [0,T] \times U$.*

Then $V(\cdot, x)$ is $(1/2)-$Hölder continuous for all $x \in I\!\!R^n$.

Proof. Let $0 \leq t < s \leq T$ be fixed. By the dynamic programming principle, we have :

$$\begin{aligned} |V(t,x) - V(s,x)| &= \left| \inf_{\nu \in \mathcal{U}} E_{t,x}\left[V(s,X_s)\right] - V(s,x) \right| \\ &\leq \sup_{\nu \in \mathcal{U}} E_{t,x}\left|V(s,X_s) - V(s,x)\right| \ . \end{aligned}$$

By the Lipschitz-continuity of $V(s,\cdot)$ established in Proposition 1.3, we see that :

$$|V(t,x) - V(s,x)| \leq Const \sup_{\nu \in \mathcal{U}} E_{t,x}|X_s - x| \ . \tag{1.23}$$

We shall now prove that

$$\sup_{\nu \in \mathcal{U}} E_{t,x}|X_s - x| \leq Const\ |s-t|^{1/2} \ , \tag{1.24}$$

which provides the required $(1/2)-$Hölder continuity in view of (1.23). By definition of the process X, we have

$$\begin{aligned} E_{t,x}|X_s - x|^2 &= E_{t,x}\left| \int_t^s b(r,X_r,\nu_r)dr + \int_t^s \sigma(r,X_r,\nu_r)dW_r \right|^2 \\ &\leq Const\ E_{t,x}\left[\int_t^s |h(r,X_r,\nu_r)|^2\, dr \right] \end{aligned}$$

where $h := [b^2 + \sigma^2]^{1/2}$ is continuous in (t, x, u) and Lipschitz in x uniformly in (t, u). Then

$$E_{t,x}|X_s - x|^2 \quad \leq \quad Const \left((s - t) + \int_t^s E_{t,x}|X_r - x|^2 dr \right) ,$$

which provides the estimation

$$E_{t,x}|X_s - x|^2 \quad \leq \quad Const\ |s - t| \tag{1.25}$$

by Gronwall's lemma, and (1.24) follows. □

Remark 1.2 When f and/or k are non-zero, the conditions required on f and k in order to obtain the $(1/2)-$Hölder continuity of the value function can be deduced from the reduction of Remark 4 at the end of Section 1.1.

Remark 1.3 Further regularity results can be proved for the value function under convenient conditions. Typically, one can prove that $\mathcal{L}^u V$ exists in the generalized sense, for all $u \in U$. This implies immediately that the result of Proposition 1.1 holds in the generalized sense. More technicalities are needed in order to derive the result of Proposition 1.2 in the generalized sense. We refer to [14], §IV.10, for a discussion of this issue.

1.5.2 A deterministic control problem with non-smooth value function

Let $\sigma \equiv 0$, $b(x, u) = u$, $U = [-1, 1]$, and $n = 1$. The controlled state is then the one-dimensional deterministic process defined by :

$$X_s \;=\; X_t + \int_t^s \nu_t dt \quad \text{for} \quad 0 \leq t \leq s \leq T\ .$$

Consider the deterministic control problem

$$\begin{aligned} V(t, x) \quad &:= \quad \sup_{\nu \in \mathcal{U}} (X_T)^2 \\ &= \quad \sup_{\nu \in \mathcal{U}} \left(x + \int_t^T \nu_t dt \right)^2 . \end{aligned}$$

The value function of this problem is easily seen to be given by :

$$V(t,x) \;=\; \begin{cases} (x+T-t)^2 & \text{for } x \geq 0 \;\; \text{with optimal control } \hat{u}=1\ , \\ (x-T+t)^2 & \text{for } x \leq 0 \;\; \text{with optimal control } \hat{u}=-1\ . \end{cases}$$

This function is continuous. However, a direct computation shows that it is not differentiable at $x = 0$.

1.5.3 A stochastic control problem with non-smooth value function

Let $U = I\!R$, and the controlled process $X = (Y, Z)$ be the $I\!R^2-$valued process defined by the dynamics :

$$dY_t \;=\; Z_t\sqrt{2}dW_t^1 \quad \text{and} \quad dZ_t \;=\; \nu_t dt + \sqrt{2}dW_t^2\ ,$$

where $W = (W^1, W^2)$ is a standard Brownian motion valued in $I\!R^2$. Let g be a non-negative lower semicontinuous mapping on $I\!R$, and consider the stochastic control problem

$$V(t,x) \;:=\; \sup_{\nu\in\mathcal{U}} E_{t,x}\left[g(Y_T)\right]\ .$$

Let us assume that V is smooth, and work towards a contradiction. In order to apply the results developed in this chapter, we have to amount to a minimization problem by simply working with $-V$.

1. If V is $C^{1,2}([0,T), I\!R^2)$, then it follows from Proposition 1.1 that V satisfies

$$-\frac{\partial V}{\partial t} - u\,\frac{\partial V}{\partial z} - z^2\frac{\partial^2 V}{\partial y^2} - \frac{\partial^2 V}{\partial z^2} \;\geq\; 0 \quad \text{for all} \quad u \;\in\; I\!R\ ,$$

and all $t \in [0,T)$ and $x = (y,z) \in I\!R^2$. From the arbitrariness of $u \in I\!R$, it follows that the function V is independent of the z variable, and therefore :

$$-\frac{\partial V}{\partial t}(t,y) - z^2\frac{\partial^2 V}{\partial y^2}(t,y) \;\geq\; 0 \quad \text{for all} \quad z \;\in\; I\!R\ , \tag{1.26}$$

and $(t,y) \in [0,T) \times I\!R$. Setting $z = 0$, we see that

$$V(\cdot, y) \quad \text{is non-increasing for all} \quad y \in I\!R \; . \tag{1.27}$$

Also, by sending z to infinity in (1.26), it follows that

$$V(t, \cdot) \quad \text{is concave for all} \quad t \in [0,T) \; . \tag{1.28}$$

2. Since g is non-negative, it is easily seen that

$$V(T-, y) \;:=\; \lim_{t \nearrow T} V(t,y,z) \;\geq\; g(y) \quad \text{for all} \quad (y,z) \in I\!R^2 \; . \tag{1.29}$$

This is an easy consequence of Fatou's lemma, the lower semicontinuity of g, and the continuity of Y_T in its initial condition y.

Now, it follows from (1.27) and (1.29) that :

$$V(t,y,z) \;=\; V(t,y) \;\geq\; V(T-,y) \;\geq\; g(y) \quad \text{for all} \quad (t,y,z) \in [0,T] \times I\!R^2 \; .$$

In view of (1.28), this proves that

$$V(t,y,z) \;=\; V(t,y) \;\geq\; g^{conc}(y) \quad \text{for all} \quad (t,y,z) \in [0,T] \times I\!R^2 \; , \tag{1.30}$$

where g^{conc} is the concave envelope of g, i.e. the smallest concave function whose graph lies above the graph of g.

3. Using the inequality $g \leq g^{conc}$ together with Jensen's inequality and the martingale property of Y, it follows that

$$\begin{aligned} V(t,y,z) \;&:=\; \sup_{\nu \in \mathcal{U}} \; E_{t,x}\left[g(Y_T)\right] \\ &\leq\; \sup_{\nu \in \mathcal{U}} \; E_{t,x}\left[g^{conc}(Y_T)\right] \\ &\leq\; \sup_{\nu \in \mathcal{U}} \; g^{conc}\left(E_{t,x}[(Y_T)]\right) \;=\; g^{conc}(y) \; . \end{aligned}$$

In view of (1.30), we have then proved that

$$\begin{aligned} & V \;\in\; C^{1,2}([0,T), I\!R^2) \\ \Longrightarrow \quad & V(t,y,z) \;=\; g^{conc}(y) \text{ for all } (t,y,z) \in [0,T) \times I\!R^2 \; . \end{aligned}$$

Now recall that this implication holds for any arbitrary non-negative lower semicontinuous function g. We then obtain a contradiction whenever the function g^{conc} is not $C^2(I\!R)$. Hence

$$g^{conc} \;\notin\; C^2(I\!R) \;\Longrightarrow\; V \;\notin\; C^{1,2}([0,T), I\!R^2)\;.$$

2 Stochastic control problems and viscosity solutions

2.1 Intuition behind viscosity solutions

We consider a non-linear second order partial differential equation

$$\text{(E)} \quad F\left(x, u(x), Du(x), D^2u(x)\right) \;=\; 0 \;\text{ for }\; x \in \mathcal{O}$$

where $\mathcal{O}$ is an open subset of $\mathbb{R}^n$ and F is a continuous map from $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \longrightarrow \mathbb{R}$. A crucial condition on F is the so-called *ellipticity* condition :

$$F(x,r,p,A) \;\le\; F(x,r,p,B) \quad \text{whenever} \quad A \;\ge\; B\;,$$

for all $(x,r,p) \in \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n$. The full importance of this condition will be made clear by Proposition 2.1 below.

The first step towards the definition of a notion of weak solution to (E) is the introduction of sub and supersolutions.

Definition 2.1 *A function $u : \mathcal{O} \longrightarrow \mathbb{R}$ is a classical supersolution (resp. subsolution) of (E) if $u \in C^2(\mathcal{O})$ and*

$$F\Big(x, u(x), Du(x), D^2u(x)\Big) \;\ge\; \textit{(resp. } \le\textit{)} \;\; 0 \;\; \textit{for} \;\; x \in \mathcal{O}\;.$$

The theory of viscosity solutions is motivated by the following result, whose simple proof is left to the reader.

Proposition 2.1 *Let u be a $C^2(\mathcal{O})$ function. Then the following claims are equivalents.*
(i) *u is a classical supersolution (resp. subsolution) of (E)*
(ii) *for all pairs $(x_0, \varphi) \in \mathcal{O} \times C^2(\mathcal{O})$ such that x_0 is a minimizer (resp. maximizer) of the difference $u - \varphi$) on $\mathcal{O}$, we have*

$$F\Big(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\Big) \;\ge \textit{(resp. } \le\textit{)} \;\; 0\;.$$

2.2 Definition of viscosity solutions

Before going any further, we need to introduce a new notation. For a locally bounded function $u : \mathcal{O} \longrightarrow I\!R$, we denote by u_* and u^* the lower and upper semicontinuous envelopes of u. We recall that u_* is the largest lower semicontinuous function below u, u^* is the smallest upper semicontinuous function above u, and

$$u_*(x) \;=\; \liminf_{x' \to x} u(x') \;, \qquad u^*(x) \;=\; \limsup_{x' \to x} u(x') \;.$$

We are now ready for the definition of viscosity solutions. Observe that Claim (ii) in the above proposition does not involve the regularity of u. It therefore suggests the following weak notion of solution to (E).

Definition 2.2 *Let $u : \mathcal{O} \longrightarrow I\!R$ be a locally bounded function.*
(i) *We say that u is a (discontinuous) viscosity supersolution of (E) if*

$$F\left(x_0, u_*(x_0), D\varphi(x_0), D^2\varphi(x_0)\right) \;\geq\; 0$$

for all pair $(x_0, \varphi) \in \mathcal{O} \times C^2(\mathcal{O})$ such that x_0 is a minimizer of the difference $(u_* - \varphi)$ on $\mathcal{O}$.
(ii) *We say that u is a (discontinuous) viscosity subsolution of (E) if*

$$F\left(x_0, u^*(x_0), D\varphi(x_0), D^2\varphi(x_0)\right) \;\leq\; 0$$

for all pair $(x_0, \varphi) \in \mathcal{O} \times C^2(\mathcal{O})$ such that x_0 is a maximizer of the difference $(u^* - \varphi)$ on $\mathcal{O}$.
(iii) *We say that u is a (discontinuous) viscosity solution of (E) if it is both a viscosity supersolution and subsolution of (E).*

Remark 2.1 Clearly, the above definition is not changed if the minimum or maximum are local and/or strict. Also, by a density argument, the test function can be chosen to be in $C^\infty(\mathcal{O})$.

In Section 2.6, we will show that the value function V is a viscosity solution of the HJB equation (1.11) under the conditions of Theorem 1.3 (except the smoothness assumption on V). We also want to emphasize that proving that the value function is a viscosity solution is almost as easy as proving that it is a classical solution under the assumption on V.

2.3 First properties

We now turn to two important properties of viscosity solutions : the change of variable formula and the stability result.

Proposition 2.2 *Let u be a locally bounded (discontinuous) viscosity super-solution of (E). If f is a $C^1(\mathbb{R})$ function with $Df \neq 0$ on $\mathbb{R}$, then the function $v := f^{-1} \circ u$ is a (discontinuous)*
- *viscosity super-solution, when $Df > 0$,*
- *viscosity subsolution, when $Df < 0$,*

of the equation

$$K(x, v(x), Dv(x), D^2v(x)) \;=\; 0 \quad \textit{for} \quad x \in \mathcal{O}\,,$$

where

$$K(x, r, p, A) \;:=\; F\left(x, f(r), Df(r)p, D^2f(r)pp' + Df(r)A\right)$$

We leave the easy proof of this proposition to the reader. The next result shows how limit operations with viscosity solutions can be performed very easily.

Proposition 2.3 *Let u_ε be a lower semicontinuous viscosity super-solution of the equation*

$$F_\varepsilon\left(x, Du_\varepsilon(x), D^2u_\varepsilon(x)\right) \;=\; 0 \quad \textit{for} \quad x \in \mathcal{O}\,,$$

where $(F_\varepsilon)_\varepsilon$ is a sequence of continuous functions satisfying the ellipticity condition. Suppose that $(\varepsilon, x) \longmapsto u_\varepsilon(x)$ and $(\varepsilon, z) \longmapsto F_\varepsilon(z)$ are locally *bounded, and define*

$$u_*(x) \ := \ \liminf_{(\varepsilon,x')\to(0,x)} u_\varepsilon(x') \quad \textit{and} \quad F^*(z) \ := \ \limsup_{(\varepsilon,z')\to(0,z)} F_\varepsilon(z') \ .$$

Then, u_* is a lower semicontinuous viscosity supersolution of the equation

$$F^*\Big(x, Du_*(x), D^2u_*(x)\Big) \ = \ 0 \quad \textit{for} \quad x \in \mathcal{O} \ .$$

A similar statement holds for subsolutions.

Proof. The fact that u_* is a lower semicontinuous function is left as an exercise for the reader. Let $\varphi \in C^2(\mathcal{O})$ and $\bar{x}$, be a strict minimizer of the difference $u_\varepsilon - \varphi$. By definition of u_*, there is a sequence $(\varepsilon_n, x_n) \in (0,1] \times \mathcal{O}$ such that

$$(\varepsilon_n, x_n) \ \longrightarrow \ (0, \bar{x}) \quad \text{and} \quad u_{\varepsilon_n}(x_n) \ \longrightarrow \ u_*(\bar{x}) \ .$$

Consider some $r > 0$ together with the closed ball $\bar{B}$ with radius r, centered at $\bar{x}$. Of course, we may choose $|x_n - \bar{x}| < r$ for all $n \geq 0$. Let $\bar{x}_n$ be a minimizer of $u_{\varepsilon_n} - \varphi$ on $\bar{B}$. We claim that

$$\bar{x}_n \ \longrightarrow \ \bar{x} \quad \text{as} \quad n \to \infty \ . \tag{2.1}$$

Before verifying this, let us complete the proof. We first deduce that $\bar{x}_n$ is an interior point of $\bar{B}$ for large n, so that $\bar{x}_n$ is a local minimizer of the difference $u_{\varepsilon_n} - \varphi$. Then :

$$F_{\varepsilon_n}\Big(\bar{x}_n, D\varphi(\bar{x}_n), D^2\varphi(\bar{x}_n)\Big) \ \geq \ 0 \ ,$$

and the required result follows by taking limits and using the definition of F^*.

