Quasi-sure Stochastic Analysis through Aggregation

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Abstract

This paper is on developing stochastic analysis simultaneously under a general family of probability measures that are not dominated by a single probability measure. The interest in this question originates from the probabilistic representations of fully nonlinear partial differential equations and applications to mathematical finance. The existing literature relies either on the capacity theory (Denis and Martini [5]), or on the underlying nonlinear partial differential equation (Peng [13]). In both approaches, the resulting theory requires certain smoothness, the so called quasi-sure continuity, of the corresponding processes and random variables in terms of the underlying canonical process. In this paper, we investigate this question for a larger class of "non-smooth" processes, but with a restricted family of non-dominated probability measures. For smooth processes, our approach leads to similar results as in previous literature, provided the restricted family satisfies an additional density property.

Key words: non-dominated probability measures, weak solutions of SDEs, uncertain volatility model, quasi-sure stochastic analysis.
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1 Introduction

It is well known that all probabilistic constructions crucially depend on the underlying probability measure. In particular, all random variables and stochastic processes are defined up to null sets of this measure. If, however, one needs to develop stochastic analysis *simultaneously* under a family of probability measures, then careful constructions are needed as the null sets of different measures do not necessarily coincide. Of course, when this family of measures is dominated by a single measure this question trivializes as we can simply work with the null sets of the dominating measure. However, we are interested exactly in the cases where there is no such dominating measure. An interesting example of this situation is provided in the study of financial markets with uncertain volatility. Then, essentially all measures are orthogonal to each other.

Since for each probability measure we have a well developed theory, for simultaneous stochastic analysis, we are naturally led to the following problem of aggregation. Given a family of random variables or stochastic processes, $X^{\mathbb{P}}$, indexed by probability measures $\mathbb{P}$, can one find an *aggregator* X that satisfies $X = X^{\mathbb{P}}$, $\mathbb{P}-$almost surely for every probability measure $\mathbb{P}$? This paper studies exactly this abstract problem. Once aggregation is achieved, then essentially all classical results of stochastic analysis generalize as shown in Section 6 below.

This probabilistic question is also closely related to the theory of second order backward stochastic differential equations (2BSDE) introduced in [3]. These type of stochastic equations have several applications in stochastic optimal control, risk measures and in the Markovian case, they provide probabilistic representations for fully nonlinear partial differential equations. A uniqueness result is also available in the Markovian context as proved in [3] using the theory of viscosity solutions. Although the definition given in [3] does not require a special structure, the non-Markovian case, however, is better understood only recently. Indeed, [17] further develops the theory and proves a general existence and uniqueness result by probabilistic techniques. The aggregation result is a central tool for this result and in our accompanying papers [15, 16, 17]. Our new approach to 2BSDE is related to the quasi sure analysis introduced by Denis and Martini [5] and the G-stochastic analysis of Peng [13]. These papers are motivated by the volatility uncertainty in mathematical finance. In such financial models the volatility of the underlying stock process is only known to stay between two given bounds $0 \le \underline{a} < \overline{a}$. Hence, in this context one needs to define probabilistic objects simultaneously for all probability measures under which the canonical process B is a square integrable martingale with absolutely continuous quadratic variation process satisfying

$$\underline{a}dt \le d\langle B\rangle_t \le \overline{a}dt.$$

Here $d\langle B\rangle_t$ is the quadratic variation process of the canonical map B. We denote the set of all such measures by $\overline{\mathcal{P}}_W$, but without requiring the bounds $\underline{a}$ and $\overline{a}$, see subsection 2.1.

As argued above, stochastic analysis under a family of measures naturally leads us to the problem of aggregation. This question, which is also outlined above, is stated precisely in Section 3, Definition 3.1. The main difficulty in aggregation originates from the fact that the above family of probability measures are not dominated by one single probability measure. Hence the classical stochastic analysis tools can not be applied simultaneously under all probability measures in this family. As a specific example, let us consider the case of the stochastic integrals. Given an appropriate integrand H, the stochastic integral $I_t^{\mathbb{P}} = \int_0^t H_s dB_s$ can be defined classically under each probability measure $\mathbb{P}$. However, these processes may depend on the underlying probability measure. On the other hand we are free to redefine this integral outside the support of $\mathbb{P}$. So, if for example, we have two probability measures $\mathbb{P}^1, \mathbb{P}^2$ that are orthogonal to each other, see e.g. Example 2.1, then the integrals are immediately aggregated since the supports are disjoint. However, for uncountably many probability measures, conditions on H or probability measures are needed. Indeed, in order to aggregate these integrals, we need to construct a stochastic process I_t defined on all of the probability space so that $I_t = I_t^{\mathbb{P}}$ for all t, $\mathbb{P}-$almost surely. Under smoothness assumptions on the integrand H this aggregation is possible and a pointwise definition is provided by Karandikar [10] for càdlàg integrands H. Denis and Martini [5] uses the theory of capacities and construct the integral for *quasi-continuous* integrands, as defined in that paper. A different approach based on the underlying partial differential equation was introduced by Peng [13] yielding essentially the same results as in [5]. In Section 6 below, we also provide a construction without any restrictions on H but in a slightly smaller class than $\overline{\mathcal{P}}_W$.

For general stochastic processes or random variables, an obvious consistency condition (see Definition 3.2, below) is clearly needed for aggregation. But Example 3.3 also shows that this condition is in general not sufficient. So to obtain aggregation under this minimal condition, we have two alternatives. First is to restrict the family of processes by requiring smoothness. Indeed the previous results of Karandikar [10], Denis-Martini [5], and Peng [13] all belong to this case. A precise statement is given in Section 3 below. The second approach is to slightly restrict the class of non-dominated measures. The main goal of this paper is to specify these restrictions on the probability measures that allows us to prove aggregation under only the consistency condition (3.4).

Our main result, Theorem 5.1, is proved in Section 5. For this main aggregation result, we assume that the class of probability measures are constructed from a *separable* class of diffusion processes as defined in subsection 4.4, Definition 4.8. This class of diffusion processes is somehow natural and the conditions are motivated from stochastic optimal control. Several simple examples of such sets are also provided. Indeed, the processes obtained by a straightforward concatenation of deterministic piece-wise constant processes forms a separable class. For most applications, this set would be sufficient. However, we believe that working with general separable class helps our understanding of quasi-sure stochastic analysis.

The construction of a probability measure corresponding to a given diffusion process, however, contains interesting technical details. Indeed, given an $\mathbb{F}$-progressively measurable process α, we would like to construct a unique measure $\mathbb{P}^\alpha$. For such a construction, we start with the Wiener measure $\mathbb{P}_0$ and assume that α takes values in $\mathbb{S}_d^{>0}$ (symmetric, positive definite matrices) and also satisfy $\int_0^t |\alpha_s| ds < \infty$ for all $t \geq 0$, $\mathbb{P}_0$-almost surely. We then consider the $\mathbb{P}_0$ stochastic integral

$$X_t^\alpha := \int_0^t \alpha_s^{1/2} dB_s. \tag{1.1}$$

Classically, the quadratic variation density of X^α under $\mathbb{P}_0$ is equal to α. We then set $\mathbb{P}_S^\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1}$ (here the subscript S is for the strong formulation). It is clear that B under $\mathbb{P}_S^\alpha$ has the same distribution as X^α under $\mathbb{P}_0$. One can show that the quadratic variation density of B under $\mathbb{P}_S^\alpha$ is equal to a satisfying $a(X^\alpha(\omega)) = \alpha(\omega)$ (see Lemma 8.1 below for the existence of such a). Hence, $\mathbb{P}_S^\alpha \in \overline{\mathcal{P}}_W$. Let $\overline{\mathcal{P}}_S \subset \overline{\mathcal{P}}_W$ be the collection of all such local martingale measures $\mathbb{P}_S^\alpha$. Barlow [1] has observed that this inclusion is strict. Moreover, this procedure changes the density of the quadratic variation process to the above defined process a. Therefore to be able to specify the quadratic variation a priori, in subsection 4.2, we consider the weak solutions of a stochastic differential equation ((4.4) below) which is closely related to (1.1). This class of measures obtained as weak solutions almost provides the necessary structure for aggregation. The only additional structure we need is the uniqueness of the map from the diffusion process to the corresponding probability measure. Clearly, in general, there is no uniqueness. So we further restrict ourselves into the class with uniqueness which we denote by $\mathcal{A}_W$. This set and the probability measures generated by them, $\mathcal{P}_W$, are defined in subsection 4.2.

The implications of our aggregation result for quasi-sure stochastic analysis are given in Section 6. In particular, for a separable class of probability measures, we first construct a quasi sure stochastic integral and then prove all classical results such as Kolmogrov continuity criterion, martingale representation, Ito's formula, Doob-Meyer decomposition and the Girsanov theorem. All of them are proved as a straightforward application of our main aggregation result.

If in addition the family of probability measures is dense in an appropriate sense, then our aggregation approach provides the same result as the quasi-sure analysis. These type of results, of course, require continuity of all the maps in an appropriate sense. The details of this approach are investigated in our paper [16], see also Remark 7.5 in the context of the application to the hedging problem under uncertain volatility. Notice that, in contrast with [5], our approach provides existence of an optimal hedging strategy, but at the price of slightly restricting the family of probability measures.

The paper is organized as follows. The local martingale measures $\overline{\mathcal{P}}_W$ and a universal filtration are studied in Section 2. The question of aggregation is defined in Section 3. In the next section, we define $\mathcal{A}_W$, $\mathcal{P}_W$ and then the separable class of diffusion processes. The main aggregation result, Theorem 5.1, is proved in Section 5. The next section generalizes several classical results of stochastic analysis to the quasi-sure setting. Section 7 studies the application to the hedging problem under uncertain volatility. In Section 8 we investigate the class $\overline{\mathcal{P}}_S$ of mutually singular measures induced from strong formulation. Finally, several examples concerning weak solutions and the proofs of several technical results are provided in the Appendix.

Notations. We close this introduction with a list of notations introduced in the paper.

- $\Omega := \{\omega \in C(\mathbb{R}_+, \mathbb{R}^d) : \omega(0) = 0\}$, B is the canonical process, $\mathbb{P}_0$ is the Wiener measure on Ω.
- For a given stochastic process X, $\mathbb{F}^X$ is the filtration generated by X.
- $\mathbb{F} := \mathbb{F}^B = \{\mathcal{F}_t\}_{t\geq 0}$ is the filtration generated by B.
- $\mathbb{F}^+ := \{\mathcal{F}_t^+, t \geq 0\}$, where $\mathcal{F}_t^+ := \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$,
- $\mathcal{F}_t^{\mathbb{P}} := \mathcal{F}_t^+ \vee \mathcal{N}^{\mathbb{P}}(\mathcal{F}_t^+)$ and $\overline{\mathcal{F}}_t^{\mathbb{P}} := \mathcal{F}_t^+ \vee \mathcal{N}^{\mathbb{P}}(\mathcal{F}_\infty)$, where
$$\mathcal{N}^{\mathbb{P}}(\mathcal{G}) := \Big\{E \subset \Omega : \text{ there exists } \tilde{E} \in \mathcal{G} \text{ such that } E \subset \tilde{E} \text{ and } \mathbb{P}[\tilde{E}] = 0\Big\}.$$
- $\mathcal{N}_{\mathcal{P}}$ is the class of $\mathcal{P}-$polar sets defined in Definition 2.2.
- $\hat{\mathcal{F}}_t^{\mathcal{P}} := \bigcap_{\mathbb{P}\in\mathcal{P}} \big(\mathcal{F}_t^{\mathbb{P}} \vee \mathcal{N}_{\mathcal{P}}\big)$ is the universal filtration defined in (2.3).
- $\mathcal{T}$ is the set of all $\mathbb{F}-$stopping times τ taking values in $\mathbb{R}_+ \cup \{\infty\}$.
- $\hat{\mathcal{T}}^{\mathcal{P}}$ is set of all $\hat{\mathbb{F}}^{\mathcal{P}}-$stopping times.
- $\langle B \rangle$ is the universally defined quadratic variation of B, defined in subsection 2.1.
- $\hat{a}$ is the density of the quadratic variation $\langle B \rangle$, also defined in subsection 2.1.
- $\mathbb{S}_d$ is the set of $d \times d$ symmetric matrices.
- $\mathbb{S}_d^{>0}$ is the set of positive definite symmetric matrices.

