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ORIGINAL ARTICLE

Valuation of power plants by utility indifference and numerical computation

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Abstract This paper presents a real option valuation model of a power plant, which accounts for physical constraints and market incompleteness. Switching costs, minimum on-off times, ramp rates, or non-constant heat rates are important characteristics that can lead, if neglected, to overestimated values. The existence of non-hedgeable uncertainties is also a feature of energy markets that can impact assets value. We use the utility indifference approach to define the value of the physical asset. We derive the associated mixed optimal switching-control problem and provide a characterization of its solution by means of a coupled system of reflected Backward Stochastic Differential Equations (BSDE). We relate this system to a system of variational inequalities, and we provide a numerical comparative study by implementing BSDE simulation algorithms, and PDE finite differences schemes.

Keywords Real option · Backward stochastic differential equation · Utility indifference · Non-linear Monte Carlo methods · Finite differences for PDE

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1 Introduction

Real option valuation techniques have been introduced to capture the value of flexibility or optionality embedded in investments, and go beyond the traditional notions of Discounted Cash Flows (DCF) and Net Present Value (NPV), where expected future cash flows are estimated and discounted at a suitable rate. The fundamental concepts of this theory are presented for example in the books by Dixit and Pindyck (1994), Trigeorgis (1996) or Schwartz and Trigeorgis (2004), and are easily illustrated on the following example. Consider the investment decision in a thermal power plant at time 0, with estimated lifetime T, capacity q and heat rate (efficiency) H. Let S_t^e and S_t^f be, respectively, the electricity spot price and fuel spot price at time t. If production is decided at time t, the profit made is $S_t^e - HS_t^f$, otherwise it is 0. In the absence of any constraint, the optimal production strategy is to produce when the spark spread $S_t^e - HS_t^f$ is positive, and to turn off the plant when it is non-positive. The cash flows then received at each time t are $q(S_t^e - HS_t^f)^+$. At time T, total profits accumulated sum up to $\int_0^T e^{r(T-t)} q(S_t^e - HS_t^f)^+ dt$. We can then identify the value of the plant to the price of an option on (S^e, S^f) paying a stream of call options. Option pricing methods, well developed in the financial markets, can be used to price this option and estimate the power plant value.

The analysis is less straightforward in the presence of production constraints such as switching costs or minimum on/off times. In this case the optimal production strategy is not obvious and the payoff associated to the plant is complex. In addition, electricity markets may exhibit some incompleteness and risk neutral pricing is no longer the unambiguous valuation method.

One way of dealing with market incompleteness is to select a pricing measure among the set of equivalent martingale measure. The real option valuation problem thus takes the form of an optimal impulse control problem. Recent works have tackled this problem under production constraint. Deng and Oren (2003), Tseng and Barz (2002) or Gardner and Zhuang (2000) propose different methods to take production constraints into account, based on Stochastic Dynamic Programming. Hamadène and Jeanblanc (2007) study the starting and stopping problem of a power plant subject to start-up and shut-down costs in the Backward Stochastic Differential Equation (BSDE) framework. Ludkovski (2005), Carmona and Ludkovski (2006), Pham et al. (2007) and Djehiche et al. (2007) studied a generalization to multiple mode switching, while Hamadène and Hdhiri (2005), Hdhiri (2006) considered a generalization with discontinuous processes.

An alternative method is the utility indifference pricing. This well known method has been extensively studied for the pricing of European and American options, see for example Hobson (2003), Henderson and Hobson (2004), El Karoui and Rouge (2000), Kobylanski et al. (2002), Mania and Schweizer (2005). In this paper, we propose an extension of the utility indifference pricing method to the case of a regime switching asset, that allows for the introduction of portfolio constraints, market incompleteness and production constraints. The main tool of our analysis is the theory of Backward Stochastic Differential Equations (BSDE), which unifies the two methodologies introduced by Dixit and Pindyck, dynamic programming, on the one hand, and risk neutral pricing of contingent claims, on the other hand, see, e.g. El Karoui and Rouge (2000), El Karoui et al. (1997). Kobylanski et al. (2002) shows how the price of an American option can be related to the solution of a Reflected BSDE. Our main result provides a characterization of the indifference price of the production asset as the initial value of a coupled system of reflected BSDEs.

The paper is organized as follows. Section 3 gives a formal description of the problem and formulates the optimal control problem arising in the definition of the utility-based valuation. In Sect. 4, we state the verification result which relates this optimal control problem to a coupled system of reflected BSDEs. In Sect. 5, we provide a constructive proof of the existence of a solution to this system using a sequence of approximated optimal control problem. Section 6 gives some properties of the price in accordance with the know results on indifference prices for European and American options. In Sect. 7, we use the classical connection between BSDEs and semilinear PDEs to provide an equivalent formulation in terms of a coupled system of obstacle problems for PDEs. We finally present in Sect. 8 two alternative numerical methods to compute the indifferent price: a Monte Carlo based simulation algorithm (Bally et al. 2005; Bouchard and Touzi 2005; Gobet et al. 2004) to solve the BSDE and a finite differences scheme to solve the PDE. We provide a comparative implementation of both methods on several examples, in both cases of complete and incomplete market. In a complete market, we observe that the finite differences scheme converges faster than the BSDE-based algorithm when the dimension is small ($N = 2$). We obtain results in dimension 4 for the Monte Carlo method showing its tractability and potential advantage in higher dimension where the PDE method would be harder to implement for memory-storage reasons. In the presence of market incompleteness, the non-linearities in the equations characterizing the indifferent price highly increase the computational complexity. While there exists a proof a convergence for the finite differences scheme (see Chaumont et al. 2005), no such result exists for the BSDE-based algorithm. We observe that the finite differences scheme still converges, at a higher cost in terms of computational time though, and that a straightforward adaptation of the BSDE-based algorithm for quadratic BSDEs becomes quickly untractable.

2 Notations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $\mathbb{F} := \{\mathcal{F}_t, t \geq 0\}$ which satisfies the usual conditions. Let $T > 0$ be a given fixed maturity, and $\{W_t, 0 \leq t \leq T\}$ a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with values in $\mathbb{R}^n$. We denote by $\mathbb{E}[.]$ the expectation operator under $\mathbb{P}$ and $\mathbb{E}_t[.] := \mathbb{E}[.|\mathcal{F}_t]$ the conditional expectation operator with respect to $\mathcal{F}_t$. Expectation under another probability measure $\mathbb{Q}$ will be denoted by $\mathbb{E}^{\mathbb{Q}}[.]$.

We will make use of the following notation throughout the article. For a subset K of $\mathbb{R}^n$, we denote by $L^\infty(K)$ the set of all bounded $\mathcal{F}_T$-measurable K-valued random variables, and by $\mathcal{H}^2(K)$ the set of all $\mathbb{F}$-adapted K-valued processes C such that: $\mathbb{E}\left(\int_0^T C_t^2 dt\right) < \infty$. The subset of all continuous processes in $\mathcal{H}^2(K)$ is denoted $\mathcal{H}^2_0(K)$. The set of all $\mathbb{F}$-adapted, K-valued and bounded processes is denoted by $\mathcal{H}^\infty(K)$. Similarly, $\mathcal{H}^\infty_0(K)$ consists of all continuous processes of $\mathcal{H}^\infty(K)$. The set of all $\mathbb{F}$-adapted, K-valued, continuous, non-decreasing processes, starting from 0 is denoted $\mathcal{J}(K)$.

The set $\mathcal{M}_n(K)$ is the collection of all $n \times n$ matrices with entries in K. For a matrix $M \in \mathcal{M}_n(K)$, we denote by M^* its transpose. Given two vectors $x, y \in \mathbb{R}^d$, we denote by $x \cdot y$ the Euclidean scalar product, by $|x| = \sqrt{x \cdot x}$ the Euclidean norm, and by $\mathrm{diag}[x]$ the diagonal matrix with diagonal elements given by the components of x. Finally, for $x, y \in \mathbb{R}$, we shall use the notations $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$.

3 Problem formulation

Throughout the paper we consider an agent whose preferences are described by the exponential utility function:

$$U(x) := -e^{-\eta x}, \quad x \in \mathbb{R},$$

where the parameter $\eta > 0$ corresponds to the constant absolute risk aversion level of the agent. This agent is allowed to manage a physical asset and to invest on a financial market.

3.1 Input and output commodity market

We consider a financial market on which are traded the input and output commodities, and containing a non-risky financial asset, whose price process is normalized to unity, by the usual change of numéraire. The financial market is defined by a multidimensional stochastic price process S with values in $\mathbb{R}^N$, solution of the multivariate stochastic differential equation:

$$dS_t = \hat{\mu}(t, S_t)dt + \hat{\Sigma}(t, S_t)dW_t,$$

where $\hat{\mu}(t, S_t) = \mathrm{diag}[S_t]\mu_t$, $\hat{\Sigma}(t, S_t) = \mathrm{diag}[S_t]\Sigma_t$, and the stochastic processes (μ, Σ), valued, respectively, in $\mathbb{R}^N$ and $\mathcal{M}_N(\mathbb{R})$, are bounded predictable processes. We also suppose that Σ has full rank and Σ^{-1} is bounded.

3.2 Management strategies

The physical asset can be in M different modes. We denote by ψ_t^i, $1 \leq i \leq M$, the instantaneous rate of benefit in mode i. Throughout this paper, we assume that $\psi^i \in \mathcal{H}^\infty(\mathbb{R})$. This is indeed a restriction due to the fact that our analysis is based on the approach of quadratic BSDEs developed by Kobylanski (see Kobylanski 2000), which requires the boundedness of the terminal condition. A possible extension to unbounded terminal conditions and instantaneous gains may be obtained by following the recent paper by Briand and Hu (2005). For the sake of simplicity in the notation, we also introduce a fictitious mode $i = 0$ with $\psi^0 \equiv 0$ corresponding to the absence of the physical asset.

Example 1 (linear production cost) The simplest example has two states off (1) and on (2), no maintenance costs, i.e. $\psi^1 \equiv 0$, and a linear production cost function of the type $\widetilde{\psi}_t^2 = q\left(S_t^1 - HS_t^2\right)$, where S^1 is the electricity spot price, S^2 the gas spot price, q is a constant production capacity, and H is a constant heat rate. As we require ψ^2 to be bounded, we can define $\psi_t^2 = h\left(\widetilde{\psi}_t^2\right)$ where the function h is the threshold function: $h(x) := x\mathbf{1}_{[\underline{C},\overline{C}]} + \underline{C}\mathbf{1}_{(-\infty,\underline{C}]} + \overline{C}\ \mathbf{1}_{[\overline{C},\infty)}$.

In addition to the benefit rate functions ψ^i, the production asset is characterized by an horizon T, a terminal payoff $\chi \in L^\infty(\mathbb{R})$ at time T, switching costs $C_{i,j} \geq 0$ when switching from mode i to $j \neq i$ and minimal times δ_i in each mode. In words, this means that switching the production asset from mode i to mode $j \neq i$ at some time t induces the cost $C_{i,j}$, and implies that the production regime cannot be changed before time $t + \delta_j$. We suppose throughout the paper the conditions:

$$\forall n \geq 1, \quad \forall(i_0, \ldots, i_n), \quad C_{i_0,i_1} + \cdots + C_{i_{n-1},i_n} + \delta_{i_0} + \cdots + \delta_{i_n} > 0, \tag{1}$$

$$\forall i, j, k, \quad C_{i,j} + C_{j,k} \geq C_{i,k}. \tag{2}$$

Condition (1) implies that a management strategy with infinitely many switches either impossible or non-optimal. Condition (2) is a natural condition on the structure of switching costs.