It remains to prove Claim (2.1). Recall that $(x_n)_n$ is valued in the compact set $\bar{B}$. Then, there is a subsequence, still named $(x_n)_n$, which converges to some $\tilde{x} \in \bar{B}$. We only have to prove that $\tilde{x} = \bar{x}$. Using the fact that $\bar{x}_n$ is a minimizer of $u_{\varepsilon_n} - \varphi$ on $\bar{B}$, together with the definition of u_*, we see that

$$\begin{aligned} 0 \;=\; (u_* - \varphi)(\bar{x}) &= \lim_{n\to\infty} \left(u_{\varepsilon_n} - \varphi\right)(x_n) \\ &\geq \liminf_{n\to\infty} \left(u_{\varepsilon_n} - \varphi\right)(\bar{x}_n) \\ &\geq (u_* - \varphi)(\tilde{x}) \; . \end{aligned}$$

We now obtain (2.1) from the fact that $\bar{x}$ is a strict minimizer of the difference $(u_* - \varphi)$. $\square$

Observe that the passage to the limit in partial differential equations written in the classical or the generalized sense usually appeals to much more technicalities, as one has to ensure convergence of all the partial derivatives involved in the equation. The above stability result provides a general method to pass to the limit when the equation is written in the viscosity sense, and its proof turns out to be remarkably simple.

A possible application of the stability result is to establish the convergence of numerical schemes. In view of the simplicity of the above statement, the notion of viscosity solutions provides a nice framework for such a numerical issue. The reader interested in this issue can consult [3].

The main difficulty in the theory of viscosity is the interpretation of the equation in the viscosity sense. First, by weakening the notion of solution to the second order nonlinear PDE (E), we are enlarging the set of solutions, and one has to guarantee that uniqueness still holds (in some convenient class of functions). This issue will be discussed in the subsequent Section 2.4. We conclude this section by the following result whose proof is trivial in the classical case, but needs some technicalities when stated in the viscosity sense.

Proposition 2.4 *Let $A \subset I\!R^n$ and $B \subset I\!R^p$ be two open subsets, and let $u : A \times B \longrightarrow I\!R$ be a lower semicontinuous viscosity supersolution of the equation :*

$$F\left(x, y, u(x,y), D_y u(x,y), D_y^2 u(x,y)\right) \;\geq\; 0 \quad on \quad A \times B \; ,$$

where F is a continuous elliptic operator. Assume further that

$$r \longmapsto F(x, y, r, p, A) \qquad \text{is non-increasing.} \tag{2.2}$$

Then, for all fixed $x_0 \in A$, the function $v(y) := u(x_0, y)$ is a viscosity supersolution of the equation :

$$F\left(x_0, y, v(y), Dv(y), D^2v(y)\right) \;\geq\; 0 \quad \text{on} \quad B\,.$$

If u is continuous, the above statement holds without Condition (2.2).

A similar statement holds for the subsolution property.

Proof. Fix $x_0 \in A$, set $v(y) := u(x_0, y)$, and let $y_0 \in B$ and $f \in C^2(B)$ be such that

$$(v - f)(y_0) \;<\; (v - f)(y) \quad \text{for all} \quad y \in J \setminus \{y_0\}\,, \tag{2.3}$$

where J is an arbitrary compact subset of B containing y_0 in its interior. For each integer n, define

$$\varphi_n(x, y) \;:=\; f(y) - n|x - x_0|^2 \quad \text{for} \quad (x, y) \in A \times B\,,$$

and let (x_n, y_n) be defined by

$$(u - \varphi_n)(x_n, y_n) \;=\; \min_{I \times J}(u - \varphi_n)\,,$$

where I is a compact subset of A containing x_0 in its interior. We claim that

$$(x_n, y_n) \;\longrightarrow\; (x_0, y_0) \quad \text{as} \quad n \longrightarrow \infty\,. \tag{2.4}$$

Before proving this, let us complete the proof. Since (x_0, y_0) is an interior point of $A \times B$, it follows from the viscosity property of u that

$$\begin{aligned} 0 \;&\leq\; F\left(x_n, y_n, u(x_n, y_n), D_y\varphi_n(x_n, y_n), D_y^2\varphi_n(x_n, y_n)\right) \\ &=\; F\left(x_n, y_n, u(x_n, y_n), Df(y_n), D^2f(y_n)\right)\,, \end{aligned}$$

and the required result follows by sending n to infinity.

We now turn to the proof of (2.4). Since the sequence $(x_n, y_n)_n$ is valued in the compact subset $A \times B$, we have $(x_n, y_n) \longrightarrow (\bar{x}, \bar{y}) \in A \times B$, after passing to a subsequence. Observe that

$$\begin{aligned} u(x_n, y_n) - f(y_n) &\leq u(x_n, y_n) - f(y_n) + n|x_n - x_0|^2 \\ &= (u - \varphi_n)(x_n, y_n) \\ &\leq (u - \varphi_n)(x_0, y_0) \;=\; u(x_0, y_0) - f(y_0) \,. \end{aligned}$$

Taking the limits, it follows from the lower semicontinuity of u that

$$u(\bar{x}, \bar{y}) - f(\bar{y}) \;\leq\; u(\bar{x}, \bar{y}) - f(\bar{y}) + \liminf_{n \to \infty} n|x_n - x_0|^2 \;\leq\; u(x_0, y_0) - f(y_0) \,.$$

Then, we must have $\bar{x} = x_0$, and

$$(v - f)(\bar{y}) \;=\; u(x_0, \bar{y}) - f(\bar{y}) \;\leq\; (v - f)(y_0) \,,$$

which concludes the proof of (2.4) in view of (2.3). □

2.4 Comparison result and uniqueness

We first state, without proof, a general *comparison result* for second order non-linear equations, see [7].

Theorem 2.1 *Let $\mathcal{O}$ be an open bounded subset of $I\!R^N$ and let F be an elliptic operator satisfying*

(i) *there exists a constant $\gamma > 0$ such that for all $(x, p, M) \in \overline{\mathcal{O}} \times I\!R^N \times \mathcal{S}^n$,*

$$F(x, r, p, M) - F(x, s, p, M) \;\geq\; \gamma(r - s), \qquad r \geq s$$

(ii) *there exists a function $\omega : I\!R_+ \longrightarrow I\!R_+$ with $\omega(0+) = 0$ such that*

$$F(y, r, \alpha(x - y), N) - F(x, r, \alpha(x - y), M) \;\leq\; \omega\left(\alpha|x - y|^2 + |x - y|\right) \quad (2.5)$$

for all $x, y \in \overline{\mathcal{O}}$, $r \in I\!R^N$ and $(M, N, \alpha) \in \mathcal{S}^n \times \mathcal{S}^n \times I\!R^*_+$ satisfying :

$$-3\alpha I_{2n} \;\leq\; \begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix} \leq 3\alpha \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix} .$$

Let $\underline{U}$ be an upper-semicontinuous viscosity subsolution of (E), and $\overline{U}$ a lower-semicontinuous viscosity supersolution of (E). Then

$$\sup_{\bar{\mathcal{O}}}(\underline{U} - \overline{U}) \;=\; \sup_{\partial\mathcal{O}}(\underline{U} - \overline{U}) \; .$$

We list below two interesting examples of operators F which satisfy the conditions of the above theorem:

(i) $F(x, r, p, A) = \gamma r + H(p)$ for some continuous function $H \; : \mathbb{R}^n \longrightarrow \mathbb{R}$, and $\gamma > 0$.

(ii) $F(x, r, p, A) = -\mathrm{Tr}\,(\sigma\sigma'(x)A) + \gamma r$, where $\sigma : \mathbb{R}^n \longrightarrow \mathcal{S}^n$ is a Lipschitz function, and $\gamma > 0$. To see that Theorem 2.1 applies to this equation, we only need to check that Condition (ii) holds. So suppose that $(M, N, \alpha) \in \mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^*_+$ satisfy (2.5). We claim that

$$\mathrm{Tr}[AA'M - BB'N] \;\le\; 3\alpha|A - B|^2 \;=\; \sum_{i,j=1}^{n} (A - B)^2_{ij} \; .$$

To see this, observe that the matrix

$$C \;:=\; \begin{pmatrix} BB' & BA' \\ AB' & AA' \end{pmatrix}$$

is a non-negative matrix in $\mathcal{S}^n$. From the right hand-side inequality of (2.5), this implies that

$$\begin{aligned} \mathrm{Tr}[AA'M - BB'N] \;&=\; \mathrm{Tr}\left[C \begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix}\right] \\ &\le\; 3\alpha \mathrm{Tr}\left[C \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}\right] \\ &=\; 3\alpha \mathrm{Tr}\left[(A - B)(A' - B')\right] \;=\; 3\alpha|A - B|^2 \; . \end{aligned}$$

Remark 2.2 In the above example (i), the condition $\gamma > 0$ is not needed when H is a convex and $H(D\varphi(x)) \le \alpha < 0$ for some $\varphi \in C^1(\mathcal{O})$. This result can be found in [2].

We finally turn to time-evolution problems in unbounded domains defined by the equation

$$\frac{\partial u}{\partial t} + G\Big(t, x, Du(t,x), D^2u(t,x)\Big) \;=\; 0 \quad \text{on} \quad Q \;:=\; [0,T) \times I\!R^n \;, \quad (2.6)$$

where G is elliptic and continuous. For $\gamma > 0$, set

$$\begin{aligned} G^{+\gamma}(t,x,p,A) &:= \sup\left\{G(s,y,p,A) \;:\; (s,y) \in B_Q(t,x;\gamma)\right\} \;, \\ G^{-\gamma}(t,x,p,A) &:= \inf\left\{G(s,y,p,A) \;:\; (s,y) \in B_Q(t,x;\gamma)\right\} \;, \end{aligned}$$

where $B_Q(t,x;\gamma)$ is the collection of elements (s,y) in Q such that $|t-s|^2 + |x-y|^2 \le \gamma^2$. The following result is reported from [14] (Theorem V.8.1 and Remark V.8.1).

Theorem 2.2 *Suppose that*

$$\begin{aligned} &\limsup_{\varepsilon\searrow 0} \left\{G^{+\gamma_\varepsilon}(t_\varepsilon, x_\varepsilon, p_\varepsilon, A_\varepsilon) - G^{-\gamma_\varepsilon}(s_\varepsilon, y_\varepsilon, p_\varepsilon, B_\varepsilon)\right\} \\ &\le\; Const\,(|t_0 - s_0| + |x_0 - y_0|)\,[1 + |p_0| + \alpha\,(|t_0 - s_0| + |x_0 - y_0|)] \end{aligned} \quad (2.7)$$

for all sequences $(t_\varepsilon, x_\varepsilon)$, $(s_\varepsilon, y_\varepsilon) \in [0,T) \times I\!R^n$, $p_\varepsilon \in I\!R^n$, *and* $\gamma_\varepsilon \ge 0$ *with :*

$$((t_\varepsilon, x_\varepsilon), (s_\varepsilon, y_\varepsilon), p_\varepsilon, \gamma_\varepsilon) \;\longrightarrow\; ((t_0, x_0), (s_0, y_0), p_0, 0) \quad \text{as} \quad \varepsilon \searrow 0 \;,$$

and symmetric matrices $(A_\varepsilon, B_\varepsilon)$ *with*

$$-KI_{2n} \;\le\; \begin{pmatrix} A_\varepsilon & 0 \\ 0 & -B_\varepsilon \end{pmatrix} \;\le\; 2\alpha \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}$$

for some α *independent of* ε.
Let $\underline{U}$ *be an upper semicontinuous viscosity subsolution of* (2.6)*, and* $\overline{U}$ *a lower semicontinuous viscosity supersolution of* (2.6)*. Then*

$$\sup_{\bar{Q}}(\underline{U} - \overline{U}) \;=\; \sup_{I\!R^n}(\underline{U} - \overline{U})(T, \cdot)$$

A sufficient condition for (2.7) to hold is that $f(\cdot,\cdot,u)$, $k(\cdot,\cdot,u)$, $b(\cdot,\cdot,u)$, and $\sigma(\cdot,\cdot,u) \in C^1(\bar{Q})$ with

$$\begin{aligned} \|b_t\|_\infty + \|b_x\|_\infty + \|\sigma_t\|_\infty + \|\sigma_x\|_\infty &\;<\; \infty \\ |b(t,x,u)| + |\sigma(t,x,u)| &\;\le\; Const(1 + |x| + |u|) \;; \end{aligned}$$

see [14], Lemma V.8.1.

2.5 Useful applications

We conclude this section by two consequences of the above comparison results, which are trivial properties in the classical case.

Lemma 2.1 *Let $\mathcal{O}$ be an open interval of $\mathbb{R}$, and $U : \mathcal{O} \longrightarrow \mathbb{R}$ be a lower semicontinuous supersolution of the equation $DU \geq 0$ on $\mathcal{O}$. Then U is nondecreasing on $\mathcal{O}$.*

Proof. For each $\varepsilon > 0$, define $W(x) := U(x) + \varepsilon x$; $x \in \mathcal{O}$. Then W satisfies in the viscosity sense $DW \geq \varepsilon$ in $\mathcal{O}$, i.e. for all $(x_0, \varphi) \in \mathcal{O} \times C^1(\mathcal{O})$ such that

$$(W - \varphi)(x_0) \;=\; \min_{x \in \mathcal{O}} (W - \varphi)(x), \tag{2.8}$$

we have $D\varphi(x_0) \geq \varepsilon$. This proves that φ is strictly increasing in a neighborhood $\mathcal{V}$ of x_0. Let $(x_1, x_2) \subset \mathcal{V}$ be an open interval containing x_0. We intend to prove that

$$W(x_1) \;<\; W(x_2)\;, \tag{2.9}$$

which provides the required result from the arbitrariness of $x_0 \in \mathcal{O}$.