- $\overline{\mathcal{P}}_W$ is the set of measures defined in subsection 2.1.
- $\overline{\mathcal{P}}_S \subset \overline{\mathcal{P}}_W$ is defined in the Introduction, see also Lemma 8.1.
- $\overline{\mathcal{P}}_{\text{MRP}} \subset \overline{\mathcal{P}}_W$ are the measures with the martingale representation property, see (2.2).
- Sets $\mathcal{P}_W$, $\mathcal{P}_S$, $\mathcal{P}_{\text{MRP}}$ are defined in subsection 4.2 and section 8, as the subsets of $\overline{\mathcal{P}}_W$, $\overline{\mathcal{P}}_S$, $\overline{\mathcal{P}}_{\text{MRP}}$ with the additional requirement of weak uniqueness.
- $\overline{\mathcal{A}}$ is the set of integrable, progressively measurable processes with values in $\mathbb{S}_d^{>0}$.
- $\overline{\mathcal{A}}_W := \bigcup_{\mathbb{P}\in\overline{\mathcal{P}}_W} \overline{\mathcal{A}}_W(\mathbb{P})$ and $\overline{\mathcal{A}}_W(\mathbb{P})$ is the set of diffusion matrices satisfying (4.1).
- $\mathcal{A}_W$, $\mathcal{A}_S$, $\mathcal{A}_{\text{MRP}}$ are defined as above using $\mathcal{P}_W$, $\mathcal{P}_S$, $\mathcal{P}_{\text{MRP}}$, see section 8.
- Sets $\Omega^a_{\hat{\tau}}$, $\Omega^{a,b}_{\hat{\tau}}$ and the stopping time θ^{ab} are defined in subsection 4.3.
- Function spaces $\mathbb{L}^0$, $\mathbb{L}^p(\mathbb{P})$, $\hat{\mathbb{L}}^p$, and the integrand spaces $\mathbb{H}^0$, $\mathbb{H}^p(\mathbb{P}^a)$, $\mathbb{H}^2_{loc}(\mathbb{P}^a)$, $\hat{\mathbb{H}}^p$, $\hat{\mathbb{H}}^2_{loc}$ are defined in Section 6.

2 Non-dominated mutually singular probability measures

Let $\Omega := C(\mathbb{R}_+, \mathbb{R}^d)$ be as above and $\mathbb{F} = \mathbb{F}^B$ be the filtration generated by the canonical process B. Then it is well known that this natural filtration $\mathbb{F}$ is left-continuous, but is not right-continuous. This paper makes use of the right-limiting filtration $\mathbb{F}^+$, the $\mathbb{P}-$completed filtration $\mathbb{F}^{\mathbb{P}} := \{\mathcal{F}^{\mathbb{P}}_t, t \geq 0\}$, and the $\mathbb{P}-$augmented filtration $\overline{\mathbb{F}}^{\mathbb{P}} := \{\overline{\mathcal{F}}^{\mathbb{P}}_t, t \geq 0\}$, which are all right continuous.

2.1 Local martingale measures

We say a probability measure $\mathbb{P}$ is a local martingale measure if the canonical process B is a local martingale under $\mathbb{P}$. It follows from Karandikar [10] that there exists an $\mathbb{F}-$progressively measurable process, denoted as $\int_0^t B_s dB_s$, which coincides with the Itô's integral, $\mathbb{P}-$almost surely for all local martingale measure $\mathbb{P}$. In particular, this provides a pathwise definition of

$$\langle B\rangle_t := B_t B_t^T - 2\int_0^t B_s dB_s \quad \text{and} \quad \hat{a}_t := \overline{\lim_{\varepsilon\downarrow 0}} \frac{1}{\varepsilon}[\langle B\rangle_t - \langle B\rangle_{t-\varepsilon}].$$

Clearly, $\langle B\rangle$ coincides with the $\mathbb{P}-$quadratic variation of B, $\mathbb{P}-$almost surely for all local martingale measure $\mathbb{P}$.

Let $\overline{\mathcal{P}}_W$ denote the set of all local martingale measures $\mathbb{P}$ such that

$$\mathbb{P}\text{-almost surely, } \langle B\rangle_t \text{ is absolutely continuous in } t \text{ and } \hat{a} \text{ takes values in } \mathbb{S}_d^{>0}, \tag{2.1}$$

where $\mathbb{S}_d^{>0}$ denotes the space of all $d \times d$ real valued positive definite matrices. We note that, for different $\mathbb{P}_1, \mathbb{P}_2 \in \overline{\mathcal{P}}_W$, in general $\mathbb{P}_1$ and $\mathbb{P}_2$ are mutually singular, as we see in the next simple example. Moreover, there is no dominating measure for $\overline{\mathcal{P}}_W$.

Example 2.1 Let $d = 1$, $\mathbb{P}_1 := \mathbb{P}_0 \circ (\sqrt{2}B)^{-1}$, and $\Omega_i := \{\langle B \rangle_t = (1+i)t, t \geq 0\}$, $i = 0, 1$. Then, $\mathbb{P}_0, \mathbb{P}_1 \in \overline{\mathcal{P}}_W$, $\mathbb{P}_0(\Omega_0) = \mathbb{P}_1(\Omega_1) = 1$, $\mathbb{P}_0(\Omega_1) = \mathbb{P}_1(\Omega_0) = 0$, and Ω_0 and Ω_1 are disjoint. That is, $\mathbb{P}_0$ and $\mathbb{P}_1$ are mutually singular. $\Box$

In many applications, it is important that $\mathbb{P} \in \overline{\mathcal{P}}_W$ has martingale representation property (MRP, for short), i.e. for any $(\overline{\mathbb{F}}^{\mathbb{P}}, \mathbb{P})$-local martingale M, there exists a unique ($\mathbb{P}$-almost surely) $\overline{\mathbb{F}}^{\mathbb{P}}$-progressively measurable $\mathbb{R}^d$ valued process H such that

$$\int_0^t |\hat{a}_s^{1/2} H_s|^2 ds < \infty \quad \text{and} \quad M_t = M_0 + \int_0^t H_s dB_s, \quad t \geq 0, \quad \mathbb{P}\text{-almost surely.}$$

We thus define

$$\overline{\mathcal{P}}_{\text{MRP}} := \left\{ \mathbb{P} \in \overline{\mathcal{P}}_W : B \text{ has MRP under } \mathbb{P} \right\}. \tag{2.2}$$

The inclusion $\overline{\mathcal{P}}_{\text{MRP}} \subset \overline{\mathcal{P}}_W$ is strict as shown in Example 9.3 below.

Another interesting subclass is the set $\overline{\mathcal{P}}_S$ defined in the Introduction. Since in this paper it is not directly used, we postpone its discussion to Section 8.

2.2 A universal filtration

We now fix an arbitrary subset $\mathcal{P} \subset \overline{\mathcal{P}}_W$. By a slight abuse of terminology, we define the following notions introduced by Denis and Martini [5].

Definition 2.2 (i) *We say that a property holds $\mathcal{P}$-quasi-surely, abbreviated as $\mathcal{P}$-q.s., if it holds $\mathbb{P}$-almost surely for all $\mathbb{P} \in \mathcal{P}$.*
(ii) *Denote $\mathcal{N}_{\mathcal{P}} := \cap_{\mathbb{P} \in \mathcal{P}} \mathcal{N}^{\mathbb{P}}(\mathcal{F}_\infty)$ and we call $\mathcal{P}$-polar sets the elements of $\mathcal{N}_{\mathcal{P}}$.*
(iii) *A probability measure $\mathbb{P}$ is called absolutely continuous with respect to $\mathcal{P}$ if $\mathbb{P}(E) = 0$ for all $E \in \mathcal{N}_{\mathcal{P}}$.*

In the stochastic analysis theory, it is usually assumed that the filtered probability space satisfies the *usual hypotheses*. However, the key issue in the present paper is to develop stochastic analysis tools simultaneously for non-dominated mutually singular measures. In this case, we do not have a good filtration satisfying the usual hypotheses under all the measures. In this paper, we shall use the following universal filtration $\hat{\mathbb{F}}^{\mathcal{P}}$ for the mutually singular probability measures $\{\mathbb{P}, \mathbb{P} \in \mathcal{P}\}$:

$$\hat{\mathbb{F}}^{\mathcal{P}} := \{\hat{\mathcal{F}}_t^{\mathcal{P}}\}_{t \geq 0} \quad \text{where} \quad \hat{\mathcal{F}}_t^{\mathcal{P}} := \bigcap_{\mathbb{P} \in \mathcal{P}} \left(\mathcal{F}_t^{\mathbb{P}} \vee \mathcal{N}_{\mathcal{P}} \right) \text{ for } t \geq 0. \tag{2.3}$$

Moreover, we denote by $\mathcal{T}$ (resp. $\hat{\mathcal{T}}^{\mathcal{P}}$) the set of all $\mathbb{F}$-stopping times τ (resp., $\hat{\mathbb{F}}^{\mathcal{P}}$-stopping times $\hat{\tau}$) taking values in $\mathbb{R}_+ \cup \{\infty\}$.

Remark 2.3 Notice that $\mathbb{F}^+ \subset \mathbb{F}^{\mathbb{P}} \subset \overline{\mathbb{F}}^{\mathbb{P}}$. The reason for the choice of this completed filtration $\mathbb{F}^{\mathbb{P}}$ is as follows. If we use the small filtration $\mathbb{F}^+$, then the crucial aggregation result of Theorem 5.1 below will not hold true. On the other hand, if we use the augmented filtrations $\overline{\mathbb{F}}^{\mathbb{P}}$, then Lemma 5.2 below does not hold. Consequently, in applications one will not be able to check the consistency condition (5.2) in Theorem 5.1, and thus will not be able to apply the aggregation result. See also Remarks 5.3 and 5.6 below. However, this choice of the completed filtration does not cause any problems in the applications. □

We note that $\hat{\mathbb{F}}^{\mathcal{P}}$ is right continuous and all $\mathcal{P}$-polar sets are contained in $\hat{\mathcal{F}}_0^{\mathcal{P}}$. But $\hat{\mathbb{F}}^{\mathcal{P}}$ is not complete under each $\mathbb{P} \in \mathcal{P}$. However, thanks to the Lemma 2.4 below, all the properties we need still hold under this filtration.

For any sub-$\sigma-$algebra $\mathcal{G}$ of $\mathcal{F}_\infty$ and any probability measure $\mathbb{P}$, it is well-known that an $\overline{\mathcal{F}}_\infty^{\mathbb{P}}$-measurable random variable X is $[\mathcal{G} \vee \mathcal{N}^{\mathbb{P}}(\mathcal{F}_\infty)]-$measurable if and only if there exists a $\mathcal{G}$-measurable random variable $\tilde{X}$ such that $X = \tilde{X}$, $\mathbb{P}$-almost surely. The following result extends this property to processes and states that one can always consider any process in its $\mathbb{F}^+$-progressively measurable version. Since $\mathbb{F}^+ \subset \hat{\mathbb{F}}^{\mathcal{P}}$, the $\mathbb{F}^+$-progressively measurable version is also $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable. This important result will be used throughout our analysis so as to consider any process in its $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable version. However, we emphasize that the $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable version depends on the underlying probability measure $\mathbb{P}$.

Lemma 2.4 *Let $\mathbb{P}$ be an arbitrary probability measure on the canonical space $(\Omega, \mathcal{F}_\infty)$, and let X be an $\overline{\mathbb{F}}^{\mathbb{P}}$-progressively measurable process. Then, there exists a unique ($\mathbb{P}$-almost surely) $\mathbb{F}^+$-progressively measurable process $\tilde{X}$ such that $\tilde{X} = X$, $\mathbb{P}-$almost surely. If, in addition, X is càdlàg $\mathbb{P}$-almost surely, then we can choose $\tilde{X}$ to be càdlàg $\mathbb{P}$-almost surely.*

The proof is rather standard but it is provided in Appendix for completeness. We note that, the identity $\tilde{X} = X$, $\mathbb{P}$-almost surely, is equivalent to that they are equal $dt \times d\mathbb{P}$-almost surely. However, if both of them are càdlàg, then clearly $\tilde{X}_t = X_t$, $0 \le t \le 1$, $\mathbb{P}$-almost surely.

3 Aggregation

We are now in a position to define the problem.

Definition 3.1 Let $\mathcal{P} \subset \overline{\mathcal{P}}_W$, and let $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ be a family of $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable processes. An $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable process X is called a $\mathcal{P}$-*aggregator* of the family $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ if $X = X^{\mathbb{P}}$, $\mathbb{P}$-almost surely for every $\mathbb{P} \in \mathcal{P}$.

Clearly, for any family $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ which can be aggregated, the following consistency condition must hold.

Definition 3.2 We say that a family $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ satisfies the *consistency condition* if, for any $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}$, and $\hat{\tau} \in \hat{\mathcal{T}}^{\mathcal{P}}$ satisfying $\mathbb{P}_1 = \mathbb{P}_2$ on $\hat{\mathcal{F}}^{\mathcal{P}}_{\hat{\tau}}$ we have

$$X^{\mathbb{P}_1} = X^{\mathbb{P}_2} \;\; \textit{on } [0, \hat{\tau}],\;\; \mathbb{P}_1 - \textit{almost surely.} \tag{3.4}$$

Example 3.3 below shows that the above condition is in general not sufficient. Therefore, we are left with following two alternatives.

- Restrict the range of aggregating processes by requiring that there exists a sequence of $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable processes $\{X^n\}_{n\geq 1}$ such that $X^n \to X^{\mathbb{P}}$, $\mathbb{P}$-almost surely as $n \to \infty$ for all $\mathbb{P} \in \mathcal{P}$. In this case, the $\mathcal{P}$-aggregator is $X := \overline{\lim}_{n\to\infty} X^n$. Moreover, the class $\mathcal{P}$ can be taken to be the largest possible class $\overline{\mathcal{P}}_W$. We observe that the aggregation results of Karandikar [10], Denis-Martini [5], and Peng [13] all belong to this case. Under some regularity on the processes, this condition holds.
- Restrict the class $\mathcal{P}$ of mutually singular measures so that the consistency condition (3.4) is sufficient for the largest possible family of processes $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$. This is the main goal of the present paper.