In order to define the set of admissible management strategies of the production asset, we need to introduce the functions:

$$\overline{\delta}_i(t) := (t + \delta_i) \wedge T, \quad 1 \leq i \leq M.$$

Definition 1 A management strategy of the production asset is an $\mathbb{F}$-adapted càdlàg pure jump process $\{\xi_t,\ t \in [0, T]\}$ with values in $\{1, \ldots, M\}$, with jump times $(\theta_n, n \geq 0)$ and states $(\xi^n,\ n \geq 0)$, such that, for all $n \geq 0$, $\overline{\delta}_{\xi^n}(\theta_n) \leq \theta_{n+1}$. In this setting, we have:

$$\xi_t = \sum_{n\geq 0} \xi^n \mathbf{1}_{\{\theta_n \leq t < \theta_{n+1}\}}.$$

An admissible management strategy is such that $N(\xi) := \inf\{n \geq 0,\ \theta_n = T\} < \infty$ a.s., i.e. which is composed a.s. of a finite number of switches. We denote by $\mathcal{X}_0$ the set of such admissible strategies. Given a management strategy $\xi \in \mathcal{X}_0$, we denote by $\mathcal{X}_t(\xi)$ the set of all admissible strategies ξ' such that $\xi' = \xi$ on $[0, t]$.

We will also make use of the following notation. The set of all $\mathbb{F}$-stopping times with values in $[t, T]$ will be denoted by $\mathcal{T}_t$. Given a management strategy $\xi \in \mathcal{X}_0$, we define the sequence $(\theta_n^* := \overline{\delta}_{\xi^n}(\theta_n), n \geq 0)$ of the switching times increased by the minimal times. We also define the sequence $(C_n^* := C_{\xi^n,\xi^{n+1}})$ of the switching costs. Conditions (1)–(2) ensure that a management strategy ξ such that $\mathbb{P}(N(\xi) = \infty) > 0$ is either not possible (presence of minimal times) or not optimal (presence of switching costs). This justifies the choice of the admissible set $\mathcal{X}_0$. Without loss of generality, we suppose that the power plant has just been switched to mode 1 at time 0 ($\theta_0 = 0$ and

$\xi_0 = 1$). Given a management strategy of the plant $\xi \in \mathcal{X}_0$, we define the instantaneous cash flow at time $t \in [0, T]$:

$$dB_t^{\xi} := \psi_t^{\xi_t} dt - C_{\xi_{t-},\xi_t},$$

with the convention that $C_{i,i} = 0$ for all i.

3.3 Investment strategies

In addition to the production activity, the producer is allowed to invest continuously in the financial market. We shall denote by π_t the amount invested in the market at time t. We suppose that the cash flows generated by the management strategies are invested in the portfolio. By the usual self-financing condition, the wealth process X is defined for any $t \in [0, T]$ by:

$$X_t^{x,\pi,\xi} := x + \int_0^t \sum_{i=1}^N \pi_u^i \frac{dS_u^i}{S_u^i} + B_T^{\xi} = x + \int_0^t \pi_u \cdot (\mu_u du + \Sigma_u dW_u) + B_T^{\xi},$$

where x denotes the initial capital. In order to account for possible portfolio constraints, we assume that the process π takes values in some given closed convex subset K of $\mathbb{R}^N$. We follow the definition of Hu et al. (2005) of admissible investment strategies on the financial market.

Definition 2 An investment strategy is an $\mathbb{F}-$predictable K-valued process $\pi = \{\pi_t,\ 0 \le t \le T\}$ with $\mathbb{E}\int_0^T \left|\Sigma_t^* \pi_t\right|^2 dt < \infty$ a.s. such that, for all $\xi \in \mathcal{X}_0$, the family $\left\{e^{-\eta X_\tau^{0,\pi,\xi}} : \tau \text{ stopping time with values in } [0, T]\right\}$ is uniformly integrable. We denote by $\mathcal{A}_0$ the collection of all such investment strategies. For all stopping time τ and $\pi^0 \in \mathcal{A}_0$, we denote by $\mathcal{A}_\tau(\pi^0)$ the subset of $\mathcal{A}_0$ consisting of all investment strategies $\pi \in \mathcal{A}_0$ such that $\pi = \pi^0$ on $[0, \tau]$.

We will denote by $X^{x,\pi}$ the wealth process of an agent which only invests on the financial markets and does not own the physical asset: $dX_t^{x,\pi} = \pi_t \cdot (\mu_t dt + \Sigma_t dW_t)$.

Example 2 (Incomplete market.) Let $K = \mathbb{R}^k \times \{0\}^{N-k}$, for some $k \in \{1, \dots, N\}$. Then only the first k components of S represent prices of financial assets which can be traded by the producer.

3.4 Utility valuation of the production asset

The variable χ represents some terminal payoff associated with the presence of the power plant. For instance, it may represent the dismantling cost of the power plant. Similarly, we introduce the random variable $\chi' \in L^\infty$ as the terminal payoff in the absence of the power plant, that may be different from χ. Let

$$V_0(x) := \sup_{(\xi,\pi)\in\mathcal{X}_0\times\mathcal{A}_0} \mathbb{E}\left[U\left(X_T^{x,\pi,\xi}+\chi\right)\right], \tag{3}$$

$$v_0(x) := \sup_{\pi\in\mathcal{A}_0} \mathbb{E}\left[U\left(X_T^{x,\pi}+\chi'\right)\right], \tag{4}$$

be the indirect utility function of the manager, respectively, in the presence and absence of the power plant. Then, the utility valuation of the power plant is defined by:

$$p_0(x) := \sup\{p \geq 0 \;:\; V_0(x-p) \geq v_0(x)\}, \tag{5}$$

i.e. the highest price the agent is ready to pay for the purchase of the power plant. In the context of the exponential utility, we can write:

$$v_0(x) \;=\; -e^{-\eta(x+y_0)} \quad \text{and} \quad V_0(x) \;=\; -e^{-\eta(x+\overline{Y}^1_0)},$$

where y_0 and $\overline{Y}^1_0$ are independent of the initial capital x. Then the value of the plant is given by:

$$p_0 = \overline{Y}^1_0 - y_0. \tag{6}$$

The main result of this paper provides a characterization of $(y_0, \overline{Y}^1_0)$ by means of a coupled system of reflected Backward Stochastic Differential Equations.

4 A verification result

In this section we relate v_0 and V_0 to the solution of a coupled system of reflected BSDEs.

4.1 Quadratic BSDEs and optimal investment decision

The analysis of this paper appeals to the notion of quadratic BSDEs. We first provide a brief description and mention a useful result.

Given a quadratic generator $f \;:\; \Omega\times[0,T]\times\mathbb{R}^n \;\longrightarrow\; \mathbb{R}$, satisfying for all $t\in[0,T]$ and $z\in\mathbb{R}^n$:

$$|f(t,z)| \leq a_0 + b_0|z|^2 \quad \text{and} \quad \left|\frac{\partial f}{\partial z}(t,z)\right| \leq a_1 + b_1|z| \text{ a.s.}, \tag{7}$$

for some non-negative constants $a_0,\ a_1,\ b_0,\ b_1$, and a bounded $\mathcal{F}_T$-measurable random variable ζ, we consider the following BSDE:

$$Y_t = \zeta - \int_t^T f(u,Z_u)du - \int_t^T Z_u\cdot\Sigma_u dW_u. \tag{8}$$

Since Σ^{-1} is bounded, the existence of a unique solution $(Y, Z) \in \mathcal{H}_0^\infty(\mathbb{R}) \times \mathcal{H}_0^2(\mathbb{R}^N)$ to this BSDE follows from the results of Kobylanski (2000). In addition, the process Z exhibits the following property:

Lemma 1 *The process* $\int_0^\cdot Z dW$ *is a BMO martingale on* $[0, T]$, *i.e. for all stopping time* τ *with values in* $[0, T]$, $\mathbb{E}_\tau[\int_\tau^T |Z_s|^2 ds] < \infty$. *As a consequence, the stochastic exponential* $\mathcal{E}\left(\int_0^\cdot Z dW\right)$ *is a uniformly integrable martingale.*

Proof See for example Hu et al. (2005) or Briand and Hu (2005). □

In the sequels, we will denote by

$$\mathcal{E}_{t,T}^f[\zeta] := Y_t$$

the value at time t of the Y component of the solution of BSDE (8). This notation is inspired by the notion of non-linear expectation introduced by Peng (1997, 2004). However, in our case, we do not deal with a non-linear expectation per say since $f_t(0)$ may not be 0, and no properties of non-linear expectations will be used in what follows.

For $0 \le i \le M$, we introduce the random functions:

$$\begin{aligned} f_t^i(z) := \frac{\eta}{2}\left|\Sigma_t^* z - \frac{1}{\eta}\Sigma_t^{-1}\mu_t - \Pi_t\left(\Sigma_t^* z - \frac{1}{\eta}\Sigma_t^{-1}\mu_t\right)\right|^2 \\ + z \cdot \mu_t - \frac{1}{2\eta}\left|\Sigma_t^{-1}\mu_t\right|^2 - \psi_t^i, \end{aligned} \tag{9}$$

where $\Pi_t(x)$ is the orthogonal projection of x on the closed convex set $\Sigma_t^* K$, the image of K by Σ_t^*. We are then able to relate the value function v_0 to the initial value of a quadratic BSDE, as was proved by Hu et al. (2005):

Proposition 1 (Hu et al. 2005) *The indirect utility function of the manager in the absence of the production asset is given by*:

$$v_0(x) = -\exp\left(-\eta\left(x + \mathcal{E}_{0,T}^{f^0}[\chi']\right)\right).$$

4.2 Optimal management-investment decision

This section relates the value function V_0 to the initial value of a system of reflected BSDEs. We consider the coupled system of Reflected BSDEs (RBSDE) (Y_t^i, Z_t^i, K_t^i), for $t \in [0, T]$ and $1 \le i \le M$:

$$Y_t^i = \chi - \int_t^T f_u^i\left(Z_u^i\right) du - \int_t^T Z_u^i \cdot \Sigma_u dW_u + \left(K_T^i - K_t^i\right), \tag{10}$$

$$Y_t^i \ge \max_{j \ne i}\left\{\overline{Y}_t^j - C_{i,j}\right\}, \tag{11}$$

$$\overline{Y}_t^i = \mathcal{E}_{t,\overline{\delta}_i(t)}^{f^i}\left[Y_{\overline{\delta}_i(t)}^i\right], \tag{12}$$

$$K^i \in \mathcal{J}(\mathbb{R}), \quad \int_0^T \left(Y_t^i - \max_{j \neq i}\left\{\overline{Y}_t^j - C_{i,j}\right\}\right) dK_t^i = 0. \tag{13}$$

The existence and uniqueness of a solution $(Y^i, Z^i, K^i) \in \mathcal{H}_0^\infty(\mathbb{R}) \times \mathcal{H}_0^2(\mathbb{R}^N) \times \mathcal{J}(\mathbb{R})$, $1 \leq i \leq M$ to the system of coupled RBSDEs (10)–(13) will be discussed in the subsequent section.