To prove (2.9), suppose to the contrary that $W(x_1) \geq W(x_2)$, and the consider the function $v(x) = W(x_2)$ which solves the equation

$$Dv \;=\; 0 \quad \text{on} \quad (x_1, x_2)\;.$$

together with the boundary conditions $v(x_1) = v(x_2) = W(x_2)$. Observe that W is a lower semicontinuous viscosity supersolution of the above equation. From the comparison theorem of Remark 2.2, this implies that

$$\sup_{[x_1,x_2]} (v - W) \;=\; \max\left\{(v - W)(x_1), (v - W)(x_2)\right\} \;\leq\; 0\;.$$

Hence $W(x) \geq v(x) = W(x_2)$ for all $x \in [x_1, x_2]$. Applying this inequality at $x_0 \in (x_1, x_2)$, and recalling that the test function φ is strictly increasing on $[x_1, x_2]$, we get :

$$(W - \varphi)(x_0) \;>\; (W - \varphi)(x_2),$$

contradicting (2.8). □

Lemma 2.2 *Let $\mathcal{O}$ be an open interval of $I\!R$, and $U : \mathcal{O} \longrightarrow I\!R$ be a lower semicontinuous supersolution of the equation $-D^2U \geq 0$ on $\mathcal{O}$. Then U is concave on $\mathcal{O}$.*

Proof. Let $a < b$ be two arbitrary elements in $\mathcal{O}$, and consider some $\varepsilon > 0$ together with the function

$$v(s) \quad := \quad \frac{U(a)\left[e^{\sqrt{\varepsilon}(b-s)}-1\right]+U(b)\left[e^{\sqrt{\varepsilon}(s-a)}-1\right]}{e^{\sqrt{\varepsilon}(b-a)}-1} \text{ for } a \leq s \leq b \ .$$

Clearly, v solves the equation

$$(\varepsilon v - D^2 v)(t,s) \quad = \quad 0 \text{ on } (a,b) \ .$$

Since U is lower semicontinuous it is bounded from below on the interval $[a,b]$. Therefore, by possibly adding a constant to U, we can assume that $U \geq 0$, so that U is a lower semicontinuous viscosity supersolution of the above equation. It then follows from the comparison theorem 2.2 that :

$$\sup_{[a,b]} (v-U) \quad = \quad \max\left\{(v-U)(a), (v-U)(b)\right\} \ \leq \ 0 \ .$$

Hence,

$$U(s) \quad \geq \quad v(s) \quad = \quad \frac{U(a)\left[e^{\sqrt{\varepsilon}(b-s)}-1\right]+U(b)\left[e^{\sqrt{\varepsilon}(s-a)}-1\right]}{e^{\sqrt{\varepsilon}(b-a)}-1} \ ,$$

and by sending ε to zero, we see that

$$U(s) \quad \geq \quad [U(b)-U(a)]\,\frac{s-a}{b-a} + U(a)$$

for all $s \in [a,b]$. Let λ be an arbitrary element of the interval [0,1], and set $s := \lambda a + (1-\lambda)b$. The last inequality takes the form :

$$U(\lambda a + (1-\lambda)b) \quad \geq \quad \lambda U(a) + (1-\lambda)U(b) \ ,$$

proving the concavity of U. $\square$

2.6 The HJB equation in the viscosity sense

We now turn to the stochastic control problem introduced in Section 1.1. The chief goal of this paragraph is to use the notion of viscosity solutions in order to relax the smoothness condition on the value function V in the statement of Propositions 1.1 and 1.2. Notice that the following proofs are obtained by slight modification of the corresponding proofs in the smooth case.

Remark 2.3 Recall that the general theory of viscosity applies for nonlinear partial differential equations on an open domain $\mathcal{O}$. This indeed ensures that the optimizer in the definition of viscosity solutions is an interior point. In the setting of control problems with finite horizon, the time variable moves forward so that the zero boundary is not relevant. We shall then write the Hamilton-Jacobi-Bellman equation on the domain $[0,T)\times I\!R^n$. Although this is not an open domain, the general theory of viscosity solutions is still valid.

Proposition 2.5 *Assume that V is locally bounded on $[0,T)\times I\!R^n$, and let the coefficients $k(\cdot,\cdot,u)$ and $f(\cdot,\cdot,u)$ be continuous in (t,x) for all fixed $u\in U$. Then, the value function V is a (discontinuous) viscosity subsolution of the equation*

$$-\frac{\partial V}{\partial t}(t,x) - H\left(t,x,V(t,x),DV(t,x),D^2V(t,x)\right) \;\le\; 0 \qquad (2.10)$$

on $[0,T)\times I\!R^n$.

Proof. Let $(t,x)\in Q:=[0,T)\times I\!R^n$ and $\varphi\in C^2(Q)$ be such that

$$0 \;=\; (V^*-\varphi)(t,x) \;=\; \max_Q\,(V^*-\varphi)\,. \qquad (2.11)$$

Let $(t_n,x_n)_n$ be a sequence in Q such that

$$(t_n,x_n) \;\longrightarrow\; (t,x) \quad\text{and}\quad V(t_n,x_n) \;\longrightarrow\; V^*(t,x)\,.$$

Since φ is smooth, notice that

$$\eta_n \;:=\; V(t_n,x_n)-\varphi(t_n,x_n) \;\longrightarrow\; 0\,.$$

Next, let $u \in U$ be fixed, and consider the constant control process $\nu = u$. We shall denote by X^n the associated state process with initial data $X^n_{t_n} = x_n$. Finally, for all $n > 0$, we define the stopping time :

$$\theta_n \;:=\; \inf\left\{s > t_n \;:\; (s - t_n, X^n_s - x_n) \not\in [0, h_n) \times \alpha B\right\} \;,$$

where $\alpha > 0$ is some given constant, B denotes the unit ball of $I\!R^n$, and

$$h_n \;:=\; \sqrt{\eta_n}\mathbf{1}_{\{\eta_n \neq 0\}} + n^{-1}\mathbf{1}_{\{\eta_n = 0\}} \;.$$

Notice that $\theta_n \longrightarrow t$ as $n \longrightarrow \infty$.

1. From the dynamic programming principle, it follows that :

$$0 \;\geq\; E_{t_n,x_n}\left[V(t_n, x_n) - \beta(t_n, \theta_n)V(\theta_n, X^n_{\theta_n}) - \int_{t_n}^{\theta_n} \beta(t_n, r) f(r, X^n_r, \nu_r)dr\right] \;.$$

Now, in contrast with the proof of Proposition 1.1, the value function is not known to be smooth, and therefore we can not apply Itô's lemma to V. The main trick of this proof is to use the inequality $V \leq V^* \leq \varphi$ on Q, implied by (2.11), so that we can apply Itô's lemma to the smooth test function φ :

$$\begin{aligned}
0 \;&\geq\; \eta_n + E_{t_n,x_n}\left[\varphi(t_n, x_n) - \beta(t_n, \theta_n)\varphi(\theta^n_h, X^n_{\theta_n}) - \int_{t_n}^{\theta_n} \beta(t_n, r) f(r, X^n_r, \nu_r)dr\right] \\
&=\; \eta_n - E_{t_n,x_n}\left[\int_{t_n}^{\theta_n} \beta(t_n, r)(\varphi_t + \mathcal{L}^{\cdot}\varphi - f)(r, X^n_r, u)dr\right] \\
&\quad - E_{t_n,x_n}\left[\int_{t_n}^{\theta_n} \beta(t_n, r)D\varphi(r, X^n_r)\sigma(r, X^n_r, u)dW_r\right] \;,
\end{aligned}$$

where φ_t denotes the partial derivative with respect to t.

2. We now continue exactly along the lines of the proof of Proposition 1.1. Observe that $\beta(t_n, r)D\varphi(r, X^n_r)\sigma(r, X^n_r, u)$ is bounded on the stochastic interval $[t_n, \theta_n]$. Therefore, the second expectation on the right hand-side of the last inequality vanishes, and :

$$\frac{\eta_n}{h_n} - E_{t_n,x_n}\left[\frac{1}{h_n}\int_{t_n}^{\theta_n} \beta(t_n, r)(\varphi_t + \mathcal{L}^{\cdot}\varphi - f)(r, X_r, u)dr\right] \;\leq\; 0 \;.$$

We now send n to infinity. The a.s. convergence of the random value inside the expectation is easily obtained by the mean value Theorem; recall that for $n \geq N(\omega)$ sufficiently large, $\theta_n(\omega) = h_n$. Since the random variable $h_n^{-1} \int_t^{\theta_n} \beta(t_n, r)(\mathcal{L}^{\cdot}\varphi - f)(r, X_r^n, u)dr$ is essentially bounded, uniformly in n, on the stochastic interval $[t_n, \theta_n]$, it follows from the dominated convergence theorem that :

$$-\frac{\partial \varphi}{\partial t}(t,x) - \mathcal{L}^u\varphi(t,x) - f(t,x,u) \quad \leq \quad 0 \; ,$$

which is the required result, since $u \in U$ is arbitrary. □

We next wish to show that V satisfies the nonlinear partial differential equation (2.10) with equality, in the viscosity sense. This is also obtained by a slight modification of the proof of Proposition 1.2.

Proposition 2.6 *Assume that the value function V is locally bounded on $[0,T) \times I\!R^n$. Let the function H be continuous, and $\|k^+\|_\infty < \infty$. Then, V is a (discontinuous) viscosity supersolution of the equation*

$$-\frac{\partial V}{\partial t}(t,x) - H\left(t, x, V(t,x), DV(t,x), D^2V(t,x)\right) \quad \geq \quad 0 \qquad (2.12)$$

on $[0,T) \times I\!R^n$.

Proof. Let $(t_0, x_0) \in Q := [0,T) \times I\!R^n$ and $\varphi \in C^2(Q)$ be such that

$$0 \;=\; (V_* - \varphi)(t_0,x_0) \;<\; (V_* - \varphi)(t,x) \quad \text{for} \quad (t,x) \in Q \setminus \{(t_0,x_0)\} \qquad (2.13)$$

In order to prove the required result, we assume to the contrary that

$$h(t_0,x_0) \;:=\; \frac{\partial \varphi}{\partial t}(t_0,x_0) + H\left(t_0, x_0, \varphi(t_0,x_0), D\varphi(t_0,x_0), D^2\varphi(t_0,x_0)\right) \;>\; 0,$$

and work towards a contradiction.

1. Since H is continuous, there exists an open neighborhood of (t_0,x_0) :

$$\mathcal{N}_\eta \;:=\; \{(t,x) \;:\; (t-t_0, x-x_0) \in (-\eta,\eta) \times \eta B \text{ and } h(t,x) > 0\} \; ,$$

for some $\eta > 0$. From (2.13), it follows that

$$3\gamma e^{\eta\|k^+\|_\infty} \;:=\; \min_{\partial\mathcal{N}_\eta} (V-\varphi) \;>\; 0\;. \tag{2.14}$$

Next, let $(t_n, x_n)_n$ be a sequence in $\mathcal{N}_h$ such that

$$(t_n, x_n) \;\longrightarrow\; (t_0, x_0) \quad \text{and} \quad V(t_n, x_n) \;\longrightarrow\; V_*(t_0, x_0)\;.$$

Since $(V-\varphi)(t_n, x_n) \longrightarrow 0$, we can assume that the sequence (t_n, x_n) also satisfies :

$$|(V-\varphi)(t_n, x_n)| \;\leq\; \gamma \quad \text{for all} \quad n \geq 1\;. \tag{2.15}$$

Finally, we introduce a $\gamma-$optimal control $\tilde{\nu}^n$ for the problem $V(t_n, x_n)$, i.e.

$$J(t_n, x_n, \tilde{\nu}^n) \;\leq\; V(t_n, x_n) + \gamma\;. \tag{2.16}$$

We shall denote by $\tilde{X}^n$ and $\tilde{\beta}^n$ the controlled process and the discount factor defined by the control $\tilde{\nu}^n$ and the initial data $\tilde{X}^n_{t_n} = x_n$.

3. Consider the stopping time

$$\theta_n \;:=\; \inf\left\{s > t_n \;:\; (s, \tilde{X}^n_s) \notin \mathcal{N}_\eta\right\}\;,$$

and observe that, by continuity of the state process, $(\theta_n, \tilde{X}^n_{\theta_n}) \in \partial\mathcal{N}_\eta$, so that :

$$(V-\varphi)(\theta_n, \tilde{X}^n_{\theta_n}) \;\geq\; (V_*-\varphi)(\theta_n, \tilde{X}^n_{\theta_n}) \;\geq\; 3\gamma e^{-\eta\|k^+\|_\infty} \tag{2.17}$$

by (2.14). We now use the inequality $V \geq V_*$, together with (2.17) and (2.15) to see that :

$$\begin{aligned}
&\tilde{\beta}^n(t_n, \theta_n)V(\theta_n, \tilde{X}^n_{\theta_n}) - V(t_n, x_n)\\
&\geq \int_{t_n}^{\theta_n} d[\tilde{\beta}^n(t_n, r)\varphi(r, \tilde{X}^n_r)] + 3\gamma e^{\eta\|k^+\|_\infty}\tilde{\beta}^n(t_n, \theta_n) - \gamma\\
&\geq \int_{t_n}^{\theta_n} d[\tilde{\beta}^n(t_n, r)\varphi(r, \tilde{X}^n_r)] + 2\gamma\;.
\end{aligned}$$

By Itô's lemma, this provides :

$$V(t_n, x_n) \ \leq \ E_{t_n, x_n}\left[\tilde{\beta}^n(t_n, \theta_n)V(\theta_n, \tilde{X}^n_{\theta_n}) - \int_{t_n}^{\theta_n}(\varphi_t + \mathcal{L}^{\tilde{\nu}^n_r}\varphi)(r, \tilde{X}^n_r)dr\right] - 2\gamma \ ,$$

where the stochastic term has zero mean, as its integrand is bounded on the stochastic interval $[t_n, \theta_n]$. Observe also that $(\varphi_t + \mathcal{L}^{\tilde{\nu}_r}\varphi)(r, \tilde{X}^n_r) + f(r, \tilde{X}^n_r, \tilde{\nu}^n_r) \geq h(r, \tilde{X}^n_r) \geq 0$ on the stochastic interval $[t_n, \theta_n]$. We therefore deduce that :

$$\begin{aligned} V(t_n, x_n) \ &\leq \ -2\gamma + E_{t_n, x_n}\left[\int_{t_n}^{\theta_n}\tilde{\beta}^n(t_n, r)f(r, \tilde{X}_r, \tilde{\nu}_r) + \tilde{\beta}^n(t_n, \theta_n)V(\theta_n, \tilde{X}^n_{\theta_n})\right] \\ &\leq \ -2\gamma + J(t_n, x_n, \tilde{\nu}) \\ &\leq \ V(t_n, x_n) - \gamma \ , \end{aligned}$$

where the last inequality follows by (2.16). This completes the proof. □

As a consequence of Propositions 2.5 and 2.6, we have the main result of this section :

Theorem 2.3 *Let the conditions of Propositions 2.5 and 2.6 hold. Then, the value function V is a (discontinuous) viscosity solution of the Hamilton-Jacobi-Bellman equation*

$$-\frac{\partial V}{\partial t}(t, x) - H\left(t, x, V(t, x), DV(t, x), D^2V(t, x)\right) \ = \ 0 \qquad (2.18)$$

on $[0, T) \times I\!R^n$.

3 Hedging contingent claims under portfolio constraints

3.1 Problem formulation

3.1.1 The financial market

Given a finite time horizon $T > 0$, we shall consider throughout these notes a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a standard Brownian motion $W = \{(W_t^1, \ldots, W_t^d),\ 0 \le t \le T\}$ valued in $I\!R^d$, and generating the ($P-$augmented) filtration $I\!F$. We denote by ℓ the Lebesgue measure on $[0, T]$.

The financial market consists of a non-risky asset S^0 normalized to unity, i.e. $S^0 \equiv 1$, and d risky assets with price process $S = (S^1, \ldots, S^d)$ whose dynamics is defined by a stochastic differential equation. More specifically, given a vector process μ valued in $I\!R^d$, and a matrix-valued process σ valued in $\mathcal{M}_{I\!R}(d)$, the price process S^i is defined as the unique strong solution of the stochastic differential equation :

$$S_0^i \ = \ s^i \ , \qquad dS_t^i \ = \ S_t^i \left[b_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j \right] \ ; \tag{3.1}$$

here b and σ are assumed to be bounded $I\!F-$adapted processes.