We close this section by constructing an example in which the consistency condition is not sufficient for aggregation.

Example 3.3 Let $d = 2$. First, for each $x, y \in [1, 2]$, let $\mathbb{P}^{x,y} := \mathbb{P}_0 \circ (\sqrt{x}B^1, \sqrt{y}B^2)^{-1}$ and $\Omega_{x,y} := \{\langle B^1\rangle_t = xt, \langle B^2\rangle_t = yt, t \geq 0\}$. Cleary for each (x, y), $\mathbb{P}^{x,y} \in \overline{\mathcal{P}}_W$ and $\mathbb{P}^{x,y}[\Omega_{x,y}] = 1$. Next, for each $a \in [1, 2]$, we define

$$\mathbb{P}_a[E] := \frac{1}{2}\int_1^2 (\mathbb{P}^{a,z}[E] + \mathbb{P}^{z,a}[E])dz \quad \text{for all} \quad E \in \mathcal{F}_\infty.$$

We claim that $\mathbb{P}_a \in \overline{\mathcal{P}}_W$. Indeed, for any $t_1 < t_2$ and any bounded $\mathcal{F}_{t_1}$-measurable random variable η, we have

$$2\mathbb{E}^{\mathbb{P}_a}[(B_{t_2} - B_{t_1})\eta] = \int_1^2 \{\mathbb{E}^{\mathbb{P}^{a,z}}[(B_{t_2} - B_{t_1})\eta] + \mathbb{E}^{\mathbb{P}^{z,a}}[(B_{t_2} - B_{t_1})\eta]\}dz = 0.$$

Hence $\mathbb{P}_a$ is a martingale measure. Similarly, one can easily show that $I_2 dt \leq d\langle B\rangle_t \leq 2I_2 dt$, $\mathbb{P}_a$-almost surely, where I_2 is the 2×2 identity matrix.

For $a \in [1,2]$ set

$$\Omega_a := \{\langle B^1\rangle_t = at, t \geq 0\} \cup \{\langle B^2\rangle_t = at, t \geq 0\} \supseteq \cup_{z\in[1,2]} [\Omega_{a,z} \cup \Omega_{z,a}]$$

so that $\mathbb{P}_a[\Omega_a] = 1$. Also for $a \neq b$, we have $\Omega_a \cap \Omega_b = \Omega_{a,b} \cup \Omega_{b,a}$ and thus $\mathbb{P}_a[\Omega_a \cap \Omega_b] = \mathbb{P}_b[\Omega_a \cap \Omega_b] = 0$.

Now let $\mathcal{P} := \{\mathbb{P}_a, a \in [1,2]\}$ and set $X^a_t(\omega) = a$ for all t, ω. Notice that, for $a \neq b$, $\mathbb{P}_a$ and $\mathbb{P}_b$ disagree on $\mathcal{F}_0^+ \subset \hat{\mathcal{F}}_0^{\mathcal{P}}$. Then the consistency condition (3.4) holds trivially. However, we claim that there is no $\mathcal{P}$-aggregator X of the family $\{X^a, a \in [1,2]\}$. Indeed, if there is X such that $X = X^a$, $\mathbb{P}_a$-almost surely for all $a \in [1,2]$, then for any $a \in [1,2]$,

$$1 = \mathbb{P}_a[X^a_. = a] = \mathbb{P}_a[X_. = a] = \frac{1}{2}\int_1^2 \Big(\mathbb{P}^{a,z}[X_. = a] + \mathbb{P}^{z,a}[X_. = a]\Big)dz.$$

Let λ_n the Lebesgue measure on $[1,2]^n$ for integer $n \geq 1$. Then, we have

$$\lambda_1\Big(\{z : \mathbb{P}^{a,z}[X_. = a] = 1\}\Big) = \lambda_1\Big(\{z : \mathbb{P}^{z,a}[X_. = a] = 1\}\Big) = 1, \quad \text{for all } a \in [1,2].$$

Set $A_1 := \{(a,z) : \mathbb{P}^{a,z}[X_. = a] = 1\}$, $A_2 := \{(z,a) : \mathbb{P}^{z,a}[X_. = a] = 1\}$ so that $\lambda_2(A_1) = \lambda_2(A_2) = 1$. Moreover, $A_1 \cap A_2 \subset \{(a,a) : a \in (0,1]\}$ and $\lambda_2(A_1 \cap A_2) = 0$. Now we directly calculate that $1 \geq \lambda_2(A_1 \cup A_2) = \lambda_2(A_1) + \lambda_2(A_2) - \lambda_2(A_1 \cap A_2) = 2$. This contradiction implies that there is no aggregator. □

4 Separable classes of mutually singular measures

The main goal of this section is to identify a condition on the probability measures that yields aggregation as defined in the previous section. It is more convenient to specify this restriction through the diffusion processes. However, as we discussed in the Introduction there are technical difficulties in the connection between the diffusion processes and the probability measures. Therefore, in the first two subsections we will discuss the issue of uniqueness of the mapping from the diffusion process to a martingale measure. The separable class of mutually singular measures are defined in subsection 4.4 after a short discussion of the supports of these measures in subsection 4.3.

4.1 Classes of diffusion matrices

Let

$$\overline{\mathcal{A}} := \Big\{a : \mathbb{R}_+ \to \mathbb{S}_d^{>0} \mid \mathbb{F}\text{-progressively measurable and } \int_0^t |a_s|ds < \infty, \text{ for all } t \geq 0\Big\}.$$

For a given $\mathbb{P} \in \overline{\mathcal{P}}_W$, let

$$\overline{\mathcal{A}}_W(\mathbb{P}) := \Big\{a \in \overline{\mathcal{A}} : a = \hat{a}, \quad \mathbb{P}\text{-almost surely}\Big\}. \tag{4.1}$$

Recall that $\hat{a}$ is the density of the quadratic variation of $\langle B \rangle$ and is defined pointwise. We also define

$$\overline{\mathcal{A}}_W := \bigcup_{\mathbb{P}\in\overline{\mathcal{P}}_W} \overline{\mathcal{A}}_W(\mathbb{P}).$$

A subtle technical point is that $\overline{\mathcal{A}}_W$ is strictly included in $\overline{\mathcal{A}}$. In fact, the process

$$a_t := \mathbf{1}_{\{\hat{a}_t \geq 2\}} + \mathbf{3}\mathbf{1}_{\{\hat{a}_t < 2\}} \quad \text{is clearly in } \overline{\mathcal{A}} \setminus \overline{\mathcal{A}}_W.$$

For any $\mathbb{P} \in \overline{\mathcal{P}}_W$ and $a \in \overline{\mathcal{A}}_W(\mathbb{P})$, by the Lévy characterization, the following Itô's stochastic integral under $\mathbb{P}$ is a $\mathbb{P}$-Brownian motion:

$$W_t^{\mathbb{P}} := \int_0^t \hat{a}_s^{-1/2} dB_s = \int_0^t a_s^{-1/2} dB_s, \quad t \geq 0. \quad \mathbb{P} - \text{a.s.} \tag{4.2}$$

Also since B is the canonical process, $a = a(B_\cdot)$ and thus

$$dB_t = a_t^{1/2}(B_\cdot) dW_t^{\mathbb{P}}, \ \mathbb{P}\text{-almost surely, and } W_t^{\mathbb{P}} \text{ is a } \mathbb{P}\text{-Brownian motion.} \tag{4.3}$$

4.2 Characterization by diffusion matrices

In view of (4.3), to construct a measure with a given quadratic variation $a \in \overline{\mathcal{A}}_W$, we consider the stochastic differential equation,

$$dX_t = a_t^{1/2}(X_\cdot) dB_t, \quad \mathbb{P}_0\text{-almost surely.} \tag{4.4}$$

In this generality, we consider only weak solutions $\mathbb{P}$ which we define next. Although the following definition is standard (see for example Stroock & Varadhan [18]), we provide it for specificity.

Definition 4.1 Let a be an element of $\overline{\mathcal{A}}_W$.
(i) For $\mathbb{F}-$stopping times $\tau_1 \leq \tau_2 \in \mathcal{T}$ and a probability measure $\mathbb{P}^1$ on $\mathcal{F}_{\tau_1}$, we say that $\mathbb{P}$ is a *weak solution of* (4.4) *on* $[\tau_1, \tau_2]$ *with initial condition* $\mathbb{P}^1$, *denoted as* $\mathbb{P} \in \mathcal{P}(\tau_1, \tau_2, \mathbb{P}^1, a)$, if the followings hold:

1. $\mathbb{P} = \mathbb{P}^1$ on $\mathcal{F}_{\tau_1}$;
2. The canonical process B_t is a $\mathbb{P}$-local martingale on $[\tau_1, \tau_2]$;
3. The process $W_t := \int_{\tau_1}^t a_s^{-1/2}(B_\cdot) dB_s$, defined $\mathbb{P}-$almost surely for all $t \in [\tau_1, \tau_2]$, is a $\mathbb{P}$-Brownian Motion.

(ii) We say that the equation (4.4) has *weak uniqueness on* $[\tau_1, \tau_2]$ *with initial condition* $\mathbb{P}^1$ if any two weak solutions $\mathbb{P}$ and $\mathbb{P}'$ in $\mathcal{P}(\tau_1, \tau_2, \mathbb{P}^1, a)$ satisfy $\mathbb{P} = \mathbb{P}'$ on $\mathcal{F}_{\tau_2}$.
(iii) We say that (4.4) *has weak uniqueness* if (ii) holds for any $\tau_1, \tau_2 \in \mathcal{T}$ and any initial condition $\mathbb{P}^1$ on $\mathcal{F}_{\tau_1}$.

We emphasize that the stopping times in this definition are $\mathbb{F}$-stopping times.

Note that, for each $\mathbb{P} \in \overline{\mathcal{P}}_W$ and $a \in \overline{\mathcal{A}}_W(\mathbb{P})$, $\mathbb{P}$ is a weak solution of (4.4) on $\mathbb{R}_+$ with initial value $\mathbb{P}(B_0 = 0) = 1$. We also need uniqueness of this map to characterize the measure $\mathbb{P}$ in terms of the diffusion matrix a. Indeed, if (4.4) with a has weak uniqueness, we let $\mathbb{P}^a \in \overline{\mathcal{P}}_W$ be the unique weak solution of (4.4) on $\mathbb{R}_+$ with initial condition $\mathbb{P}^a(B_0 = 0) = 1$, and define,

$$\mathcal{A}_W := \left\{a \in \overline{\mathcal{A}}_W : \text{ (4.4) has weak uniqueness}\right\}, \qquad \mathcal{P}_W := \{\mathbb{P}^a : a \in \mathcal{A}_W\}. \tag{4.5}$$

We also define

$$\mathcal{P}_{\text{MRP}} := \overline{\mathcal{P}}_{\text{MRP}} \cap \mathcal{P}_W, \quad \mathcal{A}_{\text{MRP}} := \{a \in \mathcal{A}_W : \mathbb{P}^a \in \mathcal{P}_{\text{MRP}}\}. \tag{4.6}$$

For notational simplicity, we denote

$$\mathbb{F}^a := \mathbb{F}^{\mathbb{P}^a}, \quad \overline{\mathbb{F}}^a := \overline{\mathbb{F}}^{\mathbb{P}^a}, \quad \text{for all} \quad a \in \mathcal{A}_W. \tag{4.7}$$

It is clear that, for each $\mathbb{P} \in \mathcal{P}_W$, the weak uniqueness of the equation (4.4) may depend on the version of $a \in \overline{\mathcal{A}}_W(\mathbb{P})$. This is indeed the case and the following example illustrates this observation.