We can however provide intuition on these processes. First, Eq. (12) must be understood as the value at time t of the Y component of BSDE:

$$\overline{Y}_s^{i,t} = Y_{\overline{\delta}_i(t)}^i - \int_s^T f_u^i\left(\overline{Z}_u^{i,t}\right) du - \int_s^T \overline{Z}_u^{i,t} \cdot \Sigma_u dW_u$$

for $t \leq t \leq \overline{\delta}_i(t)$. The quantity $\overline{Y}_t^i$ is then defined as $\overline{Y}_t^i := \overline{Y}_t^{i,t}$. Remark that if there exists a solution to the system, the processes $\overline{Y}^i$ are adapted, bounded and continuous (cf. a priori estimates and monotone stability property of quadratic BSDEs, e.g. in Kobylanski 2000). In accordance with Proposition 1, Eq. (12) characterizes the value of an optimal investment problem between times t and $\overline{\delta}_i(t)$ when terminal value is $Y_{\overline{\delta}_i(t)}^i$ and the agent benefits from revenues ψ_s^i at time s, i.e. the physical asset is in mode i during the time interval. If process Y^i characterizes the optimal utility of the agent when there is no constraint on the first switching time, then $\overline{Y}^i$ characterizes the optimal utility when the physical asset has just been switched to mode i. Conversely, if the asset is in mode i at time t and there is no constraint on the next switching time, the agent's problem is to invest optimally until the occurrence of an optimal switch, i.e. a time when some other mode provides a bigger utility, taking into account the switching cost. This gives the intuition behind the reflected BSDE with obstacle equal to the biggest utility in alternatives modes, adjusted of the switching costs.

The main result of this section confirms this intuition and provides a characterization of the value function V_0, defined in (3), in terms of the component $\overline{Y}^1$ of the solution of the RBSDE (10), (11), and (13). The reason why the component $\overline{Y}^1$ appears in the proposition is only the result of the assumption made previously that the plant has just been switched to mode 1 at time 0. If the plant was switched to mode i_0 at time 0, it would be replaced by $\overline{Y}^{i_0}$; if the plant was in mode i_0 with no constraint on the first time switch at time 0, it would be replaced by Y^{i_0}.

Proposition 2 *Suppose that there exists a solution to* (10)–(13)*, then the value of the optimal problem* (3) *is given by*:

$$V_0(x) = U\left(x + \overline{Y}_0^1\right).$$

Moreover, define the management strategy $\hat{\xi}$ *by* $\hat{\theta}_0 = 0$, $\hat{\xi}_0 = 1$, *and*:

$$\hat{\theta}_{n+1} = \inf\left\{ t \geq \overline{\delta}_{\hat{\xi}^n}(\hat{\theta}_n),\ Y_t^{\hat{\xi}^n} = \max_{j \neq \hat{\xi}^n} \left\{ \overline{Y}_t^j - C_{\hat{\xi}^n, j} \right\} \right\} \wedge T,$$
$$\hat{\xi}^{n+1} = \min\left\{ j \neq \hat{\xi}^n,\ Y_{\hat{\theta}_{n+1}}^{\hat{\xi}^n} = \overline{Y}_{\hat{\theta}_{n+1}}^j - C_{\hat{\xi}^n, j} \right\},$$

for $n \geq 0$, $1 \leq i \leq M$, *and the investment strategy* $\hat{\pi}$ *by*:

$$\hat{\pi}_t = (\Sigma_t^*)^{-1} \Pi_t \left((\eta \Sigma_t)^{-1} \mu_t - \Sigma_t^* \overline{Z}_t^{\hat{\xi}^n, \hat{\theta}_n} \right) \quad \text{for } \hat{\theta}_n \leq t < \overline{\delta}_{\hat{\xi}^n}(\hat{\theta}_n),$$
$$\hat{\pi}_t = (\Sigma_t^*)^{-1} \Pi_t \left((\eta \Sigma_t)^{-1} \mu_t - \Sigma_t^* Z_t^{\hat{\xi}^n} \right) \quad \text{for } \overline{\delta}_{\hat{\xi}^n}(\hat{\theta}_n) \leq t < \hat{\theta}_{n+1},$$

for $1 \leq i \leq M$ *and* $n \geq 0$. *Then* $(\hat{\xi}, \hat{\pi})$ *defines an optimal management-investment strategy.*

To prove this result, we use a verification argument as in El Karoui et al. (1997) and Hu et al. (2005). Let $n \in \mathbb{N}$ and $(\xi, \pi) \in \mathcal{X}_0 \times \mathcal{A}_0$. Define the family of processes:

$$R_t^{n,\xi,\pi}(\xi', \pi') := U\left(X_t^{0,\pi'} + B_t^{\xi'} + Y_t^{\xi^n} \right), \tag{14}$$

for $t \in [0, T]$ and $(\xi', \pi') \in \mathcal{X}_{\theta_n}(\xi) \times \mathcal{A}_{\theta_n}(\pi)$. For the sake of simplicity, we will write $R_t^{n,\xi,\pi} := R_t^{n,\xi,\pi}(\xi', \pi')$ whenever $t \leq \theta_n$ as the latter quantity only depend on (ξ, π). Observe that, since ψ^i and Y^i are bounded, the process $R^{n,\xi,\pi}(\xi', \pi')$ is of class D and is thus well defined and integrable. We start with the following lemma:

Lemma 2 *Assume that the coupled system of RBSDEs* (10)–(13) *has a solution with bounded processes* Y^i. *Let* $n \in \mathbb{N}$, $(\xi, \pi) \in \mathcal{X}_0 \times \mathcal{A}_0$ *be fixed.*

(i) *For every* $(\xi', \pi') \in \mathcal{X}_{\theta_n^*}(\xi) \times \mathcal{A}_{\theta_n^*}(\pi)$, *the process* $\left\{ R_t^{n,\xi,\pi}(\xi', \pi'), \theta_n^* \leq t \leq \theta'_{n+1} \right\}$ *is a super-martingale and*:

$$\mathbb{E}_{\theta_n^*}\left[R_{\theta'_{n+1}}^{n,\xi,\pi}(\xi', \pi') \right] \leq e^{\eta C_n'^*} R_{\theta_n^*}^{n,\xi,\pi}. \tag{15}$$

(ii) *Let* $(\hat{\xi}, \hat{\pi}) \in \mathcal{X}_{\theta_n^*}(\xi) \times \mathcal{A}_{\theta_n^*}(\pi)$ *such that*:

$$\hat{\theta}_{n+1} := \inf\left\{ t \geq \theta_n^*,\ dK_t^{\xi^n} > 0 \right\} \wedge T, \tag{16}$$
$$\hat{\pi}_t := \pi_t^0 \mathbf{1}_{[0,\theta_n^*)}(t) + (\Sigma_t^*)^{-1} \Pi_t \left[(\eta \Sigma_t)^{-1} \mu_t - \Sigma_t^* Z_t^{\xi^n} \right] \mathbf{1}_{\left[\theta_n^*, \hat{\theta}_{n+1}\right]}(t). \tag{17}$$

Then, the process $\left\{ R^{n,\xi,\pi}(\hat{\xi}, \hat{\pi}),\ \theta_n^* \leq t < \hat{\theta}_{n+1} \right\}$ *is a martingale and*:

$$\mathbb{E}_{\theta_n^*}\left[R_{\hat{\theta}_{n+1}}^{n,\xi,\pi}(\hat{\xi}, \hat{\pi}) \right] = e^{\eta C_n'^*} R_{\theta_n^*}^{n,\xi,\pi}.$$

Proof (i) Since the processes Y^i, ψ^i are bounded, and $\pi' \in \mathcal{A}_0$, we only need to check that the process $R^{n,\xi,\pi}(\xi',\pi')$ is a local super-martingale on $[\theta_n^*, \theta'_{n+1}]$. On this interval, we can decompose this process into:

$$R_t^{n,\xi,\pi}(\xi',\pi') = R_{\theta_n^*}^{n,\xi,\pi} M_t^{n,\pi}(\pi') A_t^{n,\xi,\pi}(\xi',\pi'),$$

where $M_t^{n,\pi}(\pi')$ is a local martingale defined by $M_{\theta_n^*}^{n,\pi}(\pi') = 1$,

$$\frac{dM_t^{n,\pi}(\pi')}{M_t^{n,\pi}(\pi')} = -\eta(\pi'_t + Z_t^{\xi^n}) \cdot \Sigma_t dW_t,$$

and $A^{n,\xi,\pi}(\xi',\pi')$ is a bounded variation process defined by $A_{\theta_n^*}^{n,\xi,\pi}(\xi',\pi') = 1$ and

$$\frac{dA_t^{n,\xi,\pi}(\xi',\pi')}{A_t^{n,\xi,\pi}(\xi',\pi')} = \left(-\eta f_t^{\xi^n}(Z_t^{\xi^n}) - \eta\pi'_t \cdot \mu_t + \frac{\eta^2}{2}\left|\Sigma_t^*\left(\pi'_t + Z_t^{\xi^n}\right)\right|^2\right)dt$$
$$-\eta dB_t^{\xi'} + \eta dK_t^{\xi^n}.$$

Observing that $dB_t^{\xi'} = \psi_t^{\xi^n} dt - C_n'^* \mathbf{1}_{\{t=\theta'_{n+1}\}}$ for $\theta_n^* \le t \le \theta'_{n+1}$, that K^{ξ^n} is non-decreasing, and that:

$$f_t^i(Z_t^i) = \inf_{\pi \in K} -\psi_t^i - \pi \cdot \mu_t + \frac{\eta}{2}\left|\Sigma_t^*\left(\pi + Z_t^i\right)\right|^2, \tag{18}$$

for all $1 \le i \le M$, we deduce that $A^{n,\xi,\pi}(\xi',\pi')$ is non-decreasing on $[\theta_n^*, \theta'_{n+1})$. Therefore $R^{n,\xi,\pi}(\xi',\pi')$ is a local super-martingale. Property (15) follows from the fact that $\Delta A_{\theta'_{n+1}}^{n,\xi,\pi}(\xi',\pi') = \eta C_n'^* A_{\theta'_{n+1}}^{n,\xi,\pi}(\xi',\pi')$.

(ii) (a) In this step, we show that $R^{n,\xi,\pi}(\hat{\xi},\hat{\pi})$ is a martingale on $[\theta_n^*, \hat{\theta}_{n+1})$. Observe that the process $\hat{\pi}_t$ defined by (16) is the (unique) minimizer of the problem (18). From this and the definition of $\hat{\theta}_{n+1}$, $A_t^{n,\xi,\pi}(\hat{\xi},\hat{\pi}) = A_{\theta_n^*}^{n,\xi,\pi}$ for $t \in [\theta_n^*, \hat{\theta}_{n+1})$. Then $R_t^{n,\xi,\pi}(\xi',\pi') = R_{\theta_n^*}^{n,\xi,\pi} M_t^{n,\pi}(\pi') A_{\theta_n^*}^{n,\xi,\pi}$ is a local martingale. By Lemma 1, it follows that the process $\int Z^i \cdot \Sigma dW$ is a BMO martingale, for all $1 \le i \le M$. In order to show that $R^{n,\xi,\pi}(\hat{\tau},\hat{\pi})$ is a martingale on $[\theta_n^*, \hat{\theta}_{n+1})$, it is sufficient to prove that the process $\int \hat{\pi} \cdot \Sigma dW^1$ is a also a BMO martingale, as it is proved in Hu et al. (2005). Observe that, for $t \in [\theta_n^*, \hat{\theta}_{n+1}]$,

$$\left|\Sigma_t^*\hat{\pi}_t\right|^2 = \left|\Pi_t\left(\eta^{-1}\Sigma_t^{-1}\mu_t - \Sigma_t^* Z_t^{\xi^n}\right)\right|^2 \le 2\eta^{-2}\left|\Pi_t\left(\Sigma_t^{-1}\mu_t\right)\right|^2 + 2\left|\Pi_t\left(\Sigma_t^* Z_t^{\xi^n}\right)\right|^2.$$

We then deduce that, for all stopping time τ with values in $[\theta_n^*, \hat{\theta}_{n+1})$,

$$\mathbb{E}\left[\int_\tau^T \left|\Sigma_t^*\hat{\pi}_t\right|^2 dt \middle| \mathcal{F}_\tau\right] \le c_1 + 2\mathbb{E}\left[\int_\tau^T \left|\Pi_t\left(\Sigma_t^* Z_t^{\xi^n}\right)\right|^2 dt \middle| \mathcal{F}_\tau\right] \le c_1 + 2c_0,$$

for some constant c_1. Since the latter bound does not depend on the arbitrary stopping time τ, this shows that the process $\int \hat{\pi} \cdot \Sigma dW$ is a BMO martingale on $[\theta_n^*, \hat{\theta}_{n+1}]$.