Remark 3.1 The normalization of the non-risky asset to unity is, as usual, obtained by discounting, i.e. taking the non-risky asset as a *numéraire*.

In the financial literature, σ is known as the *volatility* process. We assume it to be invertible so that the *risk premium* process

$$\lambda_t^0 \ := \ \sigma_t^{-1} b_t \ , \qquad 0 \le t \le T \ ,$$

is well-defined. Throughout these notes, we shall make use of the process

$$Z_t^0 \ := \ \mathcal{E}\left(- \int_0^t \lambda_r^{0\prime} dW_r \right) \ := \ \exp\left(- \int_0^t \lambda_r^{0\prime} dW_r - \frac{1}{2} \int_0^t |\lambda_r^0|^2 dr \right) \ .$$

Standing Assumption. The volatility process σ satisfies :

$$E\left[\exp\frac{1}{2}\int_0^T |\sigma\sigma'|^{-1}\right] \;<\; \infty \quad \text{and} \quad \sup_{[0,T]} |\sigma\sigma'|^{-1} \;<\; \infty \quad P-\text{a.s.}$$

Since b is bounded, this condition ensures that the process λ_0 satisfies the Novikov condition

$$E[\exp\int_0^T |\lambda^0|^2/2] \;<\; \infty \;,$$

and we have $E[Z_T^0] = 1$. The process Z_0 is then a martingale, and induces the probability measure P_0 defined by :

$$P^0(A) \;:=\; E\left[Z_t^0 \mathbf{1}_A\right] \quad \text{for all} \quad A \in \mathcal{F}_t \;, \quad 0 \le t \le T \;.$$

Clearly P^0 is equivalent to the original probability measure P. By Girsanov Theorem, the process

$$W_t^0 \;\;:=\;\; W_t + \int_0^t \lambda_t^0 dt \;, \quad 0 \le t \le T \;,$$

is a standard Brownian motion under P^0.

3.1.2 Portfolio and wealth process

Let X_t denote the wealth at time t of some investor on the financial market. We assume that the investor allocates continuously his wealth between the non-risky asset and the risky assets. We shall denote by π_t^i the proportion of wealth invested in the $i-th$ risky asset. This means that

$\pi_t^i X_t$ is the amount invested at time t in the $i-th$ risky asset.

The remaining proportion of wealth $1 - \sum_{i=1}^d \pi_t^i$ is invested in the non-risky asset.

The *self-financing condition* states that the variation of the wealth process is only affected by the variation of the price process. Under this condition, the wealth process satisfies :

$$
\begin{aligned}
dX_t &= X_t \sum_{i=1}^{d} \pi_t^i \frac{dS_t^i}{S_t^i} \\
&= X_t \pi_t' [b_t dt + \sigma_t dW_t] \;=\; X_t \pi_t' \sigma_t dW_t^0 \;. \qquad (3.2)
\end{aligned}
$$

Hence, the investment strategy π should be restricted so that the above stochastic differential equation has a well-defined solution. Also π_t should be based on the information available at time t. This motivates the following definition.

Definition 3.1 *An investment strategy is an $\mathbb{F}-$adapted process π valued in $\mathbb{R}^d$ and satisfying $\int_0^T |\sigma_t' \pi_t|^2 dt < \infty$ $P-$a.s.*
We shall denote by $\mathcal{A}$ the set of all investment strategies.

Clearly, given an initial capital $x \geq 0$ together with an investment strategy π, the stochastic differential equation (3.2) has a unique solution

$$
X_t^{x,\pi} \;:=\; x\mathcal{E}\left(\int_0^t \pi_r' \sigma_r dW_r^0\right) \;, \quad 0 \leq t \leq T \;.
$$

We then have the following trivial, but very important, observation :

$$
X^{x,\pi} \quad \text{is a } P^0\text{−supermartingale} \;, \qquad (3.3)
$$

as a non-negative local martingale under P^0.

3.1.3 Problem formulation

Let K be a closed convex subset of $\mathbb{R}^d$ containing the origin, and define the set of constrained strategies :

$$
\mathcal{A}_K \;:=\; \{\pi \in \mathcal{A} \;:\; \pi \in K \;\ell \otimes P - \text{a.s.}\} \;.
$$

The set K represents some constraints on the investment strategies.

Example 3.1 *Incomplete market*: taking $K = \{x \in \mathbb{R}^d \;:\; x^i = 0\}$, for some integer $1 \leq i \leq d$, means that trading on the $i-$th risky asset is forbidden.

Example 3.2 *No short-selling constraint* : taking $K = \{x \in I\!R^d \ : \ x^i \geq 0\}$, for some integer $1 \leq i \leq d$, means that the financial market does not allow to sell short the $i-$th asset.

Example 3.3 *No borrowing constraint* : taking $K = \{x \in I\!R^d \ : \ x^1 + \ldots + x^d \leq 1\}$ means that the financial market does not allow to sell short the non-risky asset or, in other word, borrowing from the bank is not available.

Now, let G be a non-negative $\mathcal{F}_T-$ measurable random variable. The chief goal of these notes is to study the following stochastic control problem

$$V_0 \ := \ \inf \{x \in I\!R \ : \ X_T^{x,\pi} \ \geq \ G \ P - \text{a.s. for some } \pi \in \mathcal{A}_K\} \ . \quad (3.4)$$

The random variable G is called a *contingent claim* in the financial mathematics literature, or a *derivative asset* in the financial engineering world. Loosely speaking, this is a contract between two counterparts stipulating that the seller has to pay G at time T to the buyer. Therefore, V_0 is the minimal initial capital which allows the seller to face without risk the payment G at time T, by means of some clever investment strategy on the financial market.

Observe that the above stochastic control problem does not fit in the class of stochastic control problems introduced in Section 1.1. We will therefore pass to a dual formulation of the problem which turns out to be in the class of stochastic control problems introduced in Section 1.1.

The main step towards the dual formulation of the problem is an existence result for the problem V_0 under very mild conditions, i.e. $X_T^{V_0,\pi} \geq G \ P-$a.s. for some constrained investment strategy $\pi \in \mathcal{A}_K$. We say that π is an optimal *hedging* strategy for the contingent claim G.

The existence result will in turn be obtained by means of some representation result which is now known as the *optional decomposition theorem* (in the framework of these notes, we can even call it a *predictable decomposition theorem*).

3.2 Existence of optimal hedging strategies and dual formulation

In this section, we concentrate on the duality approach to the problem of super-replication under portfolio constraints V_0. The main ingredient is a stochastic representation theorem. We therefore start by recalling the problem solution in the unconstrained case. This corresponds to the so-called *complete market* framework. In the general constrained case, the proof relies on the same arguments except that : we need to use a more advanced stochastic representation result, namely *the optional decomposition theorem.*

Remark 3.2 *local martingale representation theorem.*
(i) Theorem. Let Y be a local $P-$local martingale. Then there exists an $I\!F-$adapted $I\!R^d-$valued process ϕ such that

$$Y_t \; = \; Y_0 + \int_0^t \phi_r' dW_r \;\; 0 \leq t \leq T \quad \text{and} \quad \int_0^T |\phi|^2 \; < \; \infty \;\; P-\text{a.s.}$$

(see e.g. Dellacherie and Meyer VIII 62).
(ii) We shall frequently need to apply the above theorem to a $Q-$local martingale Y, for some equivalent probability measure Q defined by the density $(dQ/dP) = Z_T := \mathcal{E}\left(-\int_0^T \lambda_r' dW_r\right)$, with Brownian motion $W^Q := W + \int_0^{\cdot} \lambda_r dr$. To do this, we first apply the local martingale representation theorem to the$P-$local martingale ZY. The result is $ZY = Y_0 + \int_0^{\cdot} \phi dW$ for some adapted process ϕ with $\int_0^T |\phi|^2 < \infty$. Applying Itô's lemma, one can easily check that we have :

$$Y_t \; = \; Y_0 + \int_0^t \psi_r' dW_r^Q \;\; 0 \leq t \leq T \quad \text{where} \quad \psi \; := \; (Z)^{-1}\phi + \lambda Y \; .$$

Since Z and Y are continuous processes on the compact interval $[0, T]$, it is immediately checked that $\int_0^T |\psi|^2 < \infty$ $Q-$a.s.

3.2.1 Complete market : the unconstrained Black-Scholes world

In this paragraph, we consider the unconstrained case $K = \mathbb{R}^d$. The following result shows that V_0 is obtained by the same rule than in the celebrated Black-Scholes model, which was first developed in the case of constant coefficients μ and σ.

Theorem 3.1 *Assume that $G > 0$ $P-$a.s. Then :*
(i) $V_0 = E_0[G]$
(ii) *if $E_0[G] < \infty$, then $X_T^{V_0,\pi} = G$ $P-$a.s. for some $\pi \in \mathcal{A}$.*

Proof. 1. Set $F := \{x \in \mathbb{R} : X_T^{x,\pi} \geq G$ for some $\pi \in \mathcal{A}\}$. From the P^0-supermartingale property of the wealth process (3.3), it follows that $x \geq E^0[G]$ for all $x \in F$. This proves that $V_0 \geq E^0[G]$. Observe that this concludes the proof of (i) in the case $E^0[G] = +\infty$.
2. We then concentrate on the case $E^0[G] < \infty$. Define

$$Y_t \ := \ E^0[G|\mathcal{F}_t] \quad \text{for} \quad 0 \leq t \leq T \ .$$

Apply the local martingale representation theorem to the P^0-martingale Y, see Remark 3.2. This provides

$$Y_t \ = \ Y_0 + \int_0^t \psi_r' dW_r^0 \quad \text{for some process } \psi \text{ with } \quad \int_0^T |\psi|^2 < \infty \ .$$

Now set $\pi := (Y\sigma')^{-1}\psi$. Since Y is a positive continuous process, it follows from the second condition in Standing Assumption that $\pi \in \mathcal{A}$, and $Y = Y_0 \mathcal{E}\left(\int_0^\cdot \pi_r' \sigma_r dW_r^0\right) = X^{Y(0),\pi}$. The statement of the theorem follows from the observation that $Y_T = G$. □

Remark 3.3 Statement (ii) in the above theorem implies that existence holds for the control problem V_0, i.e. there exists an optimal trading strategy. But it provides a further information, namely that the optimal hedging strategy allows to *attain* the contingent claim G. Hence, in the unconstrained setting, all (positive) contingent claims are attainable. This is the reason for calling this financial market *complete*.

Remark 3.4 The proof of Theorem 3.1 suggests that the optimal hedging strategy π is such that the P^0- martingale Y has the stochastic representation $Y = E[G] + \int_0^{\cdot} Y\pi'\sigma dW^0$. In the Markov case, we have $Y_t = v(t, S_t)$. Assuming that v is smooth, it follows from an easy application of Itô's lemma that

$$\Delta_t^i \;:=\; \frac{\pi_t^i X_t^{V_0,\pi}}{S_t^i} \;=\; \frac{\partial v}{\partial s^i}(t, S_t)\;.$$

We now focus on the positivity condition in the statement of Theorem 3.1, which rules out the main example of contingent claims, namely European call options $[S_T^i - K]^+$, and European put options $[K - S_T^i]^+$. Indeed, since the portfolio process is defined in terms of proportion of wealth, the implied wealth process is strictly positive. Then, it is clear that such contingent claims can not be attained, in the sense of Remark 3.3, and there is no hope for Claim (ii) of Theorem 3.1 to hold in this context. However, we have the following easy consequence.

Corollary 3.1 *Let G be a non-negative contingent claim. Then*
(i) *For all $\varepsilon > 0$, there exists an investment strategy $\pi_\varepsilon \in \mathcal{A}$ such that $X_T^{V_0,\pi^\varepsilon}$ $= G + \varepsilon$.*
(ii) $V_0 = E^0[G]$.

Proof. Statement (i) follows from the application of Theorem 3.1 to the contingent claim $G+\varepsilon$. Now let $V_\varepsilon(0)$ denote the value of the super-replication problem for the contingent claim $G + \varepsilon$. Clearly, $V_0 \le V_\varepsilon(0) = E_0[G + \varepsilon]$, and therefore $V_0 \le E^0[G]$ by sending ε to zero. The reverse inequality holds since Part 1 of the proof of Theorem 3.1 does not require the positivity of G. $\square$

Remark 3.5 In the Markov setting of Remark 3.4 above, and assuming that v is smooth, the approximate optimal hedging strategy of Corollary 3.1 (i) is given by

$$\Delta_t^{i,\varepsilon} \;:=\; \frac{\pi_t^{i,\varepsilon} X_t^{V_\varepsilon(0),\pi^\varepsilon}}{S_t^i} \;=\; \frac{\partial}{\partial s^i}\left\{v(t, S_t) + \varepsilon\right\} \;=\; \frac{\partial v}{\partial s^i}(t, S_t)\;;$$

observe that $\Delta := \Delta^\varepsilon$ is independent of ε.

Example 3.4 *The Black and Scholes formula* : consider a financial market with a single risky asset $d = 1$, and let μ and σ be constant coefficients, so that the P^0-distribution of $\ln[S_T/S_t]$, conditionally on $\mathcal{F}_t$, is gaussian with mean $-\sigma^2(T-t)/2$ and variance $\sigma^2(T-t)$. As a contingent claim, we consider the example of a European call option, i.e. $G = [S_T - K]^+$ for some exercise price $K > 0$. Then, one can compute directly that :

$$V(t) \quad = \quad v(t, S_t)$$

where

$$\begin{aligned} v(t,s) &:= sF(d(t,s)) - KF\left(d(t,s) - \sigma\sqrt{T-t}\right) \ , \\ d(t,s) &:= (\sigma\sqrt{T-t})^{-1}\ln(K^{-1}s) + \frac{1}{2}\sigma\sqrt{T-t} \ , \end{aligned}$$

and $F(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-u^2/2}du$ is the cumulative function of the gaussian distribution. According to Remark 3.4, the optimal hedging strategy in terms of number of shares is given by :

$$\Delta(t) \quad = \quad F(d(t, S_t)) \ .$$

3.2.2 Optional decomposition theorem

We now turn to the general constrained case. The key-point in the proof of Theorem 3.1 was the representation of the P^0-martingale Y as a stochastic integral with respect to W^0; the integrand in this representation was then identified to the investment strategy. In the constrained case, the investment strategy needs to be valued in the closed convex set K, which is not guaranteed by the representation theorem. We then need to use a *more advanced* representation theorem. The results of this section were first obtained by ElKaroui and Quenez (1995) for the incomplete market case, and Cvitanić and Karatzas (1993) in our context. Notice that a general version of this result in the semimartingale case has been obtained by Föllmer and Kramkov (1997).