Example 4.2 Let $a_0(t) := 1$, $a_2(t) := 2$ and

$$a_1(t) := 1 + \mathbf{1}_E \mathbf{1}_{(0,\infty)}(t), \quad \text{where} \quad E := \left\{ \overline{\lim_{h \downarrow 0}} \frac{B_h - B_0}{\sqrt{2h \ln \ln h^{-1}}} \neq 1 \right\} \in \mathcal{F}_0^+.$$

Then clearly both a_0 and a_2 belong to $\mathcal{A}_W$. Also $a_1 = a_0$, $\mathbb{P}_0$-almost surely and $a_1 = a_2$, $\mathbb{P}^{a_2}$-almost surely. Hence, $a_1 \in \overline{\mathcal{A}}_W(\mathbb{P}_0) \cap \overline{\mathcal{A}}_W(\mathbb{P}^{a_2})$. Therefore the equation (4.4) with coefficient a_1 has two weak solutions $\mathbb{P}_0$ and $\mathbb{P}^{a_2}$. Thus $a_1 \notin \mathcal{A}_W$. $\square$

Remark 4.3 In this paper, we shall consider only those $\mathbb{P} \in \mathcal{P}_W \subset \overline{\mathcal{P}}_W$. However, we do not know whether this inclusion is strict or not. In other words, given an arbitrary $\mathbb{P} \in \overline{\mathcal{P}}_W$, can we always find one version $a \in \overline{\mathcal{A}}_W(\mathbb{P})$ such that $a \in \mathcal{A}_W$? $\square$

It is easy to construct examples in $\mathcal{A}_W$ in the Markovian context. Below, we provide two classes of path dependent diffusion processes in $\mathcal{A}_W$. These sets are in fact subsets of $\mathcal{A}_S \subset \mathcal{A}_W$, which is defined in (8.11) below. We also construct some counter-examples in the Appendix. Denote

$$\mathbf{Q} := \left\{(t, \mathbf{x}) \ : \ t \geq 0, \mathbf{x} \in C([0,t], \mathbb{R}^d)\right\}. \tag{4.8}$$

Example 4.4 (Lipschitz coefficients) Let

$$a_t := \sigma^2(t, B_\cdot) \quad \text{where} \quad \sigma : \mathbf{Q} \to \mathbb{S}_d^{>0}$$

is Lebesgue measurable, uniformly Lipschitz continuous in $\mathbf{x}$ under the uniform norm, and $\sigma^2(\cdot, \mathbf{0}) \in \overline{\mathcal{A}}$. Then (4.4) has a unique strong solution and consequently $a \in \mathcal{A}_W$. $\square$

Example 4.5 (Piecewise constant coefficients) Let $a = \sum_{n=0}^{\infty} a_n \mathbf{1}_{[\tau_n, \tau_{n+1})}$ where $\{\tau_n\}_{n\geq 0} \subset \mathcal{T}$ is a nondecreasing sequence of $\mathbb{F}-$stopping times with $\tau_0 = 0$, $\tau_n \uparrow \infty$ as $n \to \infty$, and $a_n \in \mathcal{F}_{\tau_n}$ with values in $\mathbb{S}_d^{>0}$ for all n. Again (4.4) has a unique strong solution and $a \in \mathcal{A}_W$.

This example is in fact more involved than it looks like, mainly due to the presence of the stopping times. We relegate its proof to the Appendix. □

4.3 Support of $\mathbb{P}^a$

In this subsection, we collect some properties of measures that are constructed in the previous subsection. We fix a subset $\mathcal{A} \subset \mathcal{A}_W$, and denote by $\mathcal{P} := \{\mathbb{P}^a : a \in \mathcal{A}\}$ the corresponding subset of $\mathcal{P}_W$. In the sequel, we may also say

a property holds $\mathcal{A}-$quasi surely if it holds $\mathcal{P}-$quasi surely.

For any $a \in \mathcal{A}$ and any $\hat{\mathbb{F}}^{\mathcal{P}}-$stopping time $\hat{\tau} \in \hat{\mathcal{T}}^{\mathcal{P}}$, let

$$\Omega^a_{\hat{\tau}} := \bigcup_{n\geq 1} \Big\{ \int_0^t \hat{a}_s ds = \int_0^t a_s ds, \text{ for all } t \in [0, \hat{\tau} + \frac{1}{n}] \Big\}. \tag{4.9}$$

It is clear that

$$\Omega^a_{\hat{\tau}} \in \hat{\mathcal{F}}^{\mathcal{P}}_{\hat{\tau}}, \quad \Omega^a_t \text{ is non-increasing in } t, \quad \Omega^a_{\hat{\tau}+} = \Omega^a_{\hat{\tau}}, \quad \text{and } \mathbb{P}^a(\Omega^a_\infty) = 1. \tag{4.10}$$

We next introduce the first disagreement time of any $a, b \in \mathcal{A}$, which plays a central role in Section 5:

$$\theta^{a,b} := \inf \Big\{ t \geq 0 : \int_0^t a_s ds \neq \int_0^t b_s ds \Big\},$$

and, for any $\hat{\mathbb{F}}^{\mathcal{P}}-$stopping time $\hat{\tau} \in \hat{\mathcal{T}}^{\mathcal{P}}$, the agreement set of a and b up to $\hat{\tau}$:

$$\Omega^{a,b}_{\hat{\tau}} := \{\hat{\tau} < \theta^{a,b}\} \cup \{\hat{\tau} = \theta^{a,b} = \infty\}.$$

Here we use the convention that $\inf \emptyset = \infty$. It is obvious that

$$\theta^{a,b} \in \hat{\mathcal{T}}^{\mathcal{P}}, \quad \Omega^{a,b}_{\hat{\tau}} \in \hat{\mathcal{F}}^{\mathcal{P}}_{\hat{\tau}} \quad \text{and} \quad \Omega^a_{\hat{\tau}} \cap \Omega^b_{\hat{\tau}} \subset \Omega^{a,b}_{\hat{\tau}}. \tag{4.11}$$

Remark 4.6 The above notations can be extended to all diffusion processes $a, b \in \overline{\mathcal{A}}$. This will be important in Lemma 4.12 below. □

4.4 Separability

We are now in a position to state the restrictions needed for the main aggregation result Theorem 5.1.

Definition 4.7 A subset $\mathcal{A}_0 \subset \mathcal{A}_W$ is called a *generating class of diffusion coefficients* if

(i) $\mathcal{A}_0$ satisfies the concatenation property: $a\mathbf{1}_{[0,t)} + b\mathbf{1}_{[t,\infty)} \in \mathcal{A}_0$ for $a, b \in \mathcal{A}_0$, $t \geq 0$.

(ii) $\mathcal{A}_0$ has constant disagreement times: for all $a, b \in \mathcal{A}_0$, $\theta^{a,b}$ is a constant or, equivalently, $\Omega_t^{a,b} = \emptyset$ or Ω for all $t \geq 0$.

We note that the concatenation property is standard in the stochastic control theory in order to establish the dynamic programming principle, see, e.g. page 5 in [14]. The constant disagreement times property is important for both Lemma 5.2 below and the aggregation result of Theorem 5.1 below. We will provide two examples of sets with these properties, after stating the main restriction for the aggregation result.

Definition 4.8 We say $\mathcal{A}$ is a *separable class of diffusion coefficients generated by* $\mathcal{A}_0$ if $\mathcal{A}_0 \subset \mathcal{A}_W$ is a generating class of diffusion coefficients and $\mathcal{A}$ consists of all processes a of the form,

$$a = \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} a_i^n \mathbf{1}_{E_i^n} \mathbf{1}_{[\tau_n, \tau_{n+1})}, \tag{4.12}$$

where $(a_i^n)_{i,n} \subset \mathcal{A}_0$, $(\tau_n)_n \subset \mathcal{T}$ is nondecreasing with $\tau_0 = 0$ and

- $\inf\{n : \tau_n = \infty\} < \infty$, $\tau_n < \tau_{n+1}$ whenever $\tau_n < \infty$, and each τ_n takes at most countably many values,
- for each n, $\{E_i^n, i \geq 1\} \subset \mathcal{F}_{\tau_n}$ form a partition of Ω.

We emphasize that in the previous definition the τ_n's are $\mathbb{F}-$stopping times and $E_i^n \in \mathcal{F}_{\tau_n}$. The following are two examples of generating classes of diffusion coefficients.

Example 4.9 Let $\mathcal{A}_0 \subset \overline{\mathcal{A}}$ be the class of all deterministic mappings. Then clearly $\mathcal{A}_0 \subset \mathcal{A}_W$ and satisfies both properties (the concatenation and the constant disagreement times properties) of a generating class. $\square$

Example 4.10 Recall the set $\mathbf{Q}$ defined in (4.8). Let $\mathcal{D}_0$ be a set of deterministic Lebesgue measurable functions $\sigma : \mathbf{Q} \to \mathbb{S}_d^{>0}$ satisfying,

- σ is uniformly Lipschitz continuous in $\mathbf{x}$ under $\mathbb{L}^\infty$-norm, and $\sigma^2(\cdot, \mathbf{0}) \in \overline{\mathcal{A}}$ and

- for each $\mathbf{x} \in C(\mathbb{R}_+, \mathbb{R}^d)$ and different $\sigma_1, \sigma_2 \in \mathcal{D}_0$, the Lebesgue measure of the set $A(\sigma_1, \sigma_2, \mathbf{x})$ is equal to 0, where

$$A(\sigma_1, \sigma_2, \mathbf{x}) \quad := \quad \Big\{t : \sigma_1(t, \mathbf{x}|_{[0,t]}) = \sigma_2(t, \mathbf{x}|_{[0,t]})\Big\}.$$

Let $\mathcal{D}$ be the class of all possible concatenations of $\mathcal{D}_0$, i.e. $\sigma \in \mathcal{D}$ takes the following form:

$$\sigma(t, \mathbf{x}) \quad := \quad \sum_{i=0}^{\infty} \sigma_i(t, \mathbf{x}) \mathbf{1}_{[t_i, t_{i+1})}(t), \quad (t, \mathbf{x}) \in \mathbf{Q},$$

for some sequence $t_i \uparrow \infty$ and $\sigma_i \in \mathcal{D}_0$, $i \geq 0$. Let $\mathcal{A}_0 := \{\sigma^2(t, B.) : \sigma \in \mathcal{D}\}$. It is immediate to check that $\mathcal{A}_0 \subset \mathcal{A}_W$ and satisfies the concatenation and the constant disagreement times properties. Thus it is also a generating class. $\square$

We next prove several important properties of separable classes.

Proposition 4.11 *Let $\mathcal{A}$ be a separable class of diffusion coefficients generated by $\mathcal{A}_0$. Then $\mathcal{A} \subset \mathcal{A}_W$, and $\mathcal{A}$-quasi surely is equivalent to $\mathcal{A}_0$-quasi surely. Moreover, if $\mathcal{A}_0 \subset \mathcal{A}_{MRP}$, then $\mathcal{A} \subset \mathcal{A}_{MRP}$.*

We need the following two lemmas to prove this result. The first one provides a convenient structure for the elements of $\mathcal{A}$.

Lemma 4.12 *Let $\mathcal{A}$ be a separable class of diffusion coefficients generated by $\mathcal{A}_0$. For any $a \in \mathcal{A}$ and $\mathbb{F}$-stopping time $\tau \in \mathcal{T}$, there exist $\tau \leq \tilde{\tau} \in \mathcal{T}$, a sequence $\{a_n, n \geq 1\} \subset \mathcal{A}_0$, and a partition $\{E_n, n \geq 1\} \subset \mathcal{F}_\tau$ of Ω, such that $\tilde{\tau} > \tau$ on $\{\tau < \infty\}$ and*

$$a_t = \sum_{n \geq 1} a_n(t) \mathbf{1}_{E_n} \quad \textit{for all} \quad t < \tilde{\tau}.$$

In particular, $E_n \subset \Omega_\tau^{a,a_n}$ and consequently $\cup_n \Omega_\tau^{a,a_n} = \Omega$. Moreover, if a takes the form (4.12) and $\tau \geq \tau_n$, then one can choose $\tilde{\tau} \geq \tau_{n+1}$.

The proof of this lemma is straightforward, but with technical notations. Thus we postpone it to the Appendix.

We remark that at this point we do not know whether $a \in \mathcal{A}_W$. But the notations θ^{a,a_n} and Ω_τ^{a,a_n} are well defined as discussed in Remark 4.6. We recall from Definition 4.1 that $\mathbb{P} \in \mathcal{P}(\tau_1, \tau_2, \mathbb{P}^1, a)$ means $\mathbb{P}$ is a weak solution of (4.4) on $[\tilde{\tau}_1, \tilde{\tau}_2]$ with coefficient a and initial condition $\mathbb{P}^1$.

Lemma 4.13 *Let $\tau_1, \tau_2 \in \mathcal{T}$ with $\tau_1 \leq \tau_2$, $\{a^i, i \geq 1\} \subset \overline{\mathcal{A}}_W$ (not necessarily in $\mathcal{A}_W$) and $\{E_i, i \geq 1\} \subset \mathcal{F}_{\tau_1}$ be a partition of Ω. Let $\mathbb{P}^0$ be a probability measure on $\mathcal{F}_{\tau_1}$ and $\mathbb{P}^i \in \mathcal{P}(\tau_1, \tau_2, \mathbb{P}^0, a^i)$ for $i \geq 1$. Define*

$$\mathbb{P}(E) := \sum_{i \geq 1} \mathbb{P}^i(E \cap E_i) \quad \textit{for all} \quad E \in \mathcal{F}_{\tau_2} \quad \textit{and} \quad a_t := \sum_{i \geq 1} a_t^i \mathbf{1}_{E_i}, \qquad t \in [\tau_1, \tau_2].$$

Then $\mathbb{P} \in \mathcal{P}(\tau_1, \tau_2, \mathbb{P}^0, a)$.

Proof. Clearly, $\mathbb{P} = \mathbb{P}^0$ on $\mathcal{F}_{\tau_1}$. It suffices to show that both B_t and $B_t B_t^T - \int_{\tau_1}^t a_s ds$ are $\mathbb{P}$-local martingales on $[\tau_1, \tau_2]$.