(b) We now prove that $\hat{\pi}$ is in $\mathcal{A}_0$. On $\left[0, \theta_n^*\right]$, $\hat{\pi}$ is equal to $\pi^0 \in \mathcal{A}_0$. The BMO martingale property of $\int \hat{\pi} \cdot \Sigma dW$ on $[\theta_n^*, \hat{\theta}_{n+1}]$ implies that $\mathbb{E}\left[\int_{\theta_n^*}^{\hat{\theta}_{n+1}} \left|\Sigma_t^* \hat{\pi}_t\right|^2 dt\right] < \infty$, and therefore $\mathbb{E}\left[\int_0^T \left|\Sigma_t^* \hat{\pi}_t\right|^2 dt\right] < \infty$. Using now the BMO martingale property of $\int \overline{Z}^{\xi^n} \cdot \Sigma dW$, we prove that $M^{n,\pi}(\hat{\pi})$ is a uniformly integrable martingale on $[\theta_n^*, \hat{\theta}_{n+1}]$. As $A^{n,\xi,\pi}(\hat{\xi}, \hat{\pi})$ is bounded on $[\theta_n^*, \hat{\theta}_{n+1}]$, $R^{n,\xi,\pi}(\hat{\xi}, \hat{\pi})$ is a uniformly integrable family on $[\theta_n^*, \hat{\theta}_{n+1}]$, and so is $e^{-\eta X_t^{0,\hat{\pi}}}$. Hence $\hat{\pi}$ is an admissible portfolio.

(c) We complete the proof by noticing that at time $\hat{\theta}_{n+1}$: $A^{n,\xi,\pi}_{\hat{\theta}_{n+1}}(\hat{\xi}, \hat{\pi}) = e^{\eta C_n'^*} A^{n,\xi,\pi}_{\theta_n^*}$.

□

We then deduce the proposition:

Proposition 3 *Let $n \in \mathbb{N}$, $(\xi, \pi) \in \mathcal{X}_0 \times \mathcal{A}_0$ be fixed. Then we have*:

$$\underset{(\xi',\pi') \in \mathcal{X}_{\theta_n^*}(\xi) \times \mathcal{A}_{\theta_n^*}(\pi)}{\text{ess. sup}} \mathbb{E}_{\theta_n^*}\left[U\left(X^{0,\pi'} + B^{\xi'} + \overline{Y}^{\xi'^{n+1}}\right)_{\theta'_{n+1}}\right] = R^{n,\xi,\pi}_{\theta_n^*}.$$

Proof Let $(\xi', \pi') \in \mathcal{X}_{\theta_n^*}(\xi) \times \mathcal{A}_{\theta_n^*}(\pi)$. Then, $\overline{Y}^j_{\theta'_{n+1}} \leq C_{\xi^n, j} + Y^{\xi^n}_{\theta'_{n+1}}$ for all $j \neq \xi^n$, together with the super-martingale property of $R^{n,\xi,\pi}(\xi', \pi')$, yield:

$$\begin{aligned}
\mathbb{E}_{\theta_n^*}\left[U\left(X^{0,\pi'} + B^{\xi'} + \overline{Y}^{\xi'^{n+1}}\right)_{\theta'_{n+1}}\right] &\leq \mathbb{E}_{\theta_n^*}\left[U\left(X^{0,\pi'} + B^{\xi'} + C_n'^* + Y^{\xi^n}\right)_{\theta'_{n+1}}\right] \\
&\leq \mathbb{E}_{\theta_n^*}\left[e^{-\eta C_n'^*} R^{n,\xi,\pi}_{\theta'_{n+1}}(\xi', \pi')\right] \\
&\leq R^{n,\xi,\pi}_{\theta_n^*}. \qquad (19)
\end{aligned}$$

Thus,

$$\underset{(\xi',\pi') \in \mathcal{X}_{\theta_n^*}(\xi) \times \mathcal{A}_{\theta_n^*}(\pi)}{\text{ess. sup}} \mathbb{E}_{\theta_n^*}\left[U\left(X^{0,\pi'} + B^{\xi'} + \overline{Y}^{\xi'^{n+1}}\right)_{\theta'_{n+1}}\right] \leq R^{n,\xi,\pi}_{\theta_n^*}.$$

The converse inequality is obtained by observing that (19) is in fact an equality for the choice of a pair $(\hat{\xi}, \hat{\pi})$ characterized in the previous lemma. □

We can then turn to the proof of Proposition 2:

Proof of Proposition 2. Since $V_0(x) = e^{-\eta x} V_0(0)$, we only deal with the case of a zero initial capital. Let (ξ, π) be a pair of management-investment strategies in $\mathcal{X}_0 \times \mathcal{A}_0$. Results from Hu et al. (2005) to the processes $\overline{Y}^i$, $1 \leq i \leq M$, on intervals of the form $[\theta_n, \theta_n^*]$ allow us to derive the following properties:

$$\underset{(\xi',\pi') \in \mathcal{X}_{\theta_n}(\xi) \times \mathcal{A}_{\theta_n}(\pi)}{\text{ess. sup}} \mathbb{E}_{\theta_n}\left[R^{n,\xi,\pi}_{\theta_n^*}(\xi', \pi')\right] = R^{n,\xi,\pi}_{\theta_n},$$

where the argument of the supremum depends only on π', and the supremum is attained for $\pi' = \hat{\pi}$. Using this result together with Lemma 2 we get:

$$\mathbb{E}_{\theta_n^*}\left[U\left(X^{0,\pi} + B^{\xi} + \overline{Y}^{\xi^{n+1}}\right)_{\theta_{n+1}}\right] \leq U\left(X^{0,\pi} + B^{\xi} + Y^{\xi^n}\right)_{\theta_n^*},$$

thus

$$\mathbb{E}_{\theta_n}\left[U\left(X^{0,\pi} + B^{\xi} + \overline{Y}^{\xi^{n+1}}\right)_{\theta_{n+1}}\right] \leq U\left(X^{0,\pi} + B^{\xi} + \overline{Y}^{\xi^n}\right)_{\theta_n}. \tag{20}$$

Using the fact that ξ has a finite number of switches almost surely ($N(\xi) < \infty$ a.s.), a direct iteration of these inequalities implies:

$$\mathbb{E}\left[U\left(X_T^{0,\pi} + B_T^{\xi} + \chi\right)\right] \leq U\left(X_{\theta_0}^{0,\pi} + B_{\theta_0}^{\xi} + \overline{Y}_{\theta_0}^{1}\right) = U(\overline{Y}_0^1). \tag{21}$$

We therefore get $V_0(0) \leq -e^{-\eta \overline{Y}_0^1}$. The converse inequality is obtained by observing that, first, (20) is in fact an equality for the choice of the management-investment strategy $(\hat{\xi}, \hat{\pi})$. Second, $(\hat{\xi}, \hat{\pi})$ is indeed an admissible strategy since:

$$\mathbb{E}\left[U\left(X^{0,\pi} + B^{\xi} + \overline{Y}^{\xi^{n+1}}\right)_{\theta_{n+1}}\right] = U\left(\overline{Y}_0^1\right),$$

showing that $\mathbb{P}(N(\hat{\xi}) = \infty) = 0$. $\square$

As a straightforward corollary, we obtain a characterization of the power plant value p_0.

Corollary 1 *The utility indifference price of the production asset is given by*:

$$p_0 = \overline{Y}_0^1 - \mathcal{E}_{0,T}^{f^0}\left[\chi'\right].$$

5 Existence of a solution of the RBSDE system

To prove the existence of a solution of the system of RBSDEs, we adapt the method developed in Hamadène and Jeanblanc (2007). We define the sequences of processes $Y^{i,n}$, $Z^{i,n}$, $K^{i,n}$, $n \geq 0$, for $1 \leq i \leq M$ as follows. We start from:

$$Y_t^{i,0} := \mathcal{E}_{t,T}^{f^i}\left[\chi\right]. \tag{22}$$

Given $Y^{i,n-1}$, we compute $\overline{Y}^{i,n-1}$ as:

$$\overline{Y}_t^{i,n-1} = \mathcal{E}_{t,\overline{\delta}_i(t)}^{f^i}\left[Y_{\overline{\delta}_i(t)}^{i,n-1}\right], \tag{23}$$

and $Y^{i,n}$, $Z^{i,n}$, $K^{i,n}$ as the solution of a reflected BSDE:

$$Y_t^{i,n} = \chi - \int_t^T f_u^i\left(Z_u^{i,n}\right) du - \int_t^T Z_u^{i,n} \cdot \Sigma_u dW_u + \left(K_T^{i,n} - K_t^{i,n}\right), \tag{24}$$

$$Y_t^{i,n} \geq \max_{j \neq i} \left\{\overline{Y}_t^{j,n-1} - C_{i,j}\right\}, \tag{25}$$

$$K^{i,n} \in \mathcal{J}(\mathbb{R}) \text{ and } \int_0^T \left(Y_t^{i,n} - \max_{j \neq i} \left\{\overline{Y}_t^{j,n-1} - C_{i,j}\right\}\right) dK_t^{i,n} = 0. \tag{26}$$

Given $Y^{i,n} \in \mathcal{H}_0^\infty(\mathbb{R})$, the process $\overline{Y}^{i,n}$ is defined for all t by $\overline{Y}_t^{i,n} := \overline{Y}_t^{i,n,t}$, where $(\overline{Y}^{i,n,t}, \overline{Z}^{i,n,t})$ solve the BSDE:

$$\overline{Y}_s^{i,n,t} = Y_{\overline{\delta}_i(t)}^{i,n} - \int_s^T f_u^i\left(\overline{Z}_u^{i,n,t}\right) du - \int_s^T \overline{Z}_u^{i,n,t} \cdot \Sigma_u dW_u$$

for $t \leq s \leq \overline{\delta}_i(t)$. As mentioned previously, $\overline{Y}^{i,n}$ is an adapted, bounded and continuous process. Indeed, the a priori estimate in Kobylanski (2000) shows that if $Y^{i,n} \in \mathcal{H}_0^\infty(\mathbb{R})$, then $\overline{Y}^{i,n}$ is a bounded process. In addition, the monotone stability property in Kobylanski (2000) shows that if $Y^{i,n} \in \mathcal{H}_0^\infty(\mathbb{R})$ then $\overline{Y}^{i,n}$ is continuous. The reflected BSDE $(Y^{i,n+1}, Z^{i,n+1}, K^{i,n+1})$ is thus well defined and the results of Kobylanski et al. (2002) ensure the existence of a solution in $\mathcal{H}_0^\infty(\mathbb{R}) \times \mathcal{H}_0^2(\mathbb{R}^N) \times \mathcal{J}(\mathbb{R})$. We can thus compute sequentially the processes $Y^{i,n}$, $Z^{i,n}$, $K^{i,n}$, $\overline{Y}^{i,n}$ for all n.