We first need to introduce some notations. Let

$$\delta(y) \; := \; \sup_{x \in K} \; x'y$$

be the support function of the closed convex set K. Since K contains the origin, δ is non-negative. We shall denote by

$$\tilde{K} \; := \; \mathrm{dom}(K) \;\; = \;\; \{y \in I\!R^d \; : \; \delta(y) < \infty\}$$

the effective domain of δ. For later use, observe that $\tilde{K}$ is a closed convex cone of $I\!R^d$. Recall also that, since K is closed and convex, we have the following classical results from convex analysis (see e.g. Rockafellar 1970) :

$$x \; \in \; K \quad \text{if and only if} \quad \delta(y) - x'y \; \geq \; 0 \quad \text{for all } y \in \tilde{K} \; , \tag{3.5}$$

We next denote by $\mathcal{D}$ the collection of all bounded adapted processes valued in $\tilde{K}$. For each $\nu \in \mathcal{D}$, we set

$$\beta_t^\nu \;\; := \;\; \exp\left(-\int_0^t \delta(\nu_r)dr\right) \;\; , \quad 0 \leq t \leq T \; ,$$

and we introduce the Doléans-Dade exponential

$$Z_t^\nu \; := \; \mathcal{E}\left(-\int_0^t \lambda_r^{\nu\prime} dW_r\right) \quad \text{where} \quad \lambda^\nu \; := \; \sigma^{-1}(b - \nu) \; = \; \lambda^0 - \sigma^{-1}\nu \; .$$

Since b and ν are bounded, λ^ν inherits the Novikov condition

$$E\left[\exp\left(\frac{1}{2}\int_0^T |\lambda^\nu|^2\right)\right] \;\; < \;\; \infty$$

from the first condition in Standing Assumption. We then introduce the family of probability measures

$$P^\nu(A) \; := \; E\left[Z_t^\nu \mathbf{1}_A\right] \quad \text{for all} \quad A \in \mathcal{F}_t \; , \quad 0 \leq t \leq T \; .$$

Clearly P^ν is equivalent to the original probability measure P. By Girsanov Theorem, the process

$$W_t^\nu \;\; := \;\; W_t + \int_0^t \lambda_r^\nu dr \tag{3.6}$$

$$\;\; = \;\; W^0(t) - \int_0^t \sigma_r^{-1}\nu_r dr \; , \quad 0 \leq t \leq T \; , \tag{3.7}$$

is a standard Brownian motion under P^ν.

Remark 3.6 The reason for introducing these objects is that the important property (3.3) extends to the family $\mathcal{D}$:

$$\beta_\nu X^{x,\pi} \quad \text{is a } P^\nu\text{−supermartingale for all} \quad \nu \in \mathcal{D},\ \pi \in \mathcal{A}_K\ , \qquad (3.8)$$

and $x > 0$. Indeed, by Itô's lemma together with (3.6),

$$d(X^{x,\pi}\beta^\nu) \;=\; X^{x,\pi}\beta^\nu\left[-(\delta(\nu) - \pi'\nu)dt + \pi'\sigma dW^\nu\right]\ .$$

In view of (3.5), this shows that $X^{x,\pi}\beta^\nu$ is a non-negative local P^ν−supermartingale, which provides (3.8).

Theorem 3.2 *Let Y be an $\mathbb{F}-$ adapted positive càdlàg process, and assume that*

the process $\beta^\nu Y$ is a P^ν−supermartingale for all $\nu \in \mathcal{D}$.

Then, there exists a predictable non-decreasing process C, with $C_0 = 0$, and a constrained portfolio $\pi \in \mathcal{A}_K$ such that $Y = X^{Y_0,\pi} - C$.

Proof. 1. We start by applying the Doob (unique) decomposition theorem (see e.g. Dellacherie and Meyer VII 12) to the P^0−supermartingale $Y\beta^0$ $= Y$, together with the local martingle representation theorem, under the probability measure P^0. This implies the existence of an adapted process ψ^0 and a non-decreasing predictable process C^0 satisfying $C^0_0 = 0$, $\int_0^T |\psi^0|^2 < \infty$, and :

$$Y_t \;=\; Y_0 + \int_0^t \psi^0_r dW^0_r - C^0_t\ , \qquad (3.9)$$

see Remark 3.2. Observe that

$$M^0 \;:=\; Y_0 + \int_0^\cdot \psi^0 dW^0 \;=\; Y + C^0 \;\geq\; Y \;>\; 0\ . \qquad (3.10)$$

We then define

$$\pi^0 \;:=\; \left(M^0\sigma'\right)^{-1}\psi^0\ .$$

From the second condition in Standing Assumption together with the continuity of M^0 on $[0,T]$ and the fact that $\int_0^T |\psi^0|^2 < \infty$, it follows that $\pi^0 \in \mathcal{A}$. Then $M^0 = X^{Y_0,\pi^0}$ and by (3.10),

$$Y \;=\; X^{Y_0,\pi^0} - C^0 \, .$$

In order to conclude the proof, it remains to show that the process π is valued in K.

2. By Itô's lemma together with (3.6), it follows that :

$$d(Y\beta^\nu) \;=\; M^0\beta^\nu {\pi^0}'\sigma dW^\nu - \beta^\nu \left[(Y\delta(\nu) - M^0{\pi^0}'\nu)dt + dC^0\right] \, .$$

Since $Y\beta^\nu =$ is a $P_\nu-$supermartingale, the process

$$C^\nu \;:=\; \int_0^\cdot \beta^\nu \left[(Y\delta(\nu) - M^0{\pi^0}'\nu)dt + dC^0\right]$$

is non-decreasing. In particular,

$$\begin{aligned} 0 \;&\leq\; \int_0^t (\beta^\nu)^{-1} dC^\nu \;=\; C_t^0 + \int_0^t \left(Y_r\delta(\nu_r) - M_r^0{\pi_r^0}'\nu_r\right) dr \\ &\leq\; C_t^0 + \int_0^t M_r^0 \left(\delta(\nu_r) - {\pi_r^0}'\nu_r\right) dr \quad \text{for all } \nu \in \mathcal{D} \, , \end{aligned} \tag{3.11}$$

where the last inequality follows from (3.10) and the non-negativity of the support function δ.

3. Now fix some $\nu \in \mathcal{D}$, and define the set $F_\nu := \{(t,\omega) : [-{\pi^0}'\nu + \delta(\nu)](t,\omega) < 0\}$. Consider the process

$$\nu^{(n)} \;=\; \nu 1_{F_\nu^c} + n\nu 1_{F_\nu} \, , \quad n \in I\!N \, .$$

Clearly, since $\tilde{K}$ is a cone, we have $\nu^{(n)} \in \mathcal{D}$ for all $n \in I\!N$. Writing (3.11) with $\nu^{(n)}$, we see that, whenever $\ell \otimes P[F_\nu] > 0$, the right hand-side term converges to $-\infty$ as $n \to \infty$, a contradiction. Hence $\ell \otimes P[F_\nu] = 0$ for all $\nu \in \mathcal{D}$. From (3.5), this proves that $\pi \in K$ $\ell \otimes P-$a.s. $\square$

3.2.3 Dual formulation

Let $\mathcal{T}$ be the collection of all stopping times valued in $[0, T]$, and define the family of random variables :

$$Y_\tau \ := \ \operatorname*{ess\,sup}_{\nu \in \mathcal{D}} \ E^\nu \left[G\gamma^\nu(\tau, T) | \mathcal{F}_\tau \right] \ ; \ \ \tau \ \in \ \mathcal{T} \quad \text{where} \quad \gamma^\nu(\tau, T) \ := \ \frac{\beta^\nu_T}{\beta^\nu_\tau} \ ,$$

and $E^\nu[\cdot]$ denotes the conditional expectation operator under P^ν. The purpose of this section is to prove that $V_0 = Y_0$, and that existence holds for the control problem V_0. As a by-product, we will also see that existence for the control problem Y_0 holds only in very specific situations. These results are stated precisely in Theorem 3.3. As a main ingredient, their proof requires the following (classical) dynamic programming principle.

Lemma 3.1 (Dynamic Programming). *Let $\tau \leq \theta$ be two stopping times in $\mathcal{T}$. Then :*

$$Y_\tau \ = \ \operatorname*{ess\,sup}_{\nu \in \mathcal{D}} \ E^\nu \left[Y_\theta \gamma^\nu(\tau, \theta) | \mathcal{F}_\tau \right] \ .$$

Proof. 1. Conditioning by $\mathcal{F}_\theta$, we see that

$$\begin{aligned} Y_\tau \ &\leq \ \operatorname*{ess\,sup}_{\nu \in \mathcal{D}} \ E_\nu \left[\gamma^\nu(\tau, \theta) E^\nu[G\gamma^\nu(\theta, T)|\mathcal{F}_\theta] | \, \mathcal{F}_\tau \right] \\ &\leq \ \operatorname*{ess\,sup}_{\nu \in \mathcal{D}} \ E^\nu \left[\gamma^\nu(\tau, \theta) Y_\theta | \mathcal{F}_\tau \right] \ . \end{aligned}$$

2. To see that the reverse inequality holds, fix any $\mu \in \mathcal{D}$, and let $\mathcal{D}_{\tau,\theta}(\mu)$ be the subset of $\mathcal{D}$ whose elements coincide with μ on the stochastic interval $[\tau, \theta]$. Let $(\nu_k)_k$ be a maximizing sequence of Y_θ, i.e.

$$Y_\theta \ = \ \lim_{k \to \infty} \ J^{\nu_k}_\theta \quad \text{where} \quad J^\nu_\theta \ := \ E^\nu[G\gamma^\nu(\theta, T)|\mathcal{F}_\theta] \ ;$$

the existence of such a sequence follows from the definition of the notion of essential supremum, see e.g. [20]. Also, since J^ν_θ depends on ν only through its realization on the stochastic interval $[\theta, T]$, we can assume that $\nu_k \in \mathcal{D}_{\tau,\theta}(\mu)$. We now compute that

$$Y_\tau \ \geq \ E^{\nu_k} \left[G\gamma^{\nu_k}(\tau, T) | \mathcal{F}_\tau \right] \ = \ E^\mu \left[\gamma^\mu(\tau, \theta) J^{\nu_k}_\theta | \mathcal{F}_\tau \right] \ ,$$

which implies that $Y_\tau \geq E^\mu \left[\gamma^\mu(\tau,\theta) Y_\theta | \mathcal{F}_\tau\right]$ by Fatou's lemma. $\square$

Now, observe that we may take the stopping times τ in the definition of the family $\{Y_\tau,\, \tau \in \mathcal{T}\}$ to be deterministic and thereby obtain a non-negative adapted process $\{Y_t,\, 0 \leq t \leq T\}$. A natural question is whether this process is consistent with the family $\{Y_\tau,\, \tau \in \mathcal{T}\}$ in the sense that $Y_\tau(\omega) = Y_{\tau(\omega)}(\omega)$ for a.e. $\omega \in \Omega$.

For general control problems, this is a delicate issue, which is related to the already mentioned difficulty in the proof of the dynamic programming principle of Theorem 1.2. However, in our context, it follows from the above dynamic programming principle that the family $\{Y_\tau,\, \tau \in \mathcal{T}\}$ satisfies a *supermartingale* property :

$$E^\nu \left[\beta^\nu_\theta Y_\theta | \mathcal{F}_\tau\right] \;\leq\; \beta^\nu_\tau Y_\tau \quad \text{for all } \tau, \theta \in \mathcal{T} \text{ with } \quad \tau \;\leq\; \theta\,.$$

By a classical argument, this allows to extract a process Y out of this family, which satisfies the supermartingale property in the usual sense. We only state precisely this technical point, and send the interested reader to Karatzas and Shreve (1999) Appendix D or Cvitanić and Karatzas (1993), Proposition 6.3.

Corollary 3.2 *There exists a càdlàg process $Y = \{Y_t,\, 0 \leq t \leq T\}$, consistent with the family $\{Y_\tau,\, \tau \in \mathcal{T}\}$, and such that $Y\beta^\nu$ is a $P^\nu-$supermartingale for all $\nu \in \mathcal{D}$.*

We are now able for the main result of this section.

Theorem 3.3 *Assume that $G > 0$ $P-$a.s. Then :*
(i) $V_0 = Y_0$,
(ii) *if $Y_0 < \infty$, existence holds for the problem V_0, i.e. $X_T^{V_0,\pi} \geq G$ $P-$a.s. for some $\pi \in \mathcal{A}_K$,*
(iii) *existence holds for the problem Y_0 if and only if*

$$X_T^{V_0,\hat\pi} \;=\; G \quad \text{and} \quad \beta^{\hat\nu} X^{V_0,\hat\pi} \text{ is a } P^{\hat\nu}-\text{martingale}$$

for some pair $(\hat\pi,\hat\nu) \in \mathcal{A}_K \times \mathcal{D}$.

Proof. 1. We concentrate on the proof of $Y_0 \geq V_0$ as the reverse inequality is a direct consequence of (3.8). The process Y, extracted from the family $\{Y_\tau, \tau \in \mathcal{T}\}$ in Corollary 3.2, satisfies the condition of the optional decomposition theorem 3.2. Then $Y = X^{Y_0,\pi} - C$ for some constrained portfolio $\pi \in \mathcal{A}_K$, and some predictable non-decreasing process C with $C_0 = 0$. In particular, $X_T^{Y_0,\pi} \geq Y_T = G$. This proves that $Y_0 \geq V_0$, completing the proof of (i) and (ii).
2. It remains to prove (iii). Suppose that $X_T^{V_0,\hat{\pi}} = G$ and $\beta^{\hat{\nu}} X^{V_0,\hat{\pi}}$ is a $P^{\hat{\nu}}-$martingale for some pair $(\hat{\pi}, \hat{\nu}) \in \mathcal{A}_K \times \mathcal{D}$. Then, by the first part of this proof, $Y_0 = V_0 = E^{\hat{\nu}}\left[X_T^{V_0,\hat{\pi}} \beta_T^{\hat{\nu}}\right] = E^{\hat{\nu}}\left[G\beta_T^{\hat{\nu}}\right]$, i.e. $\hat{\nu}$ is a solution of Y_0.