By a standard localization argument, we may assume without loss of generality that all the random variables below are integrable. Now for any $\tau_1 \le \tau_3 \le \tau_4 \le \tau_2$ and any bounded random variable $\eta \in \mathcal{F}_{\tau_3}$, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[(B_{\tau_4} - B_{\tau_3})\eta] &= \sum_{i\ge 1} \mathbb{E}^{\mathbb{P}^i}\left[(B_{\tau_4} - B_{\tau_3})\eta \mathbf{1}_{E_i}\right] \\ &= \sum_{i\ge 1} \mathbb{E}^{\mathbb{P}^i}\left[\mathbb{E}^{\mathbb{P}^i}\left(B_{\tau_4} - B_{\tau_3}|\mathcal{F}_{\tau_3}\right)\eta \mathbf{1}_{E_i}\right] = 0.\end{aligned}$$

Therefore B is a $\mathbb{P}$-local martingale on $[\tau_1, \tau_2]$. Similarly one can show that $B_t B_t^T - \int_{\tau_1}^t a_s ds$ is also a $\mathbb{P}$-local martingale on $[\tau_1, \tau_2]$. □

Proof of Proposition 4.11. Let $a \in \mathcal{A}$ be given as in (4.12).

(i) We first show that $a \in \mathcal{A}_W$. Fix $\theta_1, \theta_2 \in \mathcal{T}$ with $\theta_1 \le \theta_2$ and a probability measure $\mathbb{P}^0$ on $\mathcal{F}_{\theta_1}$. Set

$$\tilde{\tau}_0 := \theta_1 \quad \text{and} \quad \tilde{\tau}_n := (\tau_n \vee \theta_1) \wedge \theta_2, \quad n \ge 1.$$

We shall show that $\mathcal{P}(\theta_1, \theta_2, \mathbb{P}^0, a)$ is a singleton, that is, the (4.4) on $[\theta_1, \theta_2]$ with coefficient a and initial condition $\mathbb{P}^0$ has a unique weak solution. To do this we prove by induction on n that $\mathcal{P}(\tilde{\tau}_0, \tilde{\tau}_n, \mathbb{P}^0, a)$ is a singleton.

First, let $n = 1$. We apply Lemma 4.12 with $\tau = \tilde{\tau}_0$ and choose $\tilde{\tau} = \tilde{\tau}_1$. Then, $a_t = \sum_{i\ge 1} a_i(t)\mathbf{1}_{E_i}$ for all $t < \tilde{\tau}_1$, where $a_i \in \mathcal{A}_0$ and $\{E_i, i \ge 1\} \subset \mathcal{F}_{\tilde{\tau}_0}$ form a partition of Ω. For $i \ge 1$, let $\mathbb{P}^{0,i}$ be the unique weak solution in $\mathcal{P}(\tilde{\tau}_0, \tilde{\tau}_1, \mathbb{P}^0, a_i)$ and set

$$\mathbb{P}^{0,a}(E) := \sum_{i\ge 1} \mathbb{P}^{0,i}(E \cap E_i) \quad \text{for all} \quad E \in \mathcal{F}_{\tilde{\tau}_1}.$$

We use Lemma 4.13 to conclude that $\mathbb{P}^{0,a} \in \mathcal{P}(\tilde{\tau}_0, \tilde{\tau}_1, \mathbb{P}^0, a)$. On the other hand, suppose $\mathbb{P} \in \mathcal{P}(\tilde{\tau}_0, \tilde{\tau}_1, \mathbb{P}^0, a)$ is an arbitrary weak solution. For each $i \ge 1$, we define $\mathbb{P}^i$ by

$$\mathbb{P}^i(E) := \mathbb{P}(E \cap E_i) + \mathbb{P}^{0,i}(E \cap (E_i)^c) \quad \text{for all} \quad E \in \mathcal{F}_{\tilde{\tau}_1}.$$

We again use Lemma 4.13 and notice that $a\mathbf{1}_{E_i} + a_i \mathbf{1}_{(E_i)^c} = a_i$. The result is that $\mathbb{P}^i \in \mathcal{P}(\tilde{\tau}_0, \tilde{\tau}_1, \mathbb{P}^0, a_i)$. Now by the uniqueness in $\mathcal{P}(\tilde{\tau}_0, \tilde{\tau}_1, \mathbb{P}^0, a_i)$ we conclude that $\mathbb{P}^i = \mathbb{P}^{0,i}$ on $\mathcal{F}_{\tilde{\tau}_1}$. This , in turn, implies that $\mathbb{P}(E \cap E_i) = \mathbb{P}^{0,i}(E \cap E_i)$ for all $E \in \mathcal{F}_{\tilde{\tau}_1}$ and $i \ge 1$. Therefore, $\mathbb{P}(E) = \sum_{i\ge 1} \mathbb{P}^{0,i}(E \cap E_i) = \mathbb{P}^{0,a}(E)$ for all $E \in \mathcal{F}_{\tilde{\tau}_1}$. Hence $\mathcal{P}(\tilde{\tau}_0, \tilde{\tau}_1, \mathbb{P}^0, a)$ is a singleton.

We continue with the induction step. Assume that $\mathcal{P}(\tilde{\tau}_0, \tilde{\tau}_n, \mathbb{P}^0, a)$ is a singleton, and denote its unique element by $\mathbb{P}^n$. Without loss of generality, we assume $\tilde{\tau}_n < \tilde{\tau}_{n+1}$. Following the same arguments as above we know that $\mathcal{P}(\tilde{\tau}_n, \tilde{\tau}_{n+1}, \mathbb{P}^n, a)$ contains a unique weak solution, denoted by $\mathbb{P}^{n+1}$. Then both B_t and $B_t B_t^T - \int_0^t a_s ds$ are $\mathbb{P}^{n+1}$-local martingales on $[\tilde\tau_0, \tilde\tau_n]$ and on $[\tilde\tau_n, \tilde\tau_{n+1}]$. This implies that $\mathbb{P}^{n+1} \in \mathcal{P}(\tilde\tau_0, \tilde\tau_{n+1}, \mathbb{P}^0, a)$. On the other hand, let $\mathbb{P} \in \mathcal{P}(\tilde\tau_0, \tilde\tau_{n+1}, \mathbb{P}^0, a)$ be an arbitrary weak solution. Since we also have $\mathbb{P} \in \mathcal{P}(\tilde\tau_0, \tilde\tau_n, \mathbb{P}^0, a)$, by the uniqueness in the induction assumption we must have the equality $\mathbb{P} = \mathbb{P}^n$ on $\mathcal{F}_{\tilde\tau_n}$. Therefore, $\mathbb{P} \in \mathcal{P}(\tilde\tau_n, \tilde\tau_{n+1}, \mathbb{P}^n, a)$. Thus by uniqueness $\mathbb{P} = \mathbb{P}^{n+1}$ on $\mathcal{F}_{\tilde\tau_{n+1}}$. This proves the induction claim for $n+1$.

Finally, note that $\mathbb{P}^m(E) = \mathbb{P}^n(E)$ for all $E \in \mathcal{F}_{\tilde\tau_n}$ and $m \geq n$. Hence, we may define $\mathbb{P}^\infty(E) := \mathbb{P}^n(E)$ for $E \in \mathcal{F}_{\tilde\tau_n}$. Since $\inf\{n : \tau_n = \infty\} < \infty$, then $\inf\{n : \tilde\tau_n = \theta_2\} < \infty$ and thus $\mathcal{F}_{\theta_2} = \vee_{n\geq 1} \mathcal{F}_{\tilde\tau_n}$. So we can uniquely extend $\mathbb{P}^\infty$ to $\mathcal{F}_{\theta_2}$. Now we directly check that $\mathbb{P}^\infty \in \mathcal{P}(\theta_1, \theta_2, \mathbb{P}^0, a)$ and is unique.

(ii) We next show that $\mathbb{P}^a(E) = 0$ for all $\mathcal{A}_0-$polar set E. Once again we apply Lemma 4.12 with $\tau = \infty$. Therefore $a_t = \sum_{i\geq 1} a_i(t) \mathbf{1}_{E_i}$ for all $t \geq 0$, where $\{a_i, i \geq 1\} \subset \mathcal{A}_0$ and $\{E_i, i \geq 1\} \subset \mathcal{F}_\infty$ form a partition of Ω. Now for any $\mathcal{A}_0$-polar set E,

$$\mathbb{P}^a(E) = \sum_{i\geq 1} \mathbb{P}^a(E \cap E_i) = \sum_{i\geq 1} \mathbb{P}^{a_i}(E \cap E_i) = 0.$$

This clearly implies the equivalence between $\mathcal{A}$-quasi surely and $\mathcal{A}_0$-quasi surely.

(iii) We now assume $\mathcal{A}_0 \subset \mathcal{A}_{\text{MRP}}$ and show that $a \in \mathcal{A}_{\text{MRP}}$. Let M be a $\mathbb{P}^a$-local martingale. We prove by induction on n again that M has a martingale representation on $[0, \tau_n]$ under $\mathbb{P}^a$ for each $n \geq 1$. This, together with the assumption that $\inf\{n : \tau_n = \infty\} < \infty$, implies that M has martingale representation on $\mathbb{R}_+$ under $\mathbb{P}^a$, and thus proves that $\mathbb{P}^a \in \mathcal{A}_{\text{MRP}}$.

Since $\tau_0 = 0$, there is nothing to prove in the case of $n = 0$. Assume the result holds on $[0, \tau_n]$. Apply Lemma 4.12 with $\tau = \tau_n$ and recall that in this case we can choose the $\tilde\tau$ to be τ_{n+1}. Hence $a_t = \sum_{i\geq 1} a_i(t) \mathbf{1}_{E_i}$, $t < \tau_{n+1}$, where $\{a_i, i \geq 1\} \subset \mathcal{A}_0$ and $\{E_i, i \geq 1\} \subset \mathcal{F}_{\tau_n}$ form a partition of Ω. For each $i \geq 1$, define

$$M_t^i := [M_{t\wedge\tau_{n+1}} - M_{\tau_n}] \mathbf{1}_{E_i} \mathbf{1}_{[\tau_n,\infty)}(t) \quad \text{for all} \quad t \geq 0.$$

Then one can directly check that M^i is a $\mathbb{P}^{a_i}$-local martingale. Since $a_i \in \mathcal{A}_0 \subset \mathcal{A}_{\text{MRP}}$, there exists H^i such that $dM_t^i = H_t^i dB_t$, $\mathbb{P}^{a_i}$-almost surely. Now define $H_t := \sum_{i\geq 1} H_t^i \mathbf{1}_{E_i}$, $\tau_n \leq t < \tau_{n+1}$. Then we have $dM_t = H_t dB_t$, $\tau_n \leq t < \tau_{n+1}$, $\mathbb{P}^a$-almost surely. $\Box$

We close this subsection by the following important example.

Example 4.14 Assume $\mathcal{A}_0$ consists of all deterministic functions $a : \mathbb{R}_+ \to \mathbb{S}_d^{>0}$ taking the form $a_t = \sum_{i=0}^{n-1} a_{t_i} \mathbf{1}_{[t_i, t_{i+1})} + a_{t_n} \mathbf{1}_{[t_n,\infty)}$ where $t_i \in \mathbb{Q}$ and a_{t_i} has rational entries. This is a special case of Example 4.9 and thus $\mathcal{A}_0 \subset \mathcal{A}_W$. In this case $\mathcal{A}_0$ is countable. Let $\mathcal{A}_0 = \{a_i\}_{i\geq 1}$ and define $\hat{\mathbb{P}} := \sum_{i=1}^\infty 2^{-i} \mathbb{P}^{a_i}$. Then $\hat{\mathbb{P}}$ is a dominating probability measure of all $\mathbb{P}^a$, $a \in \mathcal{A}$, where $\mathcal{A}$ is the separable class of diffusion coefficients generated by $\mathcal{A}_0$.

Therefore, $\mathcal{A}$-quasi surely is equivalent to $\hat{\mathbb{P}}$-almost surely. Notice however that $\mathcal{A}$ is not countable. $\square$

5 Quasi-sure aggregation

In this section, we fix

$$\text{a separable class } \mathcal{A} \text{ of diffusion coefficients generated by } \mathcal{A}_0 \tag{5.1}$$

and denote $\mathcal{P} := \{\mathbb{P}^a, a \in \mathcal{A}\}$. Then we prove the main aggregation result of this paper.

For this we recall that the notion of aggregation is defined in Definition 3.1 and the notations $\theta^{a,b}$ and $\Omega_{\hat{\tau}}^{a,b}$ are introduced in subsection 4.3.

Theorem 5.1 (Quasi sure aggregation) *For $\mathcal{A}$ satisfying (5.1), let $\{X^a, a \in \mathcal{A}\}$ be a family of $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable processes. Then there exists a unique ($\mathcal{P}-$q.s.) $\mathcal{P}$-aggregator X if and only if $\{X^a, a \in \mathcal{A}\}$ satisfies the consistency condition*

$$X^a = X^b, \ \mathbb{P}^a - \textit{almost surely on } [0, \theta^{a,b}) \quad \textit{for any } a \in \mathcal{A}_0 \textit{ and } b \in \mathcal{A}. \tag{5.2}$$

Moreover, if X^a is càdlàg $\mathbb{P}^a$-almost surely for all $a \in \mathcal{A}$, then we can choose a $\mathcal{P}$-q.s. càdlàg version of the $\mathcal{P}$-aggregator X.

We note that the consistency condition (5.2) is slightly different from the condition (3.4) before. The condition (5.2) is more natural in this framework and is more convenient to check in applications. Before the proof of the theorem, we first show that, for any $a, b \in \mathcal{A}$, the corresponding probability measures $\mathbb{P}^a$ and $\mathbb{P}^b$ agree as long as a and b agree.