The processes $Y^{i,n}$ can be interpreted in terms of the value function of an optimal control problem with n possible switches. Indeed $Y^{i,0}$ defined by (22) corresponds to the maximal utility when no switch is allowed (cf. Hu et al. 2005). The problem of finding the optimal strategy with n switching times can be decomposed by dynamic programming as finding the first optimal switching time and following the optimal strategy with $n-1$ switches from that time. It is well known (e.g. Kobylanski et al. 2002) that optimal stopping problems are linked to reflected BSDEs whose barriers are the payoff. In our case, the payoff from switching is the value induced by the optimal strategy with $n-1$ switches. It is then intuitive to look for $Y^{i,n}$ as the solution of a BSDE reflected on the $Y^{j,n-1}$, $j \neq i$. To prove this result, let us introduce the following notation. Let $\xi \in \mathcal{X}_0$, $n, m \in \mathbb{N}$. We denote by $\mathcal{X}^{n,m}(\xi)$ the set of management strategies $\xi' \in \mathcal{X}_{\theta_n}(\xi)$ such that: $N(\xi') \leq n + m$. In words, $\mathcal{X}_t^{n,m}(\xi)$ is the set of all admissible management strategies in $\mathcal{X}_{\theta_n}(\xi)$ which have less than m mode switches between θ_n and T.

Proposition 4 *Let (ξ, π) be a pair of management-investment strategies in $\mathcal{X}_0 \times \mathcal{A}_0$. Then*

$$U\left(X^{0,\pi} + B^\xi + \overline{Y}^{\xi^n,m}\right)_{\theta_n} = \operatorname*{ess.\,sup}_{(\xi',\pi') \in \mathcal{X}^{n,m}(\xi) \times \mathcal{A}_{\theta_n}(\pi)} \mathbb{E}_{\theta_n}\left[U\left(X_T^{0,\pi'} + B_T^{\xi'} + \chi\right)\right].$$

Proof Consider the sequence of management strategy $\hat{\xi} \in \mathcal{X}_{\theta_n}(\xi)$ defined by:

$$\hat{\theta}_{k+1} = \inf\left\{t \geq \overline{\delta}_{\hat{\xi}_k}(\hat{\theta}_k),\ Y_t^{\hat{\xi}^k, n+m-k} = \max_{j \neq \hat{\xi}^k}\left\{\overline{Y}_t^{j, n+m-k-1} - C_{\hat{\xi}^k, j}\right\}\right\},$$
$$\hat{\xi}^{k+1} = \min\left\{j \neq \hat{\xi}^k,\ Y_{\hat{\theta}_{k+1}}^{\hat{\xi}^k, n+m-k} = \overline{Y}_{\hat{\theta}_{k+1}}^{j, n+m-k-1} - C_{\hat{\xi}^k, j}\right\},$$

for $n \leq k \leq n+m-1$ and $\hat{\theta}_{n+m+1} = T$. Consider also the investment strategy $\hat{\pi} \in \mathcal{A}_{\theta_n}(\xi)$ defined as:

$$\hat{\pi}_t = (\Sigma_t^*)^{-1}\Pi_t\left((\eta\Sigma_t)^{-1}\mu_t - \Sigma_t^* Z_t^{\hat{\xi}^k, n+m-k}\right) \quad \text{for } \hat{\theta}_k^* \leq t \leq \hat{\theta}_{k+1},$$
$$\hat{\pi}_t = (\Sigma_t^*)^{-1}\Pi_t\left((\eta\Sigma_t)^{-1}\mu_t - \Sigma_t^* \overline{Z}_t^{\hat{\xi}^k, n+m-k, \hat{\theta}_k}\right) \quad \text{for } \hat{\theta}_k \leq t \leq \hat{\theta}_k^*.$$

Following the same argument as in the proof of Proposition 2, we prove that the processes $U(X^{0,\pi'} + B^{\xi'} + \overline{Y}^{\xi'^k, n+m-k})$ and $U(X^{0,\pi'} + B^{\xi'} + Y^{\xi'^k, n+m-k})$ defined, respectively, on $[\theta'_k, \theta'^*_k]$ and $[\theta'^*_k, \theta'_{k+1}]$ are super-martingales for every $(\xi', \pi') \in \mathcal{X}^{n,m}(\xi) \times \mathcal{A}_{\theta_n}(\pi)$, and martingales with $(\pi', \xi') = (\hat{\pi}, \hat{\xi})$. The only difference with the proof of Proposition 2 lies in the fact that the number of switches is bounded by m. This implies:

$$U\left(X^{0,\pi} + B^{\xi} + \overline{Y}^{\xi^n, m}\right)_{\theta_n} = \mathbb{E}_{\theta_n}\left[U\left(X^{0,\hat{\pi}} + B^{\hat{\xi}} + Y^{\hat{\xi}^n, m}\right)_{\hat{\theta}_n^*}\right]$$
$$= \mathbb{E}_{\theta_n}\left[U\left(X^{0,\hat{\pi}} + B^{\hat{\xi}} + \overline{Y}^{\hat{\xi}^{n+1}, m-1}\right)_{\hat{\theta}_{n+1}}\right].$$

Direct iteration of this argument provides:

$$U\left(X^{0,\pi} + B^{\xi} + \overline{Y}^{\xi^n, m}\right)_{\theta_n} = \mathbb{E}_{\theta_n}\left[U\left(X_T^{0,\hat{\pi}} + B_T^{\hat{\xi}} + \chi\right)\right].$$

On the other hand, for any management-investment strategies $(\xi', \pi') \in \mathcal{X}^{n,m}(\xi) \times \mathcal{A}_{\theta_n}(\pi)$, the same super-martingale argument yields:

$$U\left(X^{0,\pi} + B^{\xi} + \overline{Y}^{\xi^n, m}\right)_{\theta_n} \geq \mathbb{E}_{\theta_n}\left[U\left(X_T^{0,\pi'} + B_T^{\xi'} + \chi\right)\right],$$

which completes the proof. □

The process $Y^{i,n}$ can thus be seen as an approximation of Y^i when the number of possible switches is restricted to n. We then deduce the following corollary:

Corollary 2 *For $i = 1, \ldots, M$, the sequences $(Y^{i,n})_{n \geq 0}$ and $(\overline{Y}^{i,n})_{n \geq 0}$ are non-decreasing.*

Proof Notice that $\mathcal{X}^{n,m}(\xi) \subset \mathcal{X}^{n,m+1}(\xi)$, so $\overline{Y}_t^{i,n} \leq \overline{Y}_t^{i,n+1}$, a.s. Since $\overline{Y}^{i,n}$ and $\overline{Y}^{i,n+1}$ are continuous processes, this implies that $\overline{Y}^{i,n} \leq \overline{Y}^{i,n+1}$, a.s. The comparison principle for quadratic reflected BSDE (Theorem 3.2 in Kobylanski et al. 2002) shows that $Y_t^{i,n} \leq Y_t^{i,n+1}$ a.s. for all t and, by continuity $Y^{i,n} \leq Y^{i,n+1}$ a.s. □

From this monotonicity property we obtain the convergence of the processes $Y^{i,n}$, $Z^{i,n}$ and $K^{i,n}$.

Proposition 5 *The sequences of processes* $(Y^{i,n}, Z^{i,n}, K^{i,n})$*,* $n \geq 0$*, converge uniformly to processes* $(\widetilde{Y}^i, \widetilde{Z}^i, \widetilde{K}^i)$ *in* $\mathcal{H}_0^\infty(\mathbb{R}) \times \mathcal{H}_0^2(\mathbb{R}^N) \times \mathcal{J}(\mathbb{R})$*. Moreover* $(\widetilde{Y}^i, \widetilde{Z}^i, \widetilde{K}^i)$ *is a solution of the coupled system of BSDEs* (10)–(13).

Proof Proof of Proposition 5. By Corollary 2, the sequence $(Y^{i,n})$ is non-decreasing. Then it converges pointwise to a process $\widetilde{Y}^i$. We now provide uniform bounds for this sequence. Let (ξ, π) be a pair of management-investment strategies in $\mathcal{X}_0 \times \mathcal{A}_0$. From the proof of Proposition 4, we also deduce:

$$
\begin{aligned}
U\left(X^{0,\pi} + B^\xi + Y^{\xi^n,m}\right)_{\theta_n^*} &= \underset{(\xi',\pi') \in \mathcal{X}^{n,m}(\xi) \times \mathcal{A}_{\theta_n^*}(\pi)}{\text{ess. sup}} \mathbb{E}_{\theta_n^*}\left[U\left(X_T^{0,\pi'} + B_T^{\xi'} + \chi\right)\right] \\
&\leq \underset{\substack{(\xi',\pi') \in \\ \mathcal{X}^{n,m}(\xi) \times \mathcal{A}_{\theta_n^*}(\pi)}}{\text{ess. sup}} \mathbb{E}_{\theta_n^*}\left[U\left(X_T^{0,\pi'} + \int_0^T \max_j \psi_t^j dt + \chi\right)\right] \\
&\leq \underset{\pi' \in \mathcal{A}_{\theta_n^*}(\pi)}{\text{ess. sup}} \mathbb{E}_{\theta_n^*}\left[U\left(X_T^{0,\pi'} + \overline{\kappa} T + \overline{\chi}\right)\right],
\end{aligned}
$$

where $\overline{\kappa}$ is a bound for $\max_j |\psi^j|$, and $\overline{\chi}$ is an upper bound for $|\chi|$. Following Hu et al. (2005), we get:

$$
U\left(X_t^{0,\pi} + \int_0^t \psi_u^i du + \overline{\Upsilon}_t^i\right) = \underset{\pi' \in \mathcal{A}_t(\pi)}{\text{ess. sup}}\, \mathbb{E}_t\left[U\left(X_T^{0,\pi'} + \int_0^T \psi_u^i du + \chi\right)\right],
$$

where $\overline{\Upsilon}_t^i = \mathcal{E}_{t,T}^g\left[\chi + \int_t^T \left(\frac{1}{2\eta}\left|\Pi_{\Sigma_u^* K}(\Sigma_u^{-1}\mu_u)\right|^2 + \psi_u^i\right) du\right]$, thus:

$$
U\left(X^{0,\pi} + \int_0^{\cdot} \psi_u^i du + \overline{\Upsilon}^i\right)_{\theta_n^*} \geq \underset{\pi' \in \mathcal{A}_{\theta_n^*}(\pi)}{\text{ess. sup}} \mathbb{E}_{\theta_n^*}\left[U\left(X_T^{0,\pi'} - \overline{\kappa} T - \overline{\chi}\right)\right],
$$

and we end up with:

$$
U\left(X^{0,\pi} + B^\xi + Y^{\xi^n,m}\right)_{\theta_n^*} \leq U\left(2(\overline{\kappa} T + \overline{\chi}) + X^{0,\pi} + \int_0^{\cdot} \psi_u^{\xi^n} du + \overline{\Upsilon}^{\xi^n}\right)_{\theta_n^*}.
$$

This being true for all management strategy ξ, we obtain:

$$Y_t^{i,n} \leq 2\overline{\kappa}T + 2\overline{\xi} + \overline{\Upsilon}_t^i.$$

On the other hand, $Y^{0,n} \geq Y^{i,0} = \overline{\Upsilon}_t^i$ because the sequence $Y^{i,n}$ is non-decreasing. Since $\overline{\Upsilon}^i \in \mathcal{H}^\infty(\mathbb{R})$, as a solution of a quadratic BSDE with bounded terminal condition, the sequence $(Y^{i,n})_{n\geq 0}$ is uniformly bounded by some constant. In particular, this implies that $\widetilde{Y}^i \in \mathcal{H}^\infty(\mathbb{R})$. Using relation (23), we deduce that the sequences $(\overline{Y}^{i,n})_{n\geq 0}$, $1 \leq i \leq M$, are uniformly bounded. We are thus in the conditions of proposition 2.4 in Kobylanski (2000), and we conclude that the sequences $(\overline{Y}^{i,n})_{n\geq 0}$, $1 \leq i \leq M$, converge to processes $\check{Y}^i \in \mathcal{H}_0^\infty(\mathbb{R})$. We are also in the conditions of theorem 4 in Kobylanski et al. (2002) and we conclude that $(Y^{i,n})_{n\geq 0}$ converges uniformly on $[0, T]$ to $\widetilde{Y}^i \in \mathcal{H}_0^\infty(\mathbb{R})$, $(Z^{i,n})_{n\geq 0}$ converges to $\widetilde{Z}^i \in \mathcal{H}_0^2(\mathbb{R}^N)$ and $(K^{i,n})_{n\geq 0}$ converges uniformly on $[0, T]$ to $\widetilde{K}^i \in \mathcal{J}(\mathbb{R})$. Moreover $(\widetilde{Y}^i, \widetilde{Z}^i, \widetilde{K}^i)$ satisfies the backward system (10)–(13). □

6 Some properties of the indifference price

In this section, we state some properties of the indifference price that are extensions to the switching problem of well known results for European and American options (cf. Kobylanski et al. 2002; Mania and Schweizer 2005). The proofs are omitted and available under request. First, the indifference price reduces to the risk neutral price in the case of a complete market.