Conversely, assume that $Y_0 = E^{\hat{\nu}}[G\beta_T^{\hat{\nu}}]$ for some $\hat{\nu} \in \mathcal{D}$. Let $\hat{\pi}$ be the solution of V_0, whose existence is established in the first part of this proof. By definition $X_T^{V_0,\hat{\pi}} - G \geq 0$. Since $\beta^{\hat{\nu}} X^{V_0,\hat{\pi}}$ is a $P^{\hat{\nu}}-$super-martingale, it follows that $E^{\hat{\nu}}\left[\beta_T^{\hat{\nu}}(X_T^{V_0,\hat{\pi}} - G)\right] \leq 0$. This proves that $X_T^{V_0,\hat{\pi}} - G = 0$ $P-$a.s. We finally see that the $P^{\hat{\nu}}-$super-martingale $\beta^{\hat{\nu}} X^{V_0,\hat{\pi}}$ has constant $P^{\hat{\nu}}-$expectation :

$$\begin{aligned} Y_0 &\geq E^{\hat{\nu}}\left[\beta_t^{\hat{\nu}} X_t^{V_0,\hat{\pi}}\right] \\ &\geq E^{\hat{\nu}}\left[E^{\hat{\nu}}\left(\beta_T^{\hat{\nu}} X_T^{V_0,\hat{\pi}} \middle| \mathcal{F}_t\right)\right] \;=\; E^{\hat{\nu}}\left[\beta_T^{\hat{\nu}} G\right] \;=\; Y_0 \;, \end{aligned}$$

and therefore $\beta^{\hat{\nu}} X^{V_0,\hat{\pi}}$ is a $P^{\hat{\nu}}-$martingale. □

3.3 Explicit solution by means of the HJB equation

3.3.1 The HJB equation as a variational inequality

In order to characterize further the super-replication cost V_0, we assume that

$$b_t \;=\; b(t, S_t) \;, \sigma_t \;=\; \sigma(t, S_t) \;,$$

where the functions b and σ are continuous, Lipschitz in s uniformly in t. Then, the price process S is Markov. We also consider a contingent claim

$$G \;=\; g(S_T) \quad \text{for some lower semicontinuous} \quad g \;:\; \mathbb{R}_+^n \longrightarrow \mathbb{R}_+ \;.$$

By a trivial change of the time origin, it follows from the dual formulation of the super-replication problem of Theorem 3.3 that :

$$V_t \;=\; V(t,S_t) \;=\; \sup_{\nu\in\mathcal{D}} E^{\nu}_{t,S_t}\left[e^{-\int_t^T \delta(\nu_r)dr} g\left(S_T\right)\right] \;.$$

By Girsanov theorem, the above value function can be written in the standard form of stochastic control problems introduced in Chapter 1.1 :

$$V(t,s) \;=\; \sup_{\nu\in\mathcal{D}} E^{0}_{t,s}\left[e^{-\int_t^T \delta(\nu_r)dr} g\left(S^\nu_T\right)\right] \;.$$

where S^ν is the controlled process defined by

$$dS^\nu_r \;=\; \mathrm{diag}[S^\nu_r]\left(\nu_r dr + \sigma(r,S^\nu_r)dW\right) \;.$$

We now apply Proposition 2.5 to the value function $-V$ (in order to recover a minimization problem). Then, the value function V is a (discontinuous) viscosity supersolution of the equation

$$\begin{aligned} 0 \;\le\; & -\frac{\partial V}{\partial t} - \frac{1}{2}\mathrm{Tr}\left[\mathrm{diag}[s]\sigma\sigma'\mathrm{diag}[s]D^2V\right] \\ & +V \inf_{u\in\tilde{K}}\left\{\delta(u) - u'\frac{\mathrm{diag}[s]DV}{V}\right\} \;; \end{aligned}$$

recall that $V>0$. Since δ is positively homogeneous, and $\tilde{K}$ is a cone, this can be written equivalently in

$$0 \;\le\; -\frac{\partial V}{\partial t} - F\left(t,s,V(t,s),DV(t,s),D^2V(t,s)\right) \tag{3.12}$$

where

$$F(t,s,r,p,A)$$

$$:= \min\left\{-\tfrac{1}{2}\mathrm{Tr}\left[\mathrm{diag}[s]\sigma\sigma'\mathrm{diag}[s]A\right] \;,\; \inf_{u\in\tilde{K}_1}\left(\delta(u) - u'\;\frac{\mathrm{diag}[s]p}{r}\right)\right\} \;, \tag{3.13}$$

and

$$\tilde{K}_1 \;:=\; \{x\in\tilde{K} \;:\; |x|=1\} \;.$$

We next observe that

$$\mathrm{int(K)} \neq \emptyset \iff F \text{ is continuous } . \tag{3.14}$$

(Exercise !). It then follows from Proposition 2.6 that under this condition, the value function V is a (discontinuous) viscosity solution of the equation

$$0 = -\frac{\partial V}{\partial t} - F\left(t, s, V(t,s), DV(t,s), D^2V(t,s)\right) . \tag{3.15}$$

3.3.2 Terminal condition

From the definition of the value function V, we have :

$$V(T,s) = g(s) \quad \text{for all} \quad s \in I\!R^d_+ .$$

However, The set $\tilde{K}$ in which the control process take values is unbounded. We are therefore faced to a *singular* control problem. As we argued before, this is the typical case where a careful analysis has to be performed in order to derive the boundary condition for V_* and V^*. Typically this situation induces a *jump* in the terminal condition so that we only have :

$$V_*(T,s) \geq V(T,s) = g(s) .$$

The purpose of this section is to prove that $V_*(T,\cdot)$ and V^* are related to the function

$$\hat{g}(s) := \sup_{u \in \tilde{K}} g(se^u)e^{-\delta(u)} \quad \text{for} \quad s \in I\!R^d_+ , \tag{3.16}$$

where se^u is the $I\!R^d$ vector with components $s^i e^{u^i}$. The main results are stated in Propositions 3.2 and 3.3 below. We first start by deriving the PDE satisfied by $V_*(T,\cdot)$, as inherited from (3.12).

Proposition 3.1 *Suppose that g is lower semi-continuous and V is locally bounded. Then $V_*(T,\cdot)$ is a viscosity super-solution of*

$$\min\left\{V_*(T,\cdot) - g \ , \ \inf_{u \in \tilde{K}_1} \left(\delta(u) - u' \, \frac{\mathrm{diag}[s]DV_*}{V_*}(T,\cdot)\right)\right\} \geq 0 .$$

Proof. 1. We first check that $V_*(T,\cdot) \geq g$. Let $(t_n, s_n)_n$ be a sequence of $[0,T) \times (0,\infty)^d$ converging to (T,s), and satisfying $V(t_n,s_n) \longrightarrow V_*(T,s)$. Since $\delta(0)=0$, it follows from the definition of V that

$$V(t_n,s_n) \quad \geq \quad E_0\left[g\left(S_{t_n,s_n}(T)\right)\right] \; .$$

Since $g \geq 0$, we may apply Fatou's lemma, and derive the required inequality using the lower semi-continuity condition on g, together with the continuity of $S_{t,s}(T)$ in (t,s).

2. It remains to prove that $V_*(T,\cdot)$ is a viscosity super-solution of

$$\delta(u)V_*(T,\cdot) - u'\mathrm{diag}[s]DV_*(T,\cdot) \;\geq\; 0 \quad \text{for all} \quad u \in \tilde{K} \; . \tag{3.17}$$

Let f be a C^2 function satisfying, for some $s_0 \in (0,\infty)^d$,

$$0 \;=\; (V_*(T,\cdot)-f)(s_0) \;\;=\;\; \min_{\mathbb{R}^d_+}\left(V_*(T,\cdot)-f\right) \; .$$

Since $V_*(T,s_0) = \liminf_{(t,s)\to(T,s_0)} V_*(t,s)$ by the lower semi-continuity of V_*, we have

$$V_*(T_n,s_n) \;\longrightarrow\; V_*(T,s_0) \quad \text{for some sequence} \quad (T_n,s_n) \;\longrightarrow\; (T,s_0) \; .$$

Define

$$\varphi_n(t,s) \quad := \quad f(s) - \frac{1}{2}|s-s_0|^2 + \frac{T-t}{T-T_n} \; ,$$

let $\bar{B} = \{s \in \mathbb{R}^d_+ \;:\; \sum_i |\ln(s^i/s_0^i)| \leq 1\}$, and choose $(\bar{t}_n, \bar{s}_n)$ such that :

$$(V_* - \varphi_n)(\bar{t}_n,\bar{s}_n) \quad = \quad \min_{[T_n,T]\times\bar{B}}\left(V_*-\varphi_n\right) \; .$$

We shall prove the following claims :

$$\bar{t}_n < T \text{ for large n} \; , \tag{3.18}$$

$$\bar{s}_n \;\longrightarrow\; s_0 \quad \text{along a subsequence, and} \quad V_*(\bar{t}_n,\bar{s}_n) \;\longrightarrow\; V_*(T,s_0) \; . \tag{3.19}$$

Admitting this, wee see that, for sufficiently large n, $(\bar{t}_n, \bar{s}_n)$ is a local minimizer of the difference $(V_* - \varphi_n)$. Then, the viscosity supersolution property, established in (3.12), holds at $(\bar{t}_n, \bar{s}_n)$, implying that

$$\delta(u)V_*(\bar{t}_n, \bar{s}_n) - u'\mathrm{diag}[s]\left(Df(\bar{s}_n) - (\bar{s}_n - s_0)\right) \;\geq\; 0 \quad \text{for all} \quad u \in \tilde{K} \;,$$

by definition of φ_n in terms of f. In view of (3.19), this provides the required inequality (3.17).

Proof of (3.18) : Observe that for all $s \in \bar{B}$,

$$(V_* - \varphi_n)(T, s) \;=\; V_*(T, s) - f(s) + \frac{1}{2}|s - s_0|^2 \;\;\geq\;\; V_*(T, s) - f(s) \;\geq\; 0 \;.$$

Then, the required result follows from the fact that :

$$\begin{aligned} \lim_{n\to\infty}(V_* - \varphi_n)(T_n, s_n) \;&=\; \lim_{n\to\infty}\left\{V_*(T_n, s_n) - f(s_n) + \frac{1}{2}|s_n - s_0|^2 - \frac{1}{T - T_n}\right\} \\ &=\; -\infty \;. \end{aligned}$$

Proof of (3.19) : Since $(\bar{s}_n)_n$ is valued in the compact subset $\bar{B}$, we have $\bar{s}_n \longrightarrow \bar{s}$ along some subsequence, for some $\bar{s} \in \bar{B}$. We now use respectively the following facts : s_0 minimizes the difference $V_*(T, \cdot) - f$, V_* is lower semi-continuous, $s_n \longrightarrow s_0$, $\bar{t}_n \geq T_n$, and $(\bar{t}_n, \bar{s}_n)$ minimizes the difference $V_* - \varphi_n$ on $[T_n, T] \times \bar{B}$. The result is :

$$\begin{aligned} 0 \;&\leq\; (V_*(T, \cdot) - f)(\bar{s}) - (V_*(T, \cdot) - f)(s_0) \\ &\leq\; \liminf_{n\to\infty} \;\left\{(V_* - \varphi_n)(\bar{t}_n, \bar{s}_n) - \frac{1}{2}|\bar{s}_n - s_0|^2\right. \\ &\qquad\qquad \left. -(V_* - \varphi_n)(T_n, s_n) + \frac{1}{2}|s_n - s_0|^2 - \frac{\bar{t}_n - T_n}{T - T_n}\right\} \\ &\leq\; -\frac{1}{2}|\bar{s} - s_0|^2 + \liminf_{n\to\infty} \;\{(V_* - \varphi_n)(\bar{t}_n, \bar{s}_n) - (V_* - \varphi_n)(T_n, s_n)\} \\ &\leq\; -\frac{1}{2}|\bar{s} - s_0|^2 + \limsup_{n\to\infty} \;\{(V_* - \varphi_n)(\bar{t}_n, \bar{s}_n) - (V_* - \varphi_n)(T_n, s_n)\} \\ &\leq\; -\frac{1}{2}|\bar{s} - s_0|^2 \;\leq\; 0 \;, \end{aligned}$$

so that all above inequalities hold with equality, and (3.19) follows. $\square$

We are now able to derive the required lower bound on the terminal condition of the singular stochastic control problem $v(t, s)$.

Proposition 3.2 *Suppose that g is lower semi-continuous and V is locally bounded. Then $V_*(T, \cdot) \geq \hat{g}$.*

Proof. Introduce the lower semi-continuous function

$$h^{(u)}(r) \;:=\; \ln\left[V_*(T, e^{x+ru})\right] - \delta(u)r \;,$$

for fixed $x \in I\!R^d$ and $y \in \tilde{K}$ (here, $e^x = (e^{x^1}, \ldots, e^{x^n})$). From the previous proposition, $h^{(u)}$ is a viscosity super-solution of the equation $-h_r^{(u)} \geq 0$, and is therefore non-increasing. In particular $h(0) \geq h(1)$, i.e. $\ln\left[V_*(T, e^x)\right] \geq \ln\left[V_*(T, e^{x+u})\right] - \delta(u)$ for all $x \in I\!R^d$ and $y \in \tilde{K}$, and

$$\begin{aligned}
\ln\left[V_*(T, e^x)\right] \;&\geq\; \sup_{u \in \tilde{K}} \left\{\ln\left[V_*(T, e^{x+u})\right] - \delta(u)\right\} \\
&\geq\; \sup_{u \in \tilde{K}} \ln\left\{g\left(e^{x+u}\right) e^{-\delta(u)}\right\} \quad \text{for all} \quad x \in I\!R^d \;.
\end{aligned}$$

□

We now turn to the reverse inequality of Proposition 3.2. In order to simplify the presentation, we shall provide an easy proof under a stronger assumption.

Proposition 3.3 *Let σ be a bounded function, and $\hat{g}$ be an upper semi-continuous function with linear growth. Suppose that V is locally bounded. Then $V^*(T, \cdot) \leq \hat{g}$.*

Proof. Suppose to the contrary that $V^*(T, s) - \hat{g}(s) =: 2\eta > 0$ for some $s \in (0, \infty)^d$. Let (T_n, s_n) be a sequence in $[0, T] \times (0, \infty)^d$ satisfying :

$$(T_n, s_n) \;\longrightarrow\; (T, s) \;, \qquad V(T_n, s_n) \longrightarrow V^*(T, s)$$

and

$$V(T_n, s_n) \;>\; \hat{g}(s) + \eta \quad \text{for all} \quad n \geq 1 \;.$$

From the (dual) definition of V, this shows the existence of a sequence $(\nu^n)_n$ in $\mathcal{D}$ such that :

$$E^0_{T_n,s_n}\left[g\left(S_T^{(n)}e^{\int_{T_n}^T \nu_r^n dr}\right)e^{-\int_{T_n}^T \delta(\nu_r^n)dr}\right] \;>\; \hat{g}(s)+\eta \quad \text{for all} \quad n\geq 1 \qquad (3.20)$$

where

$$S_T^{(n)} \;:=\; s_n\mathcal{E}\left(\int_{T_n}^T \sigma(t,S_t^{\nu^n})dW_t\right) \; .$$

We now use the sublinearity of δ to see that :

$$\begin{aligned} & E^0_{T_n,s_n}\left[g\left(S_T^{(n)}e^{\int_{T_n}^T \nu_r^n dr}\right)e^{-\int_{T_n}^T \delta(\nu_r^n)dr}\right] \\ \leq\; & E^0_{T_n,s_n}\left[g\left(S_T^{(n)}e^{\int_{T_n}^T \nu_r^n dr}\right)e^{-\delta(\int_{T_n}^T \nu_r^n dr)}\right] \\ & \leq\; E^0_{T_n,s_n}\left[\hat{g}\left(S_T^{(n)}\right)\right] \; , \end{aligned}$$

where we also used the definition of $\hat{g}$ together with the fact that $\tilde{K}$ is a closed convex cone of $I\!R^d$. Plugging this inequality in (3.20), we see that

$$\hat{g}(s)+\eta \;\leq\; E^0_{t_n,s_n}\left[\hat{g}\left(S_T^{(n)}\right)\right] \; . \qquad (3.21)$$

By easy computation, it follows from the linear growth condition on $\hat{g}$ that

$$E^0\left|\hat{g}(S_T^{(n)})\right|^2 \;\leq\; Const\left(1+e^{(T-t)\|\sigma\|_\infty^2}\right) \; .$$

This shows that the sequence $\left\{\hat{g}(S_T^{(n)}),\; n\geq 1\right\}$ is bounded in $L^2(P^0)$, and is therefore uniformly integrable. We can therefore pass to the limit in (3.21) by means of the dominated convergence theorem. The required contradiction follows from the upper semicontinuity of $\hat{g}$ together with the a.s. continuity of S_T in the initial data (t,s). $\square$

3.3.3 The Black and Scholes model under portfolio constraints

In this paragraph we report an explicit solution of the super-replication problem under portfolio constraints in the context of the Black-Scholes model. This result was obtained by Broadie, Cvitanić and Soner (1997).