Lemma 5.2 *For $\mathcal{A}$ satisfying (5.1) and $a, b \in \mathcal{A}$, $\theta^{a,b}$ is an $\mathbb{F}$-stopping time taking countably many values and*

$$\mathbb{P}^a(E \cap \Omega_{\hat{\tau}}^{a,b}) = \mathbb{P}^b(E \cap \Omega_{\hat{\tau}}^{a,b}) \quad \textit{for all} \quad \hat{\tau} \in \hat{\mathcal{T}}^{\mathcal{P}} \textit{ and } E \in \hat{\mathcal{F}}_{\hat{\tau}}^{\mathcal{P}}. \tag{5.3}$$

Proof. (i) We first show that $\theta^{a,b}$ is an $\mathbb{F}$-stopping time. Fix an arbitrary time t_0. In view of Lemma 4.12 with $\tau = t_0$, we assume without loss of generality that

$$a_t = \sum_{n \geq 1} a_n(t) \mathbf{1}_{E_n} \quad \text{and} \quad b_t = \sum_{n \geq 1} b_n(t) \mathbf{1}_{E_n} \quad \text{for all} \quad t < \tilde{\tau},$$

where $\tilde{\tau} > t_0$, $a_n, b_n \in \mathcal{A}_0$ and $\{E_n, n \geq 1\} \subset \mathcal{F}_{t_0}$ form a partition of Ω. Then

$$\{\theta^{a,b} \leq t_0\} = \bigcup_n \Big[\{\theta^{a_n, b_n} \leq t_0\} \cap E_n \Big].$$

By the constant disagreement times property of $\mathcal{A}_0$, θ^{a_n,b_n} is a constant. This implies that $\{\theta^{a_n,b_n} \le t_0\}$ is equal to either $\emptyset$ or Ω. Since $E_n \in \mathcal{F}_{t_0}$, we conclude that $\{\theta^{a,b} \le t_0\} \in \mathcal{F}_{t_0}$ for all $t_0 \ge 0$. That is, $\theta^{a,b}$ is an $\mathbb{F}$-stopping time.

(ii) We next show that $\theta^{a,b}$ takes only countable many values. In fact, by (i) we may now apply Lemma 4.12 with $\tau = \theta^{a,b}$. So we may write

$$a_t = \sum_{n\ge 1} \tilde{a}_n(t)\mathbf{1}_{\tilde{E}_n} \quad \text{and} \quad b_t = \sum_{n\ge 1} \tilde{b}_n(t)\mathbf{1}_{\tilde{E}_n} \quad \text{for all} \quad t < \tilde{\theta},$$

where $\tilde{\theta} > \theta^{a,b}$ or $\tilde{\theta} = \theta^{a,b} = \infty$, $\tilde{a}_n, \tilde{b}_n \in \mathcal{A}_0$, and $\{\tilde{E}_n, n \ge 1\} \subset \mathcal{F}_{\theta^{a,b}}$ form a partition of Ω. Then it is clear that $\theta^{a,b} = \theta^{\tilde{a}_n,\tilde{b}_n}$ on $\tilde{E}_n$, for all $n \ge 1$. For each n, by the constant disagreement times property of $\mathcal{A}_0$, $\theta^{\tilde{a}_n,\tilde{b}_n}$ is constant. Hence $\theta^{a,b}$ takes only countable many values.

(iii) We now prove (5.3). We first claim that,

$$E \cap \Omega^{a,b}_{\hat{\tau}} \in \Big[\mathcal{F}_{\theta^{a,b}} \vee \mathcal{N}^{\mathbb{P}^a}(\mathcal{F}_\infty)\Big] \quad \text{for any} \quad E \in \hat{\mathcal{F}}^{\mathcal{P}}_{\hat{\tau}}. \tag{5.4}$$

Indeed, for any $t \ge 0$,

$$\begin{aligned} E \cap \Omega^{a,b}_{\hat{\tau}} \cap \{\theta^{a,b} \le t\} &= E \cap \{\hat{\tau} < \theta^{a,b}\} \cap \{\theta^{a,b} \le t\} \\ &= \bigcup_{m\ge 1} \Big[E \cap \{\hat{\tau} < \theta^{a,b}\} \cap \{\hat{\tau} \le t - \frac{1}{m}\} \cap \{\theta^{a,b} \le t\}\Big]. \end{aligned}$$

By (i) above, $\{\theta^{a,b} \le t\} \in \mathcal{F}_t$. For each $m \ge 1$,

$$E \cap \{\hat{\tau} < \theta^{a,b}\} \cap \{\hat{\tau} \le t - \frac{1}{m}\} \in \hat{\mathcal{F}}^{\mathcal{P}}_{t-\frac{1}{m}} \subset \mathcal{F}^+_{t-\frac{1}{m}} \vee \mathcal{N}^{\mathbb{P}^a}(\mathcal{F}_\infty) \subset \mathcal{F}_t \vee \mathcal{N}^{\mathbb{P}^a}(\mathcal{F}_\infty),$$

and (5.4) follows.

By (5.4), there exist $E^{a,i}, E^{b,i} \in \mathcal{F}_{\theta^{a,b}}$, $i = 1, 2$, such that

$$E^{a,1} \subset E \cap \Omega^{a,b}_{\hat{\tau}} \subset E^{a,2}, \; E^{b,1} \subset E \cap \Omega^{a,b}_{\hat{\tau}} \subset E^{b,2}, \text{ and } \mathbb{P}^a(E^{a,2}\backslash E^{a,1}) = \mathbb{P}^b(E^{b,2}\backslash E^{b,1}) = 0.$$

Define $E^1 := E^{a,1} \cup E^{b,1}$ and $E^2 := E^{a,2} \cap E^{b,2}$, then

$$E^1, E^2 \in \mathcal{F}_{\theta^{a,b}}, \quad E^1 \subset E \subset E^2, \quad \text{and} \quad \mathbb{P}^a(E^2\backslash E^1) = \mathbb{P}^b(E^2\backslash E^1) = 0.$$

Thus $\mathbb{P}^a(E \cap \Omega^{a,b}_{\hat{\tau}}) = \mathbb{P}^a(E^2)$ and $\mathbb{P}^b(E \cap \Omega^{a,b}_{\hat{\tau}}) = \mathbb{P}^b(E^2)$. Finally, since $E^2 \in \mathcal{F}_{\theta^{a,b}}$, following the definition of $\mathbb{P}^a$ and $\mathbb{P}^b$, in particular the uniqueness of weak solution of (4.4) on the interval $[0, \theta^{a,b}]$, we conclude that $\mathbb{P}^a(E^2) = \mathbb{P}^b(E^2)$. This implies (5.3) immediately. $\square$

Remark 5.3 The property (5.3) is crucial for checking the consistency conditions in our aggregation result in Theorem 5.1. We note that (5.3) does not hold if we replace the completed $\sigma-$algebra $\mathcal{F}^a_\tau \cap \mathcal{F}^b_\tau$ with the augmented $\sigma-$algebra $\overline{\mathcal{F}}^a_\tau \cap \overline{\mathcal{F}}^b_\tau$. To see this, let $d = 1$, $a_t := 1$, $b_t := 1 + \mathbf{1}_{[1,\infty)}(t)$. In this case, $\theta^{a,b} = 1$. Let $\tau := 0$, $E := \Omega^a_1$. One can easily check that $\Omega^{a,b}_0 = \Omega$, $\mathbb{P}^a(E) = 1$, $\mathbb{P}^b(E) = 0$. This implies that $E \in \overline{\mathcal{F}}^a_0 \cap \overline{\mathcal{F}}^b_0$ and $E \subset \Omega^{a,b}_0$. However, $\mathbb{P}^a(E) = 1 \neq 0 = \mathbb{P}^b(E)$. See also Remark 2.3. □

Proof of Theorem 5.1. The uniqueness of $\mathcal{P}-$aggregator is immediate. By Lemma 5.2 and the uniqueness of weak solutions of (4.4) on $[0, \theta^{a,b}]$, we know $\mathbb{P}^a = \mathbb{P}^b$ on $\mathcal{F}_{\theta^{a,b}}$. Then the existence of the $\mathcal{P}$-aggregator obviously implies (5.2). We now assume that the condition (5.2) holds and prove the existence of the $\mathcal{P}$-aggregator.

We first claim that, without loss of generality, we may assume that X^a is càdlàg. Indeed, suppose that the theorem holds for càdlàg processes. Then we construct a $\mathcal{P}$-aggregator for a family $\{X^a, a \in \mathcal{A}\}$, not necessarily càdlàg, as follows:

- If $|X^a| \le R$ for some constant $R > 0$ and for all $a \in \mathcal{A}$, set $Y^a_t := \int_0^t X^a_s ds$. Then, the family $\{Y^a, a \in \mathcal{A}\}$ inherits the consistency condition (5.2). Since Y^a is continuous for every $a \in \mathcal{A}$, this family admits a $\mathcal{P}$-aggregator Y. Define $X_t := \overline{\lim}_{\varepsilon\to 0} \frac{1}{\varepsilon}[Y_{t+\varepsilon} - Y_t]$. Then one can verify directly that X satisfies all the requirements.

- In the general case, set $X^{R,a} := (-R) \vee X^a \wedge R$. By the previous arguments there exists $\mathcal{P}$-aggregator X^R of the family $\{X^{R,a}, a \in \mathcal{A}\}$ and it is immediate that $X := \overline{\lim}_{R\to\infty} X^R$ satisfies all the requirements.

We now assume that X^a is càdlàg, $\mathbb{P}^a$-almost surely for all $a \in \mathcal{A}$. In this case, the consistency condition (5.2) is equivalent to

$$X^a_t = X^b_t,\ 0 \le t < \theta^{a,b},\ \mathbb{P}^a\text{-almost surely} \quad \text{for any } a \in \mathcal{A}_0 \text{ and } b \in \mathcal{A}. \tag{5.5}$$

Step 1. We first introduce the following quotient sets of $\mathcal{A}_0$. For each t, and $a, b \in \mathcal{A}_0$, we say $a \sim_t b$ if $\Omega^{a,b}_t = \Omega$ (or, equivalently, the constant disagreement time $\theta^{a,b} \ge t$). Then $\sim_t$ is an equivalence relationship in $\mathcal{A}_0$. Thus one can form a partition of $\mathcal{A}_0$ based on $\sim_t$. Pick an element from each partition set to construct a quotient set $\mathcal{A}_0(t) \subset \mathcal{A}_0$. That is, for any $a \in \mathcal{A}_0$, there exists a unique $b \in \mathcal{A}_0(t)$ such that $\Omega^{a,b}_t = \Omega$. Recall the notation Ω^a_t defined in (4.9). By (4.11) and the constant disagreement times property of $\mathcal{A}_0$, we know that $\{\Omega^a_t, a \in \mathcal{A}_0(t)\}$ are disjoint.

Step 2. For fixed $t \in \mathbb{R}_+$, define

$$\xi_t(\omega) \quad := \sum_{a\in\mathcal{A}_0(t)} X^a_t(\omega)\mathbf{1}_{\Omega^a_t}(\omega) \quad \text{for all} \quad \omega \in \Omega. \tag{5.6}$$

The above uncountable sum is well defined because the sets $\{\Omega^a_t, a \in \mathcal{A}_0(t)\}$ are disjoint. In this step, we show that

$$\xi_t \text{ is } \hat{\mathcal{F}}^{\mathcal{P}}_t\text{-measurable} \quad \text{and} \quad \xi_t = X^a_t,\ \ \mathbb{P}^a\text{-almost surely for all } a \in \mathcal{A}. \tag{5.7}$$

We prove this claim in the following three sub-cases.
2.1. For each $a \in \mathcal{A}_0(t)$, by definition $\xi_t = X^a_t$ on Ω^a_t. Equivalently $\{\xi_t \neq X^a_t\} \subset (\Omega^a_t)^c$. Moreover, by (4.10), $\mathbb{P}^a((\Omega^a_t)^c) = 0$. Since $\Omega^a_t \in \mathcal{F}^+_t$ and $\mathcal{F}^a_t$ is complete under $\mathbb{P}^a$, ξ_t is $\mathcal{F}^a_t$-measurable and $\mathbb{P}^a(\xi_t = X^a_t) = 1$.
2.2. Also, for each $a \in \mathcal{A}_0$, there exists a unique $b \in \mathcal{A}_0(t)$ such that $a \sim_t b$. Then $\xi_t = X^b_t$ on Ω^b_t. Since $\Omega^{a,b}_t = \Omega$, it follows from Lemma 5.2 that $\mathbb{P}^a = \mathbb{P}^b$ on $\mathcal{F}^+_t$ and $\mathbb{P}^a(\Omega^b_t) = \mathbb{P}^b(\Omega^b_t) = 1$. Hence $\mathbb{P}^a(\xi_t = X^b_t) = 1$. Now by the same argument as in the first case, we can prove that ξ_t is $\mathcal{F}^a_t$-measurable. Moreover, by the consistency condition (5.8), $\mathbb{P}^a(X^a_t = X^b_t) = 1$. This implies that $\mathbb{P}^a(\xi_t = X^a_t) = 1$.
2.3. Now consider $a \in \mathcal{A}$. We apply Lemma 4.12 with $\tau = t$. This implies that there exist a sequence $\{a_j, j \geq 1\} \subset \mathcal{A}_0$ such that $\Omega = \cup_{j\geq 1}\Omega^{a,a_j}_t$. Then

$$\{\xi_t \neq X^a_t\} \quad = \quad \bigcup_{j\geq 1}\Big[\{\xi_t \neq X^a_t\} \cap \Omega^{a,a_j}_t\Big].$$