Proposition 6 *In the case of a complete market ($K = \mathbb{R}^N$), the value of the power plant is given by*:

$$p_0 = \sup_{\xi \in \mathcal{X}_0} \mathbb{E}^{\mathbb{Q}}\left[B_T^\xi + \chi - \chi'\right],$$

where the equivalent martingale measure $\mathbb{Q}$ is given by its density:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \Sigma_t^{-1}\mu_t \cdot dW_t - \frac{1}{2}\int_0^T \left|\Sigma_t^{-1}\mu_t\right|^2 dt\right), \tag{27}$$

In addition, the risk neutral price has the following expression when the constraints vanish.

Proposition 7 *Let $K = \mathbb{R}^N$, $\delta_i = 0$, $C_{i,j} = C > 0$ for all $i \neq j$. Then*

$$p_0^C \longrightarrow p^0 := \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \max_j \psi_t^j \, dt + \chi - \chi'\right], \quad as\ C \to 0,$$

where $\mathbb{Q}$ is the equivalent martingale measure defined in (27).

However, in the case of an incomplete market, the indifference price is a decreasing function of η:

Proposition 8 *The utility indifference value p_0^η is given by*

$$p_0^\eta = \sup_{\xi \in \mathcal{X}_0} p_0(B_T^\xi, \eta),$$

where $p_0(B_T^\xi, \eta)$ is the utility indifference price of claim B_T^ξ. It is thus a decreasing function of η and

$$\lim_{\eta \to 0} p_0^\eta = \sup_{\xi \in \mathcal{X}_0} \mathbb{E}^{\mathbb{Q}_e}[B_T^\xi], \quad \inf_{\mathbb{Q} \in \mathcal{M}(S)} \sup_{\xi \in \mathcal{X}_0} \mathbb{E}^{\mathbb{Q}}[B_T^\xi] \geq \lim_{\eta \to \infty} p_0^\eta \geq \sup_{\xi \in \mathcal{X}_0} \inf_{\mathbb{Q} \in \mathcal{M}(S)} \mathbb{E}^{\mathbb{Q}}[B_T^\xi]$$

where $\mathbb{Q}_e$ is the minimal entropy martingale measure, defined as the minimizer of $\min_{\mathbb{Q} \in \mathcal{M}(S)} \mathbb{E}^{\mathbb{Q}}[\ln(d\mathbb{Q}/d\mathbb{P})]$.

7 Relation with a PDE obstacle problem and numerical schemes

In this section, we relate the system of reflected BSDEs to an obstacle problem for PDEs, and present two alternative numerical schemes to compute the asset value, using, respectively, the PDE and BSDE representation.

7.1 An obstacle problem for PDEs

It is well known (cf. El Karoui et al. 1997) that reflected BSDEs are linked to obstacle problems for PDEs in a Markovian setting. We do not prove here that the result holds for our coupled system of reflected BSDEs but rather give the formulation of the obstacle problem. In a Markovian setting, the process Y^i can be represented by a function u^i that is a viscosity solution of the PDE:

$$0 = \min \left\{ u^i - \max_{j \neq i} (\overline{u}^j - C_{i,j}), -\mathcal{L}u^i + f^i \left(., ., \frac{\partial u^i}{\partial s} \right) \right\}, \quad \text{on} \quad [0, T] \times \mathbb{R}^N,$$

with terminal condition $u^i(T, s) = \phi(s)$, for $1 \leq i \leq M$, and for all $t_0 \in [0, T]$, $\overline{u}^i(t_0, s) = w^i(t_0, t_0, s)$, where $w^i(t_0, t, s)$ is a viscosity solution of:

$$0 = -\mathcal{L}w^i + f^i \left(., ., \frac{\partial w^i}{\partial s} \right), \quad \text{on} \quad [t_0, \overline{\delta}_i(t_0)] \times \mathbb{R}^N, \tag{28}$$

with $w^i(t_0, \overline{\delta}_i(t_0), s) = u^i(\overline{\delta}_i(t_0), s)$, for every fixed $t_0 \in [0, T]$ and $\mathcal{L}$ is the Dynkin operator associated to the diffusion process S:

$$\mathcal{L}u(t,s) := \frac{\partial u}{\partial t}(t,s) + \mu(t,s)\frac{\partial u}{\partial s}(t,s) + \frac{1}{2}\mathrm{Tr}\left(\Sigma\Sigma^*(t,s)\frac{\partial^2 u}{\partial s^2}(t,s)\right).$$

7.2 The PDE-based numerical scheme

Regarding the PDE system, we denote by $\mathcal{D}^i_{n\Delta}\phi$ the solution of the PDE (28) at time $n\Delta$ when the terminal condition at time $(n+1)\Delta$ is ϕ. We also denote by u^i_n (resp. $\overline{u}^i_n$) the approximation at time $n\Delta$ of the function u^i (resp. $\overline{u}^i$). A natural numerical scheme for solving the PDE is: $u^i_{N_0} = \phi$, $u^i_n = \max\left\{\mathcal{D}^i_{n\Delta}u^i_{n+1},\ \overline{u}^{j,\kappa_j}_n - C_{i,j}\ ;\ j \neq i\right\}$, and for $0 \le k \le \kappa_i$, $\overline{u}^{i,k}_{N_0} = \phi$, $\overline{u}^{i,0}_n = u^i_n$, $\overline{u}^{i,k}_n = \overline{\mathcal{D}}_{n\Delta}u^{i,k-1}_{n+1}$. The differential operator can be approximated by classical methods. In our numerical implementation, we use the finite differences approximation.

7.3 The BSDE-based numerical scheme

We fix a discretization step Δ, such that $T = N_0\Delta$ for some $N_0 \in \mathbb{N}$, and $\delta_i = \kappa_i\Delta$. We denote by (y^i_n, z^i_n) (resp. $(\overline{y}^i_n, \overline{z}^i_n)$) the approximation at time $n\Delta$ of the processes (Y^i, Z^i) (resp. $(\overline{Y}^i, \overline{Z}^i)$). We adapt the Euler scheme for RBSDEs proposed in Bouchard and Touzi (2005) and consider the following scheme, for $0 \le n < N_0$:

$$\begin{aligned}
z^i_n &= \frac{1}{\Delta}\left(\Sigma^*_{n\Delta}\right)^{-1}\mathbb{E}_{n\Delta}\left[y^i_{n+1}\left(W_{(n+1)\Delta} - W_{n\Delta}\right)\right],\\
y^i_n &= \max\left\{\mathbb{E}_{n\Delta}[y^i_{n+1}] - \Delta f^i_{n\Delta}(z^i_n),\ \overline{y}^{1-i,\kappa_{1-i}}_n - C_{1-i}\right\},
\end{aligned}$$

with terminal condition $y^i_{N_0} = \xi$ and, for $0 \le k \le \kappa_i$:

$$\begin{aligned}
\overline{y}^{i,k}_{N_0} &= \xi + \sum_{l=1}^{\kappa_i - k}\Delta\left(\frac{1}{2\eta}\left|\Pi(\Sigma^{-1}\mu)\right|^2 + \psi^i\right)_{(N_0-l)\Delta},\\
\overline{y}^{i,0}_n &= y^i_n + \sum_{l=1}^{\kappa_i}\Delta\left(\frac{1}{2\eta}\left|\Pi(\Sigma^{-1}\mu)\right|^2 + \psi^i\right)_{(n-l)\Delta},\\
\overline{z}^{i,k}_n &= \frac{1}{\Delta}\left(\Sigma^*_{n\Delta}\right)^{-1}\mathbb{E}_{n\Delta}\left[\overline{y}^{i,k-1}_{n+1}\left(W_{(n+1)\Delta} - W_{n\Delta}\right)\right],\\
\overline{y}^{i,k}_n &= \mathbb{E}_{n\Delta}[\overline{y}^{i,k-1}_{n+1}] - \Delta g^i_{n\Delta}(\overline{z}^{i,k}_n).
\end{aligned}$$

The main difference with Bouchard and Touzi (2005) is that, at each time $n\Delta$, we need to look forward until time $n\Delta + \delta_i$ in order to decide whether a mode switch is profitable. We therefore need to compute (an approximation of) the solution of BSDE

(12). This is done in $\kappa_i + 1$ steps and justifies the use of the vector $(\overline{y}_n^{i,k})_{0\le k\le\kappa_i}$: at each time n, $\overline{y}_n^{i,k}$ is the approximation of

$$\mathcal{E}^g_{n\Delta,(n+k)\Delta}\left[Y^i_{(n+k)\Delta} + \sum_{l=1}^{\kappa_i} \Delta\left(\frac{1}{2\eta}\left|\Pi(\Sigma^{-1}\mu)\right|^2 + \psi^i\right)_{(n+k-l)\Delta}\right].$$

The numerical approximation of the conditional expectation operator can be tackled by different methods: kernel regression methods (Carrière 1996), projection methods (Longstaff and Schwartz 2001), quantization (Bally et al. 2005), Malliavin calculus (Bouchard and Touzi 2005). In this paper we implement the projection-based method of Gobet et al. (2004).

8 Numerical implementation in a complete market

In this section, we present the valuation of a coal- and a fuel oil-fired power plant with two modes 0,1 in a complete market. Our interest here is to assess the impact of physical constraints on the value and present an example in dimension 4 solved by BSDE methods showing the potential benefits of BSDEs in high dimension.

8.1 Valuation of a coal-fired power plant in a 2D market

We consider the example of a coal-fired power plant and a one-factor model for electricity and coal prices. The process $S_t = (F_t(T), G_t(T))$ is defined by:

$$\frac{dF_t(T)}{F_t(T)} = \sigma_F e^{-a(T-t)} dW^1_t, \quad \frac{dG_t(T)}{G_t(T)} = \sigma_G e^{-b(T-t)} dW^2_t,$$

where W^1 and W^2 are independent Brownian motions under risk neutral probability. Here, $F_t(T)$ is the forward price of electricity at time t with delivery at time T and $G_t(T)$ is the forward price of coal at time t with delivery at time T. This one-factor model is classically used in energy markets (see Clewlow and Strickland 2000). We suppose that there is no correlation between the two assets, which is approximately the case for coal. We also suppose that the spot prices of electricity and gas at time t are defined by $F_t(t)$ and $G_t(t)$. In this context, the spot prices are mean-reverting if a and b are non-zero and can be expressed in terms of the forward prices: $F_t(t) = \phi_F(t, F_t(T))$ and $G_t(t) = \phi_G(t, G_t(T))$. We are then in the presence of a complete market.