Proposition 3.4 *Let $d = 1$, $\sigma(t, s) = \sigma > 0$, and consider a lower semicontinuous payoff function $g : I\!R_+ \longrightarrow I\!R$. Assume that the face-lifted payoff function $\hat{g}$ is upper semicontinuous and has linear growth. Then :*

$$V(t, s) \;=\; E^0_{t,s}\left[\hat{g}\left(S_T\right)\right] \;,$$

i.e. $V(t, s)$ is the unconstrained Black-Scholes price of the face-lifted contingent claim $\hat{g}\left(S_T\right)$.

Proof. We shall provide a "PDE" proof of this result under the additional condition that $\hat{g}$ is C^1. The original probabilistic argument (which does not require this condition) can be found in [5]. From the previous paragraphs, The function $V(t, s)$ is a (discontinuous) viscosity solution of (3.15). When σ is constant, we claim that the PDE (3.15) reduces to :

$$-\mathcal{L}V \;=\; 0 \;\text{ on }\; [0, T) \times (0, \infty)^d \;, \qquad V(T, \cdot) \;=\; \hat{g} \;, \tag{3.22}$$

and the required result follows from the Feynman-Kac representation formula.

It remains to prove (3.22). By classical arguments, see e.g. [15], the linear PDE (3.22) has a classical solution v with $v_s \in C^{1,2}$. Notice that both functions v and sv_s solve the linear PDE $-\mathcal{L}u = 0$, so that the function $w^{(u)} := \delta(u)v - usv_s$ satisfies

$$-\mathcal{L}w^{(u)} \;=\; 0 \quad\text{and}\quad w^{(u)}(T, s) \;=\; \delta(u)\hat{g}(s) - us\hat{g}_s(u) \;.$$

Now observe that $w^{(u)}(T, \cdot) \geq 0$ for all $u \in K$, by definition of $\hat{g}$. The Feynman-Kac representation formula then implies that $w^{(u)} \geq 0$ for all $u \in K$, and therefore v solves the variational inequality (3.15). We finally appeal to a uniqueness result for the variational inequality (3.15) in order to conclude that $v = V$. $\square$

3.3.4 The uncertain volatility model

In this paragraph, we study the simplest incomplete market model. The number of risky assets is now $d = 2$. We consider the case $K = I\!R \times \{0\}$ so that the second risky asset is not tradable. The contingent claim is defined by $G = g(S^1(T))$, where the payoff function $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is continuous and has polynomial growth. We finally introduce the notations :

$$\overline{\sigma}(t,s_1) := \sup_{s_2>0} [\sigma_{11}^2 + \sigma_{12}^2](t,s_1,s_2); \quad \underline{\sigma}(t,s_1) := \inf_{s_2>0} [\sigma_{11}^2 + \sigma_{12}^2](t,s_1,s_2)$$

We report the following result from Cvitanić, Pham and Touzi (1999).

Proposition 3.5
(i) *Assume that* $\overline{\sigma} < \infty$ *on* $[0,T] \times \mathbb{R}_+$. *Then* $V(t,s) = V(t,s_1)$ *is a (discontinuous) viscosity solution of the Black-Scholes-Barrenblatt equation*

$$\begin{aligned} -V_t - \tfrac{1}{2}\left[\overline{\sigma}^2 V^+_{s_1s_1} - \underline{\sigma}^2 V^-_{s_1s_1}\right] = 0 \; on \; [0,T) \times (0,\infty) \\ V(T,s_1) \;=\; g(s_1) \; for \; s_1 > 0. \end{aligned} \tag{3.23}$$

(ii) *Assume that* $\overline{\sigma} = \infty$ *and*

$$either \; g \; is \; convex \quad or \quad \underline{\sigma} \;=\; 0 \; .$$

Then $v(t,s) = g^{conc}(s_1)$, *where* g^{conc} *is the concave envelope of* g.

We only gives the main ideas for the proof of this result. First, observe that the constraints set $K = \mathbb{R} \times \{0\}$ has empty interior, and the operator F defined in (3.13) is not continuous. We then proceed as follows. The viscosity supersolution property (3.12) is still valid. We first deduce from it that V_* does not depend on the s_2 variable.

- In case (i), this proves that the value function V is a (discontinuous) viscosity supersolution of the equation

$$-V_t - \frac{1}{2}\left[\overline{\sigma}^2 V^+_{s_1s_1} - \underline{\sigma}^2 V^-_{s_1s_1}\right] \;\geq\; 0 \quad \text{on} \quad [0,T) \times (0,\infty) \; .$$

It follows from the non-negativity and the lower semicontinuity of g that the value function is lower semicontinuous, i.e. $V = V_*$. Then V does not depend on the s_2 variable. One can then proceed exactly as in the proof of Proposition 2.5 to prove that V is also a (discontinuous) viscosity subsolution of (3.23). Finally since g does not depend on the s_2 variable, we have $\hat{g} = g$.

- Case (ii) is treated by the same type of arguments as the example of Paragraph 1.5.3.

4 Hedging contingent claims under gamma constraints

In this section, we focus on an alternative constraint on the portfolio π. For simplicity, we consider a financial market with a single risky asset. Let $Y_t(\omega) := S_t^{-1}\pi_t X_t(\omega)$ denote the vector of number of shares of the risky assets held at each time t and $\omega \in \Omega$. By definition of the portfolio strategy, the investor has to adjust his strategy at each time t, by passing the number of shares from Y_t to Y_{t+dt}. His demand in risky assets at time t is then given by "dY_t".

In an equilibrium model, the price process of the risky asset would be pushed upward for a large demand of the investor. We therefore study the hedging problem with constrained portfolio adjustment. This problem turns out to present serious mathematical difficulties. The analysis of this section is reported from [23], and provides a solution of the problem in a very specific situation. We hope that this presentation will encourage some readers to attack some of the possible extensions.

4.1 Problem formulation

We consider a financial market which consists of one bank account, with constant price process $S_t^0 = 1$ for all $t \in [0,T]$, and one risky asset with price process evolving according to the Black-Scholes model :

$$S_u \; := \; S_t \mathcal{E}\left(\sigma(W_t - W_u)\right), \quad t \le u \le T.$$

Here W is a standard Brownian motion in $\mathbb{R}$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$. We shall denote by $\mathbb{F} = \{\mathcal{F}_t,\ 0 \le t \le T\}$ the P-augmentation of the filtration generated by W.

Observe that there is no loss of generality in taking S as a martingale, as one can always reduce the model to this case by judicious change of measure (P^0 in the previous chapter). On the other hand, the subsequent analysis can be easily extended to the case of a varying volatility coefficient.

We denote by $Y = \{Y_u,\ t \le u \le T\}$ the process of number of shares of risky asset S held by the agent during the time interval $[t, T]$. Then, by the self-financing condition, the wealth process induced by some initial capital x, at time t, and portfolio strategy Y is given by :

$$X_u \;=\; x + \int_t^u Y_r dS_r, \quad t \le u \le T.$$

In order to introduce constraints on the variations of the hedging portfolio Y, we restrict Y to the class of continuous semimartingales with respect to the filtration $\mathbb{F}$. Since $\mathbb{F}$ is the Brownian filtration, we define the controlled portfolio strategy $Y^{y,\alpha,\gamma}$ by :

$$Y_u^{y,\alpha,\gamma} \;=\; y + \int_t^u \alpha_r dr + \int_t^u \gamma_r \sigma dW_r, \quad t \le u \le T, \tag{4.1}$$

where $y \in \mathbb{R}$ is the time t initial portfolio and the *control* pair (α, γ) takes values in

$$\mathcal{B}_t \;:=\; (\, L_a^\infty([t, T] \times \Omega \,;\, \ell \otimes P)\,)^2 \,,$$

where $L_a^\infty([t, T] \times \Omega; \ell \otimes P)$ denotes the set of $\ell \otimes P-$essentially bounded processes on the time interval $[t, T]$. Hence a *trading strategy* is defined by the triple $\nu := (y, \alpha, \gamma)$ with $y \in \mathbb{R}$ and $(\alpha, \gamma) \in \mathcal{B}_t$. The associated wealth process, denoted by X^ν, is given by :

$$X_u^{x,\nu} \;=\; x + \int_t^u Y_r^\nu dS_r, \quad t \le u \le T, \tag{4.2}$$

where x is the time t initial capital. We now formulate the Gamma constraint in the following way. Let Γ be a positive fixed constant. Given some initial capital $x \in \mathbb{R}$, we define the set of *x-admissible* trading strategies by :

$$\mathcal{A}_t(x) \;:=\; \{\nu = (y, \alpha, \gamma) \in \mathbb{R} \times \mathcal{B}_t \;:\; \gamma_\cdot \le \Gamma \;\text{ and }\; X_\cdot^{x,\nu} \ge 0\} \;.$$

As in the previous sections, We consider the super-replication problem of some European type contingent claim $g(S_T)$:

$$v(t, S_t) \;:=\; \inf\{x \;:\; X_T^{x,\nu} \ge g(S_T) \;\text{ a.s. for some } \nu \in \mathcal{A}_t(x)\} \;. \tag{4.3}$$

4.2 The main result

Our goal is to derive the following explicit solution : $v(t, S_t)$ is the (unconstrained) Black-Scholes price of some convenient *face-lifted* contingent claim $\hat{g}(S_T)$, where the function $\hat{g}$ is defined by

$$\hat{g}(s) \;:=\; h^{conc}(s) + \Gamma s \ln s \quad \text{with} \quad h(s) \;:=\; g(s) - \Gamma s \ln s \;,$$

and h^{conc} denotes the concave envelope of h. Observe that this function can be computed easily. The reason for introducing this function is the following.

Lemma 4.1 *$\hat{g}$ is the smallest function satisfying the conditions*

$$\text{(i)} \;\; \hat{g} \;\geq\; g \;, \quad \textit{and} \quad \text{(ii)} \;\; s \longmapsto \hat{g}(s) - \Gamma s \ln s \;\; \textit{is concave.}$$

The proof of this easy result is omitted.

Theorem 4.1 *Let g be a non-negative lower semicontinuous mapping on $I\!R_+$. Assume further that*

$$s \;\longmapsto\; \hat{g}(s) - C\; s \ln s \quad \textit{is convex for some constant} \;\; C \;. \tag{4.4}$$

Then the value function (4.3) *is given by :*

$$v(t,s) \;=\; E_{t,s}\left[\hat{g}\left(S_T\right)\right] \quad \textit{for all} \quad (t,s) \in [0,T) \times (0,\infty) \;.$$

4.3 Discussion

1. We first make some comments on the model. Intuitively, we expect the optimal hedging portfolio to satisfy

$$\hat{Y}_u \;\;=\;\; v_s(u, S_u) \;,$$

where v is the minimal super-replication cost; see Section 3.2.1. Assuming enough regularity, it follows from Itô's lemma that

$$d\hat{Y}_u \;\;=\;\; A_u du + \sigma S_u v_{ss}(u, S_u) dW_u \;,$$

where $A(u)$ is given in terms of derivatives of v. Compare this equation with (4.1) to conclude that the associated *gamma* is

$$\hat{\gamma}_u \;=\; S_u\, v_{ss}(u, S_u)\ .$$

Therefore the bound on the process $\hat{\gamma}$ translates to a bound on sv_{ss}. Notice that, by changing the definition of the process γ in (4.1), we may bound v_{ss} instead of sv_{ss}. However, we choose to study sv_{ss} because it is a dimensionless quantity, i.e., if all the parameters in the problem are increased by the same factor, sv_{ss} still remains unchanged.

2. Observe that we only require an upper bound on the control γ. The similar problem with a lower bound on γ is still open, and presents some specific difficulties. In particular, it seems that the control $\int_0^t \alpha(r)dr$ has to be relaxed to the class of bounded variation processes...

3. The extension of the analysis of this section to the multi-asset framework is available; the restriction to the one-dimensional case is only made for simplicity.

4. Intuitively, we expect to obtain a similar type solution to the case of portfolio constraints. If the Black-Scholes solution happens to satisfy the gamma constraint, then it solves the problem with gamma constraint. In this case v satisfies the PDE $-\mathcal{L}v = 0$. Since the Black-Scholes solution does not satisfy the gamma constraint, in general, we expect that the function v solves the variational inequality :

$$\min\left\{-\mathcal{L}v, \Gamma - sv_{ss}\right\} \;=\; 0\ . \tag{4.5}$$

5. An important feature of the log-normal Black and Sholes model is that the variational inequality (4.5) reduces to the Black-Scholes PDE $-\mathcal{L}v = 0$ as long as the terminal condition satisfies the gamma constraint (in a weak sense). From Lemma 4.1, the *face-lifted* payoff function $\hat{g}$ is precisely the minimal function above g which satisfies the gamma constraint (in a weak sense). This explains the nature of the solution reported in Theorem 4.1, namely $v(t, S_t)$ is the Black-Scholes price of the contingent claim $\hat{g}\,(S_T)$.

6. We shall check *formally* below that the variational inequality (4.5) is the HJB equation associated to the stochastic control problem :

$$\tilde{v}(t,s) \;:=\; \sup_{\nu\in\mathcal{N}} E_{t,s}\left[g\left(S_T\right) - \frac{1}{2}\Gamma\int_t^T \nu_r[S_r^\nu]^2 dr\right] \;, \qquad (4.6)$$

where $\mathcal{N}$ is the set of all non-negative, bounded, and $\mathbb{F}-$ adapted processes, and :

$$S_u^\nu \;:=\; S_t^\nu \mathcal{E}\left(\int_t^u [\sigma^2+\nu_r]^{1/2} dW_r\right) \;, \qquad \text{for } t\le u\le T \;.$$

The above stochastic control problem is a candidate for some dual formulation of the problem $v(t,s)$ defined in (4.3). Observe, however, that the dual variables ν are acting on the diffusion coefficient of the controlled process S^ν, so that the change of measure techniques of Section 3.2 do not help to prove the duality connection between v and $\tilde{v}$.

A direct proof of some duality connection between v and $\tilde{v}$ is again an open problem. In order to obtain the PDE characterization (4.5) of v, we shall make use of an original dynamic programming principle stated directly on the initial formulation of the problem v.

7. Recall from Proposition 2.5 that the viscosity subsolution property of the value function of a minimization problem holds under very mild conditions. Applying this result to the maximization problem (4.6), t follows that $\tilde{v}$ is a (discontinuous) viscosity supersolution of :

$$\inf_{u\ge 0} \; -\mathcal{L}^u\tilde{v}(t,s) + \frac{1}{2}\Gamma s^2 u \;\ge\; 0 \quad \text{where} \quad \mathcal{L}^u\tilde{v} \;:=\; \tilde{v}_t + \frac{1}{2}(\sigma^2+u)\tilde{v}_{ss} \;.$$

Collecting terms, this provides

$$\inf_{u\ge 0} \; -\mathcal{L}^0\tilde{v}(t,s) + \frac{1}{2}u(\Gamma - \tilde{v}_{ss}) \;\;\ge\;\; 0 \;,$$

or, equivalently :

$$\min\left\{ -\mathcal{L}^0\tilde{v} \;;\; \Gamma - \tilde{v}_{ss} \right\} \;\;\ge\;\; 0 \;.$$

This suggests that (4.5) is the HJB equation associated with the stochastic control problem (4.6).