Now for each $j \geq 1$,

$$\{\xi_t \neq X^a_t\} \cap \Omega^{a,a_j}_t \subset \Big[\{\xi_t \neq X^{a_j}_t\} \cap \Omega^{a,a_j}_t\Big] \bigcup \Big[\{X^{a_j}_t \neq X^a_t\} \cap \Omega^{a,a_j}_t\Big].$$

Applying Lemma 5.2 and using the consistency condition (5.5), we obtain

$$\begin{aligned} \mathbb{P}^a\Big(\{X^{a_j}_t \neq X^a_t\} \cap \Omega^{a,a_j}_t\Big) &= \mathbb{P}^{a_j}\Big(\{X^{a_j}_t \neq X^a_t\} \cap \Omega^{a,a_j}_t\Big) \\ &= \mathbb{P}^{a_j}\Big(\{X^{a_j}_t \neq X^a_t\} \cap \{t < \theta^{a,a_j}\}\Big) = 0. \end{aligned}$$

Moreover, for $a_j \in \mathcal{A}_0$, by the previous sub-case, $\{\xi_t \neq X^{a_j}_t\} \in \mathcal{N}^{\mathbb{P}^{a_j}}(\mathcal{F}^+_t)$. Hence there exists $D \in \mathcal{F}^+_t$ such that $\mathbb{P}^{a_j}(D) = 0$ and $\{\xi_t \neq X^{a_j}_t\} \subset D$. Therefore

$$\{\xi_t \neq X^{a_j}_t\} \cap \Omega^{a,a_j}_t \subset D \cap \Omega^{a,a_j}_t \quad \text{and} \quad \mathbb{P}^a(D \cap \Omega^{a,a_j}_t) = \mathbb{P}^{a_j}(D \cap \Omega^{a,a_j}_t) = 0.$$

This means that $\{\xi_t \neq X^{a_j}_t\} \cap \Omega^{a,a_j}_t \in \mathcal{N}^{\mathbb{P}^a}(\mathcal{F}^+_t)$. All of these together imply that $\{\xi_t \neq X^a_t\} \in \mathcal{N}^{\mathbb{P}^a}(\mathcal{F}^+_t)$. Therefore, $\xi_t \in \mathcal{F}^a_t$ and $\mathbb{P}^a(\xi_t = X^a_t) = 1$.

Finally, since $\xi_t \in \mathcal{F}^a_t$ for all $a \in \mathcal{A}$, we conclude that $\xi_t \in \hat{\mathcal{F}}^{\mathcal{P}}_t$. This completes the proof of (5.7).
Step 3. For each $n \geq 1$, set $t^n_i := \frac{i}{n}, i \geq 0$ and define

$$X^{a,n} := X^a_0 \mathbf{1}_{\{0\}} + \sum_{i=1}^{\infty} X^a_{t^n_i}\mathbf{1}_{(t^n_{i-1},t^n_i]} \text{ for all } a \in \mathcal{A} \quad \text{and} \quad X^n := \xi_0 \mathbf{1}_{\{0\}} + \sum_{i=1}^{\infty} \xi_{t^n_i}\mathbf{1}_{(t^n_{i-1},t^n_i]},$$

where $\xi_{t^n_i}$ is defined by (5.6). Let $\hat{\mathbb{F}}^n := \{\hat{\mathcal{F}}^{\mathcal{P}}_{t+\frac{1}{n}}, t \geq 0\}$. By Step 2, $X^{a,n}, X^n$ are $\hat{\mathbb{F}}^n$-progressively measurable and $\mathbb{P}^a(X^n_t = X^{a,n}_t, t \geq 0) = 1$ for all $a \in \mathcal{A}$. We now define

$$X := \overline{\lim_{n\to\infty}} X^n.$$

Since $\hat{\mathbb{F}}^n$ is decreasing to $\hat{\mathbb{F}}^{\mathcal{P}}$ and $\hat{\mathbb{F}}^{\mathcal{P}}$ is right continuous, X is $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable. Moreover, for each $a \in \mathcal{A}$,

$$\{X_t = X^a_t, t \geq 0\} \bigcap \{X \text{ is càdlàg}\} \supseteq \Big[\bigcap_{n\geq 1} \{X^n_t = X^{a,n}_t, t \geq 0\}\Big] \bigcap \{X^a \text{ is càdlàg}\}.$$

Therefore $X = X^a$ and X is càdlàg, $\mathbb{P}^a$-almost surely for all $a \in \mathcal{A}$. In particular, X is càdlàg, $\mathcal{P}$-quasi surely. □

Let $\hat{\tau} \in \hat{\mathcal{T}}^{\mathcal{P}}$ and $\{\xi^a, a \in \mathcal{A}\}$ be a family of $\hat{\mathcal{F}}^{\mathcal{P}}_{\hat{\tau}}$-measurable random variables. We say an $\hat{\mathcal{F}}^{\mathcal{P}}_{\hat{\tau}}$-measurable random variable ξ is a $\mathcal{P}$-aggregator of the family $\{\xi^a, a \in \mathcal{A}\}$ if $\xi = \xi^a$, $\mathbb{P}^a$-almost surely for all $a \in \mathcal{A}$. Note that we may identify any $\hat{\mathcal{F}}^{\mathcal{P}}_{\hat{\tau}}$-measurable random variable ξ with the $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable process $X_t := \xi \mathbf{1}_{[\hat{\tau},\infty)}$. Then a direct consequence of Theorem 5.1 is the following.

Corollary 5.4 *Let $\mathcal{A}$ be satisfying* (5.1) *and $\hat{\tau} \in \hat{\mathcal{T}}^{\mathcal{P}}$. Then the family of $\hat{\mathcal{F}}^{\mathcal{P}}_{\hat{\tau}}$-measurable random variables $\{\xi^a, a \in \mathcal{A}\}$ has a unique ($\mathcal{P}$-q.s.) $\mathcal{P}$-aggregator ξ if and only if the following consistency condition holds:*

$$\xi^a = \xi^b \text{ on } \Omega^{a,b}_{\hat{\tau}}, \quad \mathbb{P}^a\text{-almost surely for any } a \in \mathcal{A}_0 \text{ and } b \in \mathcal{A}. \tag{5.8}$$

For the next result, we recall that the $\mathbb{P}$-Brownian motion $W^{\mathbb{P}}$ is defined in (4.2). As a direct consequence of Theorem 5.1, the following result defines the $\mathcal{P}$-Brownian motion.

Corollary 5.5 *For $\mathcal{A}$ satisfying* (5.1)*, the family $\{W^{\mathbb{P}^a}, a \in \mathcal{A}\}$ admits a unique $\mathcal{P}$-aggregator W. Since $W^{\mathbb{P}^a}$ is a $\mathbb{P}^a$-Brownian motion for every $a \in \mathcal{A}$, we call W a $\mathcal{P}$-universal Brownian motion.*

Proof. Let $a, b \in \mathcal{A}$. For each n, denote

$$\tau_n := \inf\Big\{t \geq 0 : \int_0^t |\hat{a}_s| ds \geq n\Big\} \wedge \theta^{a,b}.$$

Then $B_{\cdot\wedge\tau_n}$ is a $\mathbb{P}^b$-square integrable martingale. By standard construction of stochastic integral, see e.g. [11] Proposition 2.6, there exist $\mathbb{F}$-adapted simple processes $\beta^{b,m}$ such that

$$\lim_{m\to\infty} \mathbb{E}^{\mathbb{P}^b}\Big\{\int_0^{\tau_n} |\hat{a}_s^{\frac{1}{2}}(\beta^{b,m}_s - \hat{a}_s^{-\frac{1}{2}})|^2 ds\Big\} = 0. \tag{5.9}$$

Define the universal process

$$W^{b,m}_t := \int_0^t \beta^{b,m}_s dB_s.$$

Then

$$\lim_{m\to\infty} \mathbb{E}^{\mathbb{P}^b}\Big\{ \sup_{0\le t\le \tau_n} \Big|W^{b,m}_t - W^{\mathbb{P}^b}_t\Big|^2\Big\} = 0. \tag{5.10}$$

By Lemma 2.4, all the processes in (5.9) and (5.10) can be viewed as $\mathbb{F}$-adapted. Since $\tau_n \le \theta^{a,b}$, applying Lemma 5.2 we obtain from (5.9) and (5.10) that

$$\lim_{m\to\infty} \mathbb{E}^{\mathbb{P}^a}\Big\{ \int_0^{\tau_n} |\hat{a}_s^{\frac{1}{2}}(\beta^{b,m}_s - \hat{a}_s^{-\frac{1}{2}})|^2 ds\Big\} = 0, \quad \lim_{m\to\infty} \mathbb{E}^{\mathbb{P}^a}\Big\{ \sup_{0\le t\le \tau_n} \Big|W^{b,m}_t - W^{\mathbb{P}^b}_t\Big|^2\Big\} = 0.$$

The first limit above implies that

$$\lim_{m\to\infty} \mathbb{E}^{\mathbb{P}^a}\Big\{ \sup_{0\le t\le \tau_n} \Big|W^{b,m}_t - W^{\mathbb{P}^a}_t\Big|^2\Big\} = 0,$$

which, together with the second limit above, in turn leads to

$$W^{\mathbb{P}^a}_t = W^{\mathbb{P}^b}_t, \quad 0\le t\le \tau_n, \quad \mathbb{P}^a - \text{a.s.}$$

Clearly $\tau_n \uparrow \theta^{a,b}$ as $n\to\infty$. Then

$$W^{\mathbb{P}^a}_t = W^{\mathbb{P}^b}_t, \quad 0\le t< \theta^{a,b}, \quad \mathbb{P}^a - \text{a.s.}$$

That is, the family $\{W^{\mathbb{P}^a}, a\in\mathcal{A}\}$ satisfies the consistency condition (5.2). We then apply Theorem 5.1 directly to obtain the $\mathcal{P}-$aggregator W. □

The $\mathcal{P}-$Brownian motion W is our first example of a stochastic integral defined simultaneously under all $\mathbb{P}^a$, $a\in\mathcal{A}$:

$$W_t \;=\; \int_0^t \hat{a}_s^{-1/2} dB_s, \quad t\ge 0, \quad \mathcal{P}-\text{q.s.} \tag{5.11}$$

We will investigate in detail the universal integration in Section 6.

Remark 5.6 Although a and $W^{\mathbb{P}^a}$ are $\mathbb{F}$-progressively measurable, from Theorem 5.1 we can only deduce that $\hat{a}$ and W are $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable. On the other hand, if we take a version of $W^{\mathbb{P}^a}$ that is progressively measurable to the augmented filtration $\overline{\mathbb{F}}^a$, then in general the consistency condition (5.2) does not hold. For example, let $d=1$, $a_t := 1$, and $b_t := 1 + \mathbf{1}_{[1,\infty)}(t)$, $t\ge 0$, as in Remark 5.3. Set $W^{\mathbb{P}^a}_t(\omega) := B_t(\omega) + \mathbf{1}_{(\Omega^a_1)^c}(\omega)$ and $W^{\mathbb{P}^b}_t(\omega) := B_t(\omega) + [B_t(\omega) - B_1(\omega)]\mathbf{1}_{[1,\infty)}(t)$. Then both $W^{\mathbb{P}^a}$ and $W^{\mathbb{P}^b}$ are $\overline{\mathbb{F}}^a\cap\overline{\mathbb{F}}^b$-progressively measurable. However, $\theta^{a,b}=1$, but $\mathbb{P}^b(W^{\mathbb{P}^a}_0 = W^{\mathbb{P}^b}_0) = \mathbb{P}^b(\Omega^a_1) = 0$, so we do not have $W^{\mathbb{P}^a} = W^{\mathbb{P}^b}$, $\mathbb{P}^b$-almost surely on $[0,1]$. □

6 Quasi-sure stochastic analysis

In this section, we fix again a separable class $\mathcal{A}$ of diffusion coefficients generated by $\mathcal{A}_0$, and set $\mathcal{P} := \{\mathbb{P}^a : a \in \mathcal{A}\}$. We shall develop the $\mathcal{P}$-quasi sure stochastic analysis. We emphasize again that, when a probability measure $\mathbb{P} \in \mathcal{P}$ is fixed, by Lemma 2.4 there is no need to distinguish the filtrations $\mathbb{F}^+, \mathbb{F}^{\mathbb{P}}$, and $\overline{\mathbb{F}}^{\mathbb{P}}$.

We first introduce several spaces. Denote by $\mathbb{L}^0$ the collection of all $\hat{\mathcal{F}}^{\mathcal{P}}_\infty$-measurable random variables with appropriate dimension. For each $p \in [1, \infty]$ and $\mathbb{P} \in \mathcal{P}$, we denote by $\mathbb{L}^p(\mathbb{P})$ the corresponding $\mathbb{L}^p$ space under the measure $\mathbb{P}$ and

$$\hat{\mathbb{L}}^p \;:=\; \bigcap_{\mathbb{P}\in\mathcal{P}} \mathbb{L}^p(\mathbb{P}).$$

Similarly, $\mathbb{H}^0 := \mathbb{H}^0(\mathbb{R}^d)$ denotes the collection of all $\mathbb{R}^d$ valued $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable processes. $\mathbb{H}^p(\mathbb{P}^a)$ is the subset of all $H \in \mathbb{H}^0$ satisfying

$$\|H\|^p_{T,\mathbb{H}^p(\mathbb{P}^a)} := \mathbb{E}^{\mathbb{P}^a}\Big[\Big(\int_0^T |a_s^{1/2}H_s|^2 ds\Big)^{p/2}\Big] < \infty \quad \text{for all} \quad T > 0,$$

and $\mathbb{H}^2_{loc}(\mathbb{P}^a)$ is the subset of $\mathbb{H}^0$ whose elements satisfy $\int_0^T |a_s^{1/2}H_s|^2 ds < \infty$, $\mathbb{P}^a$-almost surely, for all $T \geq 0$. Finally, we define

$$\hat{\mathbb{H}}^p \;:=\; \bigcap_{\mathbb{P}\in\mathcal{P}} \mathbb{H}^p(\mathbb{P}) \quad \text{and} \quad \hat{\mathbb{H}}^2_{loc} \;:=\; \bigcap_{\mathbb{P}\in\mathcal{P}} \mathbb{H}^2_{loc}(\mathbb{P}).$$

The following two results are direct applications of Theorem 5.1. Similar results were also proved in [5, 6], see e.g. Theorem 2.1 in [5], Theorem 36 in [6] and the Kolmogorov criterion of Theorem 31 in [6].

Proposition 6.1 (Completeness) *Fix $p \geq 1$, and let $\mathcal{A}$ be satisfying* (5.1).
(i) *Let $(X_n)_n \subset \hat{\mathbb{L}}^p$ be a Cauchy sequence under each $\mathbb{P}^a$, $a \in \mathcal{A}$. Then there exists a unique random variable $X \in \hat{\mathbb{L}}^p$ such that $X_n \to X$ in $\mathbb{L}^p(\mathbb{P}^a, \hat{\mathcal{F}}^{\mathcal{P}}_\infty)$ for every $a \in \mathcal{A}$.*
(ii) *Let $(X_n)_n \subset \hat{\mathbb{H}}^p$ be a Cauchy sequence under the norm $\|\cdot\|_{T,\mathbb{H}^p(\mathbb{P}^a)}$ for all $T \geq 0$ and $a \in \mathcal{A}$. Then there exists a unique process $X \in \hat{\mathbb{H}}^p$ such that $X_n \to X$ under the norm $\|\cdot\|_{T,\mathbb{H}^p(\mathbb{P}^a)}$ for all $T \geq 0$ and $a \in \mathcal{A}$.*

Proof. (i) By the completeness of $\mathbb{L}^p(\mathbb{P}^a, \hat{\mathcal{F}}^{\mathcal{P}}_\infty)$, we may find $X^a \in \mathbb{L}^p(\mathbb{P}^a, \hat{\mathcal{F}}^{\mathcal{P}}_\infty)$ such that $X_n \to X^a$ in $\mathbb{L}^p(\mathbb{P}^a, \hat{\mathcal{F}}^{\mathcal{P}}_\infty)$. The consistency condition of Theorem 5.1 is obviously satisfied by the family $\{X^a, a \in \mathcal{A}\}$, and the result follows. (ii) can be proved by a similar argument. $\square$

Proposition 6.2 (Kolmogorov continuity criteria) *Let $\mathcal{A}$ be satisfying* (5.1)*, and X be an $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable process with values in $\mathbb{R}^n$. We further assume that for some $p > 1$, $X_t \in \hat{\mathbb{L}}^p$ for all $t \geq 0$ and satisfy*

$$\mathbb{E}^{\mathbb{P}^a}\left[|X_t - X_s|^p\right] \leq c_a|t-s|^{n+\varepsilon_a} \quad \textit{for some constants} \quad c_a, \varepsilon_a > 0.$$

Then X admits an $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable version $\tilde{X}$ which is Hölder continuous, $\mathcal{P}$-q.s. (with Hölder exponent $\alpha_a < \varepsilon_a/p$, $\mathbb{P}^a$-almost surely for every $a \in \mathcal{A}$).

Proof. We apply the Kolmogorov continuity criterion under each $\mathbb{P}^a$, $a \in \mathcal{A}$. This yields a family of $\overline{\mathbb{F}}^{\mathbb{P}^a}$-progressively measurable processes $\{X^a, a \in \mathcal{A}\}$ such that $X^a = X$, $\mathbb{P}^a$-almost surely, and X^a is Hölder continuous with Hölder exponent $\alpha_a < \varepsilon_a/p$, $\mathbb{P}^a$-almost surely for every $a \in \mathcal{A}$. Also in view of Lemma 2.4, we may assume without loss of generality that X^a is $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable for every $a \in \mathcal{A}$. Since each X^a is a $\mathbb{P}^a$-modification of X for every $a \in \mathcal{A}$, the consistency condition of Theorem 5.1 is immediately satisfied by the family $\{X^a, a \in \mathcal{A}\}$. Then, the aggregated process $\tilde{X}$ constructed in that theorem has the desired properties. □

Remark 6.3 The statements of Propositions 6.1 and 6.2 can be weakened further by allowing p to depend on a. □

We next construct the stochastic integral with respect to the canonical process B which is simultaneously defined under all the mutually singular measures $\mathbb{P}^a$, $a \in \mathcal{A}$. Such constructions have been given in the literature but under regularity assumptions on the integrand. Here we only place standard conditions on the integrand but not regularity.

Theorem 6.4 (Stochastic integration) *For $\mathcal{A}$ satisfying* (5.1)*, let $H \in \hat{\mathbb{H}}^2_{loc}$ be given. Then, there exists a unique ($\mathcal{P}$-q.s.) $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable process M such that M is a local martingale under each $\mathbb{P}^a$ and*

$$M_t = \int_0^t H_s dB_s, \ t \geq 0, \quad \mathbb{P}^a\textit{-almost surely for all} \quad a \in \mathcal{A}.$$

If in addition $H \in \hat{\mathbb{H}}^2$, then for every $a \in \mathcal{A}$, M is a square integrable $\mathbb{P}^a$-martingale. Moreover, $\mathbb{E}^{\mathbb{P}^a}[M_t^2] = \mathbb{E}^{\mathbb{P}^a}[\int_0^t |a_s^{1/2} H_s|^2 ds]$ for all $t \geq 0$.

Proof. For every $a \in \mathcal{A}$, the stochastic integral $M_t^a := \int_0^t H_s dB_s$ is well-defined $\mathbb{P}^a$-almost surely as a $\overline{\mathbb{F}}^{\mathbb{P}^a}$-progressively measurable process. By Lemma 2.4, we may assume without loss of generality that M^a is $\hat{\mathbb{F}}^{\mathcal{P}}$-adapted. Following the arguments in Corollary 5.5, in particular by applying Lemma 5.2, it is clear that the consistency condition (5.2) of Theorem 5.1 is satisfied by the family $\{M^a, a \in \mathcal{A}\}$. Hence, there exists an aggregating process

M. The remaining statements in the theorem follows from classical results for standard stochastic integration under each $\mathbb{P}^a$. $\square$

We next study the martingale representation.

Theorem 6.5 (Martingale representation) *Let $\mathcal{A}$ be a separable class of diffusion coefficients generated by $\mathcal{A}_0 \subset \mathcal{A}_{MRP}$. Let M be an $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable process which is a $\mathcal{P}-$quasi sure local martingale, that is, M is a local martingale under $\mathbb{P}$ for all $\mathbb{P} \in \mathcal{P}$. Then there exists a unique ($\mathcal{P}$-q.s.) process $H \in \hat{\mathbb{H}}^2_{loc}$ such that*

$$M_t = M_0 + \int_0^t H_s dB_s, \quad t \geq 0, \quad \mathcal{P} - q.s..$$

Proof. By Proposition 4.11, $\mathcal{A} \subset \mathcal{A}_{\text{MRP}}$. Then for each $\mathbb{P} \in \mathcal{P}$, all $\mathbb{P}-$martingales can be represented as stochastic integrals with respect to the canonical process. Hence, there exists unique ($\mathbb{P}-$almost surely) process $H^{\mathbb{P}} \in \mathbb{H}^2_{loc}(\mathbb{P})$ such that

$$M_t = M_0 + \int_0^t H^{\mathbb{P}}_s dB_s, \quad t \geq 0, \quad \mathbb{P}\text{-almost surely.}$$

Then the quadratic covariation under $\mathbb{P}^b$ satisfies

$$\langle M, B \rangle^{\mathbb{P}^b}_t = \int_0^t H^{\mathbb{P}}_s \hat{a}_s ds, \qquad t \geq 0, \quad \mathbb{P} - \text{almost surely.} \tag{6.1}$$

Now for any $a, b \in \mathcal{A}$, from the construction of quadratic covariation and that of Lebesgue integrals, following similar arguments as in Corollary 5.5 one can easily check that

$$\int_0^t H^{\mathbb{P}^a}_s \hat{a}_s ds = \langle M, B \rangle^{\mathbb{P}^a}_t = \langle M, B \rangle^{\mathbb{P}^b}_t = \int_0^t H^{\mathbb{P}^b}_s \hat{a}_s ds, \quad 0 \leq t < \theta^{a,b}, \quad \mathbb{P}^a - \text{almost surely.}$$

This implies that

$$H^{\mathbb{P}^a} \mathbf{1}_{[0,\theta^{a,b})} = H^{\mathbb{P}^b} \mathbf{1}_{[0,\theta^{a,b})}, \quad dt \times d\mathbb{P}^a - \text{almost surely.}$$

That is, the family $\{H^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ satisfies the consistency condition (5.2). Therefore, we may aggregate them into a process H. Then one may directly check that H satisfies all the requirements. $\square$

There is also $\mathcal{P}$-quasi sure decomposition of super-martingales.

Proposition 6.6 (Doob-Meyer decomposition) *For $\mathcal{A}$ satisfying (5.1), assume an $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable process X is a $\mathcal{P}$-quasi sure supermartingale, i.e., X is a $\mathbb{P}^a$-supermartingale for all $a \in \mathcal{A}$. Then there exist a unique ($\mathcal{P}$-q.s.) $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable processes M and K such that M is a $\mathcal{P}$-quasi sure local martingale and K is predictable and increasing, $\mathcal{P}$-q.s., with $M_0 = K_0 = 0$, and $X_t = X_0 + M_t - K_t$, $t \geq 0$, $\mathcal{P}$-quasi surely.*

If further X is in class (D), $\mathcal{P}$-quasi surely, i.e. the family $\{X_{\hat{\tau}}, \hat{\tau} \in \hat{\mathcal{T}}\}$ is $\mathbb{P}$-uniformly integrable, for all $\mathbb{P} \in \mathcal{P}$, then M is a $\mathcal{P}$-quasi surely uniformly integrable martingale.

Proof. For every $\mathbb{P} \in \mathcal{A}$, we apply Doob-Meyer decomposition theorem (see e.g. Dellacherie-Meyer [4] Theorem VII-12). Hence there exist a $\mathbb{P}$-local martingale $M^{\mathbb{P}}$ and a $\mathbb{P}$-almost surely increasing process $K^{\mathbb{P}}$ such that $M_0^{\mathbb{P}} = K_0^{\mathbb{P}} = 0$, $\mathbb{P}$-almost surely. The consistency condition of Theorem 5.1 follows from the uniqueness of the Doob-Meyer decomposition. Then, the aggregated processes provide the universal decomposition. □

The following results also follow from similar applications of our main result.

Proposition 6.7 (Itô's formula) *For $\mathcal{A}$ satisfying* (5.1)*, let A, H be $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable processes with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. Assume that A has finite variation over each time interval $[0,t]$ and $H \in \hat{\mathbb{H}}^2_{loc}$. For $t \geq 0$, set $X_t := A_t + \int_0^t H_s dB_s$. Then for any C^2 function $f : \mathbb{R} \to \mathbb{R}$, we have*

$$f(X_t) \;=\; f(A_0) + \int_0^t f'(X_s)(dA_s + H_s dB_s) + \frac{1}{2}\int_0^t H_s^{\mathrm{T}} \hat{a}_s H_s f''(X_s) ds, \;\; t \geq 0, \;\; \mathcal{P}\text{-q.s..}$$

Proof. Apply Itô's formula under each $\mathbb{P} \in \mathcal{P}$, and proceed as in the proof of Theorem 6.4. □

Proposition 6.8 (local time) *For $\mathcal{A}$ satisfying* (5.1)*, let A, H and X be as in Proposition 6.7. Then for any $x \in \mathbb{R}$, the local time $\{L_t^x, t \geq 0\}$ exists $\mathcal{P}$-quasi surely and is given by,*

$$2L_t^x \;=\; |X_t - x| - |X_0 - x| - \int_0^t \operatorname{sgn}(X_s - x)(dA_s + H_s dB_