The power plant can be in two modes: on (denoted 1) or off (denoted 0). The instantaneous rates of benefit of the power plant at time t are given by: $\psi^0_t := 0$, $\psi^1_t := \max\{q\,(F_t(t) - HG_t(t))\,,\ q_{\min} \le q \le q_{\max}\}$. In what follows, we make use of the notation $C_0 := C_{0,1}$ and $C_1 := C_{1,0}$. The terminal payoffs χ and χ' are set to 0. The parameters used in this example are given in Table 1 in daily values. Electricity future curve has been calibrated on data from the French market in 2004. The coal future curve is taken constant at 40 \$/ton, including carbon price, and the Euro/Dollar

Table 1 Price process parameters and power plant characteristics

$\sigma_F = 0.1$	$\sigma_G = 0.01$	$a = 0.13$	$b = 0$
$\delta_0 = 24$ h	$\delta_1 = 8$ h	$q_{\min} = 162$ MW	$q_{\max} = 553$ MW
$T = 8760$ h	$C_0 = 0$ €	$C_1 = 35530$ €	$H = 0.3627$ ton/MWh

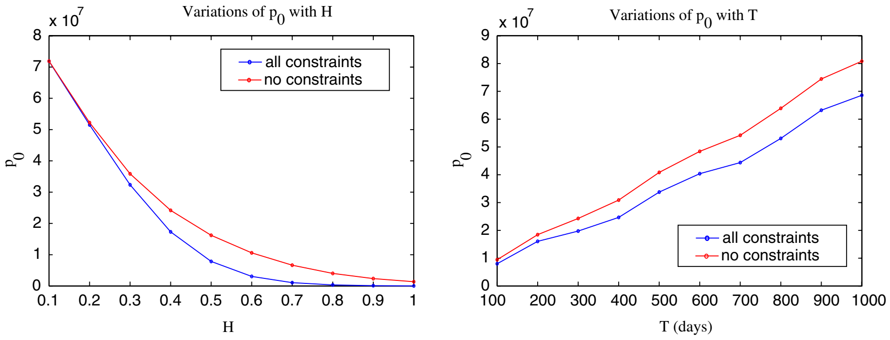

Fig. 1 Variations of the value with heat rate H (*left*) and maturity T (*right*), both in the presence and absence of constraints

exchange rate is taken constant equal to 1. Parameters of the power plant are those of a real coal-fired plant.

The preliminary change of variable

$$\xi_t = \int_0^t \sigma_F e^{-a(t-u)} dW_t^1, \quad \zeta_t = \int_0^t \sigma_G e^{-b(t-u)} dW_t^2,$$

avoids dealing with exponential coefficients of the form $e^{-a(T-t)}$, and allows the use of Brownian bridge techniques. As far as BSDE approximation is concerned, we choose to follow the methodology developed by Gobet et al. (2004) with an 8×8 grid, linear approximation inside each domain, and 25600 simulations. The time step is set to 1 h. For the PDE, we approximate the operator $\mathcal{D}^i$ by a Crank–Nicholson scheme, within a domain $[-5, 5] \times [-5, 5]$ in (ξ, ζ). In each direction, we mesh the interval with 100 steps. Time step is 1 h. In dimension two, we observed that the finite differences scheme converges about 5 to 8 times faster than the BSDE algorithm.

On this example, the coal-fired power plant price in the presence of production constraints is $p_c = 22.95 \times 10^6$ €, while the price in the absence of production constraints (i.e. $\delta_0 = \delta_1 = C_0 = C_1 = 0$) is $p_{nc} = 28.58 \times 10^6$ €. In the context of this example, we observe that the presence of production constraints reduces the power plant value by 20% over 1 year, which is very significant.

Figure 1 shows the variations of the value with respect to the heat rate H (left) and the horizon T (right). We show both the value without constraints (red lines) and with constraints (blue lines) computed by BSDE. The left plot confirms the decrease of the value when the heat rate H increases, or equivalently when the constant coal

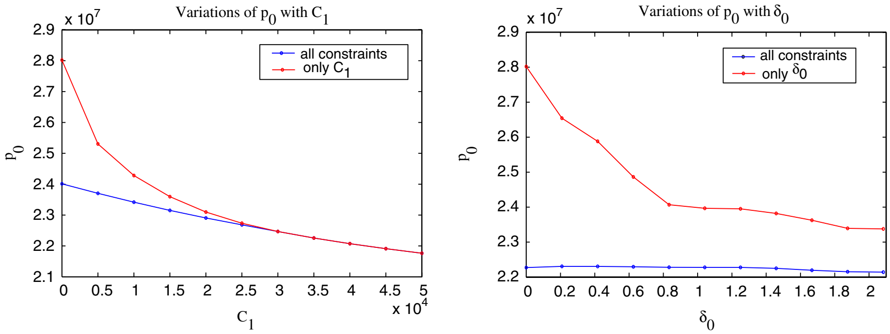

Fig. 2 Variation of the value with fixed cost C_0 (*left*) and minimal time δ_0 (*right*), both in the presence and absence of other constraints

future curve increases. In addition, the spread between the value with and without constraints increases when H takes high values, i.e. when the real option is deeper out of the money. Indeed, a higher "moneyness" of the real option implies a smaller number of switches, because production is more often profitable, and thus a smaller impact of constraints. The right plot shows the linearity with time of the decrease in value induced by the constraints: the 20% decrease on the value is in fact almost constant in time. Notice that the curves are not linear, nor convex, because of the seasonality embedded in the electricity future curve.

Finally, Fig. 2 shows the variations of the value with C_0 (left) and δ_0 (right). In both graphs, the blue curve represents the value of the power plant when the specified constraint changes, all others remaining equal to the values of Table 1. The red curve represents the value of the power plant when the specified constraint changes, all others remaining equal to 0. This allows us to assess the impact of one constraint, in the presence or absence of the others. We observe from the left figure that, as soon as the fixed cost C_0 is high enough, the presence of other constraints is negligible. This feature is also shown on the blue curve of the right figure. In the presence of other constraints, and in particular fixed costs, the minimal time δ_0 has a very low impact on the value. We also observe that the red and blue curve do not converge in the range of the plot. This is mainly due to the fact that minimal times are operational constraints that do not induce direct costs. The curves would probably converge for very high minimal times of the range of the maturity T.

As a conclusion, physical constraints induce a reduction in value that can be very significant. This reduction has a linear behavior with time but a non-linear behavior with the spread of commodities, or "moneyness" of the real option. In addition, minimal times can be neglected in the presence of high fixed costs.

8.2 Valuation of a fuel oil-fired power plant in a 4D market

In higher dimension, the finite differences method for PDEs becomes more difficult to implement because of large memory storage. On the other hand, BSDE methods do not depend so heavily on dimension, especially if simulation techniques such as Brownian bridge can be used that avoid storing all simulated trajectories. As an example of the

Table 2 Example 2—price process parameters and power plant characteristics

$\sigma_F^S = 0.1$	$\sigma_F^L = 0.05$	$\sigma_G^S = 0.51$	$\sigma_G^L = 0.05$
$a = 0.13$	$b = 0.34$	$\delta_0 = 1$ h	$\delta_1 = 8$ h
$T = 8760$ h	$C_0 = 0$ €	$C_1 = 21444$ €	$H = 0.55$ barrel/MWh
$q_{\min} = 162$ MW	$q_{\max} = 553$ MW		

potential benefits of BSDE techniques in this setting, we present the valuation of a fuel oil-fired power plant in a 2 factor model for electricity and oil prices under risk neutral probability:

$$\frac{dF_t(T)}{F_t(T)} = \sigma_F^S e^{-a(T-t)} dW_t^1 + \sigma_F^L dW_t^2, \quad \frac{dG_t(T)}{G_t(T)} = \sigma_G e^{-b(T-t)} dW_t^3 + \sigma_G^L dW_t^4.$$

This model is also classically used to take into account short and long term volatilities (cf. Clewlow and Strickland 1999; Geman 2005). For simplicity we suppose that all factors are independent. We also suppose that trading is allowed in contracts with two maturities T and $T' > T$. In this model, the spot prices are given by:

$$F_t(t) = F_0(t) e^{-\frac{1}{2}\sigma_F^2(t)t + W_t^{S,F} + W_t^{L,F}},$$

where

$$\begin{aligned} dW_t^{S,F} &= -aW_t^{S,F} dt + \sigma_F^S dW_t^1, \quad dW_t^{L,F} = \sigma_F^L dW_t^2, \\ \sigma_F^2(t) &= (\sigma_F^L)^2 + (\sigma_F^S)^2 \frac{1 - e^{-2at}}{2at} + 2\rho_F \sigma_F^S \sigma_F^L \frac{1 - e^{-at}}{at}, \end{aligned}$$

and a similar expression for $G_t(t)$. In this setting the market is complete, which justifies the use of the dynamics under risk neutral probability. The characteristics of the plant and price processes are given in Table 2. The fuel forward curve is constant and equal to 76 \$/barrel.

For this example we obtain a value of 18.4×10^6€ over 1 year with constraints, which is less than the gas-fired power plant, even though the constraints are weaker. This is due to the higher price of fuel. Figure 3 shows a trajectory of the spread $F_t(t) - HG_t(t)$ between electricity and fuel oil spot prices. The red curve shows the associated optimal switching strategy: the plant is on when the curve takes value $+50$ and it is off when the curve takes value -50. The right plot is a zoom between times 100 and 150 of the left plot. In the absence of constraints, the optimal strategy would be to run the plant when the spread is positive. In the presence of constraints, we observe that the plant only runs when the spread is significantly positive.

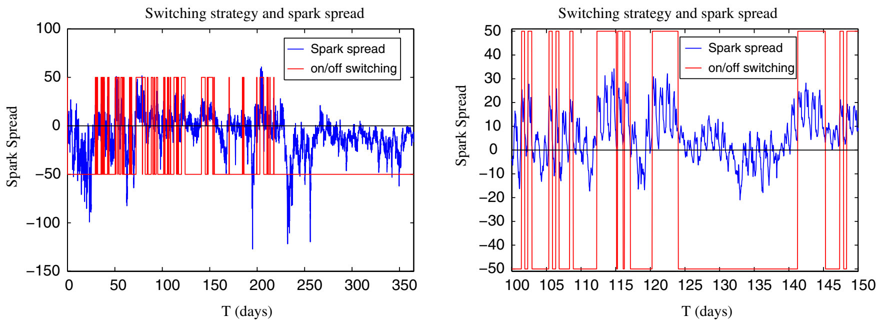

Fig. 3 One realization of the price spread $F_t(t) - HG_t(t)$ and optimal management strategy

9 Numerical implementation in an incomplete market

9.1 Source of incompleteness

In this section, we present a simple example of incomplete market. The spot price of electricity at a given time can be significantly different from the last quoted forward price for that maturity. On reason is that forward contracts are written on a delivery of a fixed electric power during a given period, typically one or several months. Forward contracts are then written on the average of spot prices during that period instead of on the spot price for a given maturity. The spot price of electricity at time t is thus no longer given by $F_t(t)$. This aspect is discussed for example in Skantze and Ilic (2000), where a general model is proposed for the relationship between the forward price $F_t(T)$ and the spot price P_T: $F_t(T) = \Psi(\mathbb{E}_t[P_T], \mathrm{Var}_t[P_T], \varepsilon_t)$, where $\mathrm{Var}_t[S_T]$ is the conditional variance of P_T and ε is a random disturbance. In particular, $F_t(t) = \Psi(P_t, 0, \varepsilon_t)$ is not necessarily equal to P_t. To take into account this specificity, we choose the simplest model: $P_t := F_t(t) + \varepsilon_t$, where ε_t is some exogenous, non-tradable, stochastic shock with dynamics:

$$d\varepsilon_t = -\kappa \varepsilon_t dt + \gamma dW_t^3,$$

and W^3 is independent of (W^1, W^2). In other words, we suppose the existence of a tradable forward contract for electricity that does not converge exactly to the spot price at maturity. The instantaneous rate of benefit in production mode is now given by: $\psi_t^1 := q\,(F_t(t) + \varepsilon_t - HG_t(t))$. Since the shock ε does not correspond to any tradable asset, the market is incomplete and the RBSDE system is non-linear. A quadratic term in $Z^{i,3}$ appears:

$$dY_t^i = \left(\frac{\eta\gamma^2}{2}(Z_t^{i,3})^2 + \mu_F e^{-a(T-t)} Z_t^{i,1} + \mu_G e^{-b(T-t)} Z_t^{i,2} - \frac{1}{2\eta}\left|\Sigma_t^{-1}\mu_t\right|^2 - \psi_t^i\right) dt$$
$$+ \sigma_F e^{-a(T-t)} Z_t^{i,1} dW_t^1 + \sigma_G e^{-b(T-t)} Z_t^{i,2} dW_t^2 + \gamma Z_t^{i,3} dW_t^3 - dK_t^i.$$

We restrict to the case where there are no delays: $\delta_0 = \delta_1 = 0$, and T is equal to 6 months. The characteristics of the power plant and the price process are those of Example 1, except that we set the drifts to $\mu_F = \mu_G = 0$. This allows us to use a smaller domain for the PDE mesh. Parameter κ is set to 0.02. We focus our analysis on the impact of η and γ on the value.