4.4 Dynamic programming and viscosity property

This paragraph is dedicated to the proof of Theorem 4.1. We shall denote

$$\hat{v}(t,s) \;:=\; E_{t,s}\left[\hat{g}\left(S_T\right)\right]\ .$$

It is easy to check that $\hat{v}$ is a smooth function satisfying

$$\mathcal{L}\hat{v} \;=\; 0 \text{ and } s\hat{v}_{ss} \;\le\; \Gamma \quad \text{on} \quad [0,T)\times(0,\infty)\ . \tag{4.7}$$

1. We start with the inequality $v \le \hat{v}$. For $t \le u \le T$, set

$$y \;:=\; \hat{v}_s(t,s)\ ,\quad \alpha_u \;:=\; \mathcal{L}\hat{v}_s(u,S_u)\ ,\quad \gamma_u \;:=\; S_u\hat{v}_{ss}(u,S_u)\ ,$$

and we claim that

$$(\alpha,\gamma) \;\in\; \mathcal{B}_t \quad \text{and} \quad \gamma \le \Gamma\ . \tag{4.8}$$

Before verifying this claim, let us complete the proof of the required inequality. Since $g \le \hat{g}$, we have

$$\begin{aligned}
g\left(S_T\right) \;&\le\; \hat{g}\left(S_T\right) \;=\; \hat{v}\left(T,S_T\right) \\
&=\; \hat{v}(t,S_t) + \int_t^T \mathcal{L}\hat{v}(u,S_u)du + \hat{v}_s(u,S_u)dS_u \\
&=\; \hat{v}(t,S_t) + \int_t^T Y_u^\nu dS_u\ ;
\end{aligned}$$

in the last step we applied Itô's formula to $\hat{v}_s$. Now, set $X_t := \hat{v}(t,S_t)$, and observe that $X_u^{X_t,\nu} = \hat{v}(u,S_u) \ge 0$ by non-negativity of the payoff function g. Hence $\nu \in \mathcal{A}_t(X_t)$, and by the definition of the super-replication problem (4.3), we conclude that $v \le \hat{v}$.

It remains to prove (4.8). The upper bound on γ follows from (4.7). As for the lower bound, it is obtained as a direct consequence of Condition (4.4). Using again (4.7) and the smoothness of $\hat{v}$, we see that $0 = (\mathcal{L}\hat{v})_s = \mathcal{L}\hat{v}_s + \sigma^2 s\hat{v}_{ss}$, so that $\alpha = -\sigma^2\gamma$ is also bounded.

2. The proof of the reverse inequality $v \ge \hat{v}$ requires much more effort. The main step is the following (*half*) dynamic programming principle.

Lemma 4.2 *Let $x \in I\!R$, $\nu \in \mathcal{A}_t(x)$ be such that $X_T^{x,\nu} \geq g(S_T)$ $P-a.s.$ Then*

$$X_\theta^{x,\nu} \;\; \geq \;\; v(\theta, S_\theta) \qquad P-a.s.$$

for all stopping times θ valued in $[t, T]$.

The obvious proof of this claim is left to the reader. We continue by stating two lemmas whose proofs rely heavily on the above dynamic programming principle, and will be reported later. We denote as usual by v_* the lower semicontinuous envelope of v.

Lemma 4.3 *The function v_* is viscosity supersolution of the equation*

$$-\mathcal{L}v_* \;\; \geq \;\; 0 \quad on \quad [0, T) \times (0, \infty) \, .$$

Lemma 4.4 *The function $s \longmapsto v_*(t,s) - \Gamma s \ln s$ is concave for all $t \in [0, T]$.*

Before proceeding to the proof of these results, let us show how the remaining inequality $v \geq \hat{v}$ follows from it. Given a trading strategy in $\mathcal{A}_t(x)$, the associated wealth process is a non-negative local martingale, and therefore a supermartingale. From this, one easily proves that $v_*(T,s) \geq g(s)$. By Lemma 4.4, $v_*(T,\cdot)$ also satisfies requirement (ii) of Lemma 4.1, and therefore

$$v_*(T,\cdot) \;\; \geq \;\; \hat{g} \, .$$

In view of Lemma 4.3, v_* is a viscosity supersolution of the equation $-\mathcal{L}v_* = 0$ and $v_*(T,\cdot) = \hat{g}$. Since $\hat{v}$ is a viscosity solution of the same equation, it follows from the classical comparison theorem that $v_* \geq \hat{v}$.

Hence, in order to complete the proof of Theorem 4.1, it remains to prove Lemmas 4.3 and 4.4.

Proof of Lemma 4.3 We split the argument in several steps.
3. We first show that the problem can be reduced to the case where the controls (α, γ) are uniformly bounded. For $\varepsilon \in (0, 1]$, set

$$\mathcal{A}_t^\varepsilon(x) \;\; := \;\; \left\{ \, \nu = (y, \alpha, \gamma) \in \mathcal{A}_t(x) \;\; : \;\; |\alpha(.)| + |\gamma(.)| \leq \varepsilon^{-1} \, \right\},$$

and

$$v^{\varepsilon}(t,S_t) \;=\; \inf\left\{x \;:\; X_T^{x,\nu} \geq g(S_T)\; P-\text{a.s. for some } \nu \in \mathcal{A}_t^{\varepsilon}(x)\right\}\;.$$

Let v_*^{ε} be the lower semicontinuous envelope of v^{ε}. It is clear that v^{ε} also satisfies the dynamic programming equation of Lemma 4.2.

Since

$$v_*(t,s) \;=\; \liminf{}_* \, v^{\varepsilon}(t,s) \;=\; \liminf_{\varepsilon\to 0,(t',s')\to(t,s)} v_*^{\varepsilon}(t',s')\;,$$

we shall prove that

$$-\mathcal{L}v^{\varepsilon} \;\geq\; 0 \quad \text{in the viscosity sense,} \tag{4.9}$$

and the statement of the lemma follows from the classical stability result of Proposition 2.3.

4. We now derive the implications of the dynamic programming principle of Lemma 4.2 applied to v^{ε}. Let $\varphi \in C^{\infty}(I\!R^2)$ and $(t_0,s_0) \in (0,T)\times(0,\infty)$ satisfy

$$0 \;=\; (v_*^{\varepsilon}-\varphi)(t_0,s_0) \;=\; \min_{(0,T)\times(0,\infty)}(v_*^{\varepsilon}-\varphi)\;;$$

in particular, we have $v_*^{\varepsilon} \geq \varphi$. Choose a sequence $(t_n,s_n)\to(t_0,s_0)$ so that $v^{\varepsilon}(t_n,s_n)$ converges to $v_*^{\varepsilon}(t_0,s_0)$. For each n, by the definition of v^{ε} and the dynamic programming, there are $x_n \in [v^{\varepsilon}(t_n,s_n), v^{\varepsilon}(t_n,s_n)+1/n]$, hedging strategies $\nu_n = (y_n,\alpha_n,\gamma_n) \in \mathcal{A}_{t_n}^{\varepsilon}(x_n)$ satisfying

$$X_{\theta_n}^{x_n,\nu_n} - v^{\varepsilon}\left(\theta_n, S_{\theta_n}\right) \;\geq\; 0$$

for every stopping time θ_n valued in $[t_n,T]$. Since $v^{\varepsilon} \geq v_*^{\varepsilon} \geq \varphi$,

$$x_n + \int_{t_n}^{\theta_n} Y_u^{\nu_n} dS_u - \varphi\left(\theta_n, S_{\theta_n}\right) \;\geq\; 0\;.$$

Observe that

$$\beta_n \;:=\; x_n - \varphi(t_n,s_n) \;\longrightarrow\; 0 \;\text{ as } n \longrightarrow \infty\;.$$

By Itô's Lemma, this provides

$$M^n_{\theta_n} \;\leq\; D^n_{\theta_n} + \beta_n \;, \tag{4.10}$$

where

$$\begin{aligned} M^n_t &:= \int_0^t \left[\varphi_s(t_n+u, S_{t_n+u}) - Y^{\nu_n}_{t_n+u}\right] dS_{t_n+u} \\ D^n_t &:= -\int_0^t \mathcal{L}\varphi(t_n+u, S_{t_n+u}) du \;. \end{aligned}$$

We now chose conveniently the stopping time θ_n. For some sufficiently large positive constant λ and arbitrary $h > 0$, define the stopping time

$$\theta_n \;:=\; (t_n + h) \wedge \inf\left\{u > t_n \;:\; |\ln(S_u/s_n)| \geq \lambda\right\} \;.$$

5. By the smoothness of $\mathcal{L}\varphi$, the integrand in the definition of M^n is bounded up to the stopping time θ_n and therefore, taking expectation in (4.10) provides :

$$-E_{t_n,s_n}\left[\int_0^{t\wedge\theta_n} \mathcal{L}\varphi(t_n+u, S_{t_n+u})du\right] \;\geq\; -\beta_n \;,$$

We now send n to infinity, divide by h and take the limit as $h \searrow 0$. The required result follows by dominated convergence. □

6. It remains to prove Lemma 4.4. The key-point is the following result, which is a consequence of Theorem 1.6.

Lemma 4.5 *Let $(\{a^n_u,\ u \geq 0\})_n$ and $(\{b^n_u,\ u \geq 0\})_n$ be two sequences of real-valued, progressively measurable processes that are uniformly bounded in n. Let (t_n, s_n) be a sequence in $[0,T) \times (0,\infty)$ converging to $(0,s)$ for some $s > 0$. Suppose that*

$$\begin{aligned} M^n_{t\wedge\tau_n} &:= \int_{t_n}^{t_n+t\wedge\tau_n} \left(z_n + \int_{t_n}^u a^n_r dr + \int_{t_n}^u b^n_r dS_r\right) dS_u \\ &\leq \beta_n + Ct\wedge\tau_n \end{aligned}$$

for some real numbers $(z_n)_n$, $(\beta_n)_n$, and stopping times $(\tau_n)_n \geq t_n$. Assume further that, as n tends to infinity,

$$\beta_n \longrightarrow 0 \quad \textit{and} \quad t\wedge\tau_n \longrightarrow t\wedge\tau_0 \quad P-a.s.,$$

where τ_0 is a strictly positive stopping time. Then :
(i) $\lim_{n\to\infty} z_n = 0$.
(ii) $\lim_{u\to 0}$ *essinf*$_{0\le r\le u}$ $b_u \le 0$*, where b be a weak limit process of $(b_n)_n$.*

Proof of Lemma 4.4 We start exactly as in the previous proof by reducing the problem to the case of uniformly bounded controls, and writing the dynamic programming principle on the value function v^ε.

By a further application of Itô's lemma, we see that :

$$M_n(t) \;=\; \int_0^t \left(z_n + \int_0^u a_r^n dr + \int_0^u b_r^n dS_{t_n+r} \right) dS_{t_n+u} \;,$$

where

$$\begin{aligned} z_n &:= \varphi_s(t_n, s_n) - y_n \\ a^n(r) &:= \mathcal{L}\varphi_s(t_n + r, S_{t_n+r}) - \alpha^n_{t_n+r} \\ b^n_r &:= \varphi_{ss}(t_n + r, S_{t_n+r}) - \frac{\gamma^n_{t_n+r}}{S_{t_n+r}} \;. \end{aligned}$$

Observe that the processes $a^n_{.\wedge\theta_n}$ and $b^n_{.\wedge\theta_n}$ are bounded uniformly in n since $\mathcal{L}\varphi_s$ and φ_{ss} are smooth functions. Also since $\mathcal{L}\varphi$ is bounded on the stochastic interval $[t_n, \theta_n]$, it follows from (4.10) that

$$M^n_{\theta_n} \;\le\; C\; t\wedge\theta_n \;+\; \beta_n$$

for some positive constant C. We now apply the results of Lemma 4.5 to the martingales M^n. The result is :

$$\lim_{n\to\infty} y_n \;=\; \varphi_s(t_0, y_0) \quad \text{and} \quad \lim_{t\to 0} \operatorname*{ess\,inf}_{0\le u\le t} \; b_t \;\le\; 0.$$

where b is a weak limit of the sequence (b_n). Recalling that $\gamma^n(t) \le \Gamma$, this provides that :

$$-s_0\varphi_{ss}(t_0, s_0) + \Gamma \ge 0 \;.$$

Hence v^ε_* is a viscosity supersolution of the equation $-s(v_*)_{ss} + \Gamma \ge 0$, and the required result follows by the stability result of Proposition 2.3. $\square$

References

[1] Avellaneda M., Levy A. and Paras A. (1995). Pricing and hedging derivative securities in markets with uncertain volatilities, *Applied Mathematical Finance* 2, 73-88.

[2] Barles G. (1987), *Solutions de viscosité des équations de Hamilton-Jacobi.* Springer Verlag.

[3] Barles G. and Souganidis P.E. (1991). Convergence of approximation schemes for fully nonlinear second order equations, *Asymptotic Analysis* 4, 271-283.

[4] Black F. and M. Scholes (1973). The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, 637-659.

[5] Broadie M., Cvitanić J. and Soner H.M. (1998). Optimal replication of contingent claims under portfolio constraints, *The Review of Financial Studies* 11, 59-79.

[6] Cheridito P., Soner M. and Touzi N. (2002). Hedging under Gamma constraint in multi-securities markets. In preparation.

[7] Crandall, M.G., Ishii, H. and Lions, P.L. (1992). User's guide to viscosity solutions of second order Partial Differential Equations, *Bull. Amer. Math. Soc.* 27, 1-67.

[8] Cvitanić J. and Karatzas I. (1993). Hedging contingent claims with constrained portfolios, *Annals of Applied Probability* 3, 652-681.

[9] Cvitanić J., Pham H. and Touzi N. (1999). Super-replication in stochastic volatility models under portfolio constraints, *Journal of Applied Probability* 36, 523-545.

[10] Dellacherie C. and Meyer P.A. (1975). *Probabilités et potentiel*, Hermann.

[11] ElKaroui N., Nguyen D.H. and Jeanblanc M. (1986). Compactification methods in the control of degenerate diffusions : existence of an optimal control, *Stochastics* 20, 169-219.

[12] El Karoui N., and Quenez M.-C. (1995). Dynamic programming and pricing of contingent claims in an incomplete market, *SIAM Journal on Control and Optimization* **33**, 29-66.

[13] Föllmer H. and Kramkov D. (1997). Optional decomposition under constraints, *Probability Theory and Related Fields* 109, 1-25.

[14] Fleming W.H. and Soner H.M. (1993). *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, New York, Heidelberg, Berlin.

[15] Friedman A. (1964). *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, N.J.

[16] Haussmann U.G. (1985). Existence of optimal Markovian controls for degenerate diffusions. Proceedings of the third Bad-Honnef Conference, *Lecture Notes in Control and Information Sciences*, Springer-Verlag.

[17] Karatzas I., and Shreve S. (1998). *Methods of Mathematical Finance*, Springer-Verlag, New York, Heidelberg, Berlin.

[18] Krylov N.V. (1980).*Controlled Diffusion Processes*. Springer Verlag, New York.

[19] Merton R.C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory* 3, 373-413. Erratum: ibid. 6 (1973), 213-214.

[20] Neveu, J. (1975) *Discrete-parameter Martingales*, North-Holland, Amsterdam.

[21] Protter, P. (1995) *Stochastic Integration and Differential Equations : a New Approach*, Springer-Verlag.

[22] Rockafellar, R.T. (1970) *Convex Analysis*,Princeton University Press, Princeton, NJ.

[23] Soner H.M. and Touzi N. (2000). Super-replication under Gamma constraints, *SIAM J.Control and Opt.* 39, 73-96.