9.2 Numerical scheme and results

We tested three methods for solving the above example: Monte Carlo method for BSDEs, numerical methods for FBSDEs, finite differences for PDEs.

BSDE algorithm. No theoretical analysis of the approximation of BSDEs with quadratic generator is available in the literature. A consistency result can be obtained, following the lines of Bouchard and Touzi (2005), by approximating the quadratic generator by a sequence of Lipschitz functions. We tested the numerical convergence of a direct Monte Carlo computation via the Gobet–Lemor–Warin method, as in the complete market case, but convergence was very hard to obtain, even on simple examples. To better understand the convergence properties of the scheme, we tried to solve 1D quadratic BSDEs with no reflexion. These equations can be expressed as exponentials of linear BSDEs, thus their solution can be easily computed. Even in this simple case, convergence is not straightforward. We started to study the examples already computed in Peng and Xu (2005) with a tree technique. The Gobet–Lemor–Warin method also gives good results on these equations. We then focused on equations with drivers of the type $f(z) = \frac{z^2}{2} - a$ and null terminal condition. We observed that the number of simulations required for the convergence of the scheme increases quickly with a. Indeed, for $a = 1$, $T = 5$ days, 10 time steps and 5 basis functions (5 interval grid with linear interpolation in each interval), 1,000 simulations are enough to get convergence. When $a = 10$ we must increase the number of simulations to 100,000. If we want to have a smaller time step, for example 100 time steps over the horizon, and $a = 10$, 500,000 simulations are needed to get convergence. In our problem, the time step is in the range of 1 h and a is around 100,000. These parameters would require millions or hundreds of millions of simulations, which is not tractable. The variance of the method increases quickly with a and the number of time steps. This behavior is also observed when using a Malliavin calculus based method (cf. Bouchard and Touzi 2005) for computing conditional expectations. A more theoretical and systematic analysis of numerical convergence of numerical schemes for quadratic BSDEs is needed to eventually solve our problem by Monte Carlo methods.

PDE algorithm. We also solved the non-linear PDE using the same scheme as Chaumont et al. (2005). This scheme is totally explicit and is proved to converge. However it imposes very strong Courant–Friedrichs–Lewy conditions. We solved the PDE on a domain $[-1, 1] \times [-1, 1]$ in (ξ, ζ) and $[-2, 2]$ in ε, meshed by $40 \times 40 \times 80$ steps. The time step has to be taken to 1/100 h (=36 s!). This is why we only computed the power plant value over an horizon of 6 months. Computational time for this example was in the range of 1 week. An implicit version of this scheme can be implemented

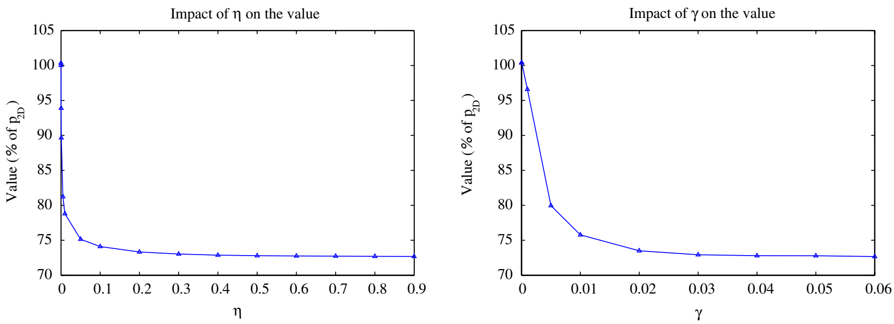

Fig. 4 Impact of risk aversion η (*left*) and non-diversified risk volatility γ (*right*) on the value, in % of the value in a complete market

and could potentially reduce the computational time. The results showed below are computed via this PDE method.

FBSDE algorithm. In the particular case where the generator is a second-order polynomial in Z, which is the case in our example, it is possible to transform the quadratic BSDE into a linear Forward-Backward SDE by means of the Girsanov theorem. The methodology developed by Delarue and Menozzi (2006) can then be followed and provides both a numerical scheme and a convergence result for this scheme. An application of Girsanov Theorem allows us to rewrite the quadratic BSDE into a linear coupled Forward Backward SDE:

$$\begin{aligned} d\varepsilon_t &= \left(-\kappa\varepsilon_t - \frac{\eta\gamma^2}{2} Z_t^{i,3}\right) dt + \gamma d\overline{W}_t^3, \\ dY_t^i &= \left(\mu_F e^{-a(T-t)} Z_t^{i,1} + \mu_G e^{-b(T-t)} Z_t^{i,2} - \frac{1}{2\eta}\left|\Sigma_t^{-1}\mu_t\right|^2 - \psi_t^i\right) dt \\ &\quad + \sigma_F e^{-a(T-t)} Z_t^{i,1} dW_t^1 + \sigma_G e^{-b(T-t)} Z_t^{i,2} dW_t^2 + \gamma Z_t^{i,3} d\overline{W}_t^3 - dK_t^i, \end{aligned}$$

where $d\overline{W}_t^3 := dW_t^3 + \frac{\eta\gamma}{2} Z_t^{i,3} dt$, and the other components of the forward process are unchanged. Numerical methods are available for these equations in Delarue and Menozzi (2006). We observed that the computational time is very high in dimension 3. As an illustration, we computed y_0 over an horizon of only 4 days. The FBSDE methods allows the use of a larger time step (1 h in our example, when the time step is 1/100 h for the PDE). However, the computational time for this example is much higher: 16 h for the FBSDE against 26 mn for the PDE.

We are then able to study the impact of the parameters γ and η on the value. The results are shown in Fig. 4.

We observe on the left plot the decrease of the value with η (Fig. 4-left, the value is expressed in percentage of the value in complete market, i.e. when $\gamma = 0$). The rapid decay of the curve indicates a large sensibility of the price with risk aversions in the range of [0, 0.1]. This is due to the fact that the benefits of the power plant over 1 year, i.e. the payoff, takes high values (in the range of millions of euros). Hence, variation of the indifference are only seen for small risk aversion coefficients. On the other hand, we observe, as expected, that the power plant value decreases when γ increases, i.e. when the magnitude of non-diversified risk increases. When the non-diversified risk vanishes, i.e. γ tends to 0, the value converges to the complete market price.

From this numerical application, we conclude that the presence of risk aversion and non-diversified risk can significantly reduce the value of the asset, up to 25%.

10 Discussion on indifference pricing for real options and conclusion

This article develops a utility indifference framework for the real option valuation of physical assets with production constraints in incomplete markets. The indifference price is characterized via a system of reflected BSDEs and numerical methods are presented, based on BSDE and PDE techniques. They show the tractability of the BSDE-based algorithm in complete market, even in dimension 4, and its limitations in the incomplete market case.

This approach provides an interesting framework for the valuation of physical assets in incomplete markets. It is first a well known extension of the arbitrage free valuation method. Second, this method takes into account the agent's preference via a utility function and, from our point of view, is more satisfactory than the risk-adjusted discount rate introduced in Dixit and Pindyck (1994). Third, in an incomplete market, it is a non-linear pricing rule in the sense that the value of a physical portfolio is not the sum of each asset's value. Non-linearity of real option prices should be an important feature to capture since it reflects the potential for risk diversification in portfolios of mixed technologies.

In our setting, the exponential utility is parameterized by a single coefficient η that could be calibrated in various ways. It can be calibrated on other more common risk criteria such as VaR: η would be set such that the distribution of Profits and Losses induced by the indifferent strategy matches a VaR target. The risk aversion coefficient would then be specific to the asset or to a class of assets. It can also be calibrated to match market prices of traded instruments. However, potential candidates such as options on spread or tolling agreements are still mainly traded OTC.

However, this methodology suffers from serious drawbacks. First of all, the computational complexity of the utility maximization problems in the presence of market incompleteness, involving the computation of non-linear PDEs or BSDEs, is such that the problem becomes untractable for large horizons or portfolio of power plants. Taking into account hourly production constraints on horizons of the range of several decades is impossible. We would suggest computing the indifference price with and without production constraints on a shorter horizon, maybe a year, derive the yearly impact of the constraints on the value, compute the indifference price without constraints on the original horizon and define a rule to extend this yearly impact over the whole horizon (for example linear in time). Because of the non-linearity of the pricing rule, the indifference value of a power plant should be computed inside a given portfolio and not in isolation. If the agent already owns a portfolio of n plants and wants to assess the value of an $n + 1$th, the indifference pricing method should assess the value of the $n + 1$ plants together and compare with the value of the original portfolio. However, computing the value of a portfolio of power plants is hard in an incomplete market because of the same non-linearity. This problem is theoretically in the scope of our analysis since a portfolio of power plants can be represented by a single asset with many states. Nonetheless, the combinatorics induces a number of states that is exponential with the number of plants and then a high number of non-linear equations to solve simultaneously. The method could be suited for the pricing of tolling agreements where the horizon is typically a year and concerns only one power plant. Second, the method faces the question of short term versus long term risk. How can we give a value to a physical asset in 10 or 20 years when there are no fuel forward prices for this horizon? Estimating mark-to-market values of physical assets with maturities way beyond market horizons (typically 5 years) seems problematic. Finally, another very important feature of real option prices that does not appear in our model is the finite depth of the markets. Another reason for non-linear prices, beyond risk diversification, is the lack of demand for the electricity produced by the power plant. Adding 10,000 MW of capacity in a portfolio is not 100 times as valuable as adding 100 MW because there is probably no demand for such production capacity. Said differently, the hypothesis of exogenous prices, independent of production is not realistic for pricing large capacities. Real option models should integrate somehow the supply-demand equilibrium.

As a conclusion, utility indifference pricing appears as an interesting framework to assess the impact of risk aversion and market incompleteness on the value of physical assets. It can be helpful to provide quantitative and qualitative understanding of these features on small examples. However the use of this technique to price portfolios of physical assets over large horizons, which is the problem usually faced by companies, seems prohibited by its computational complexity.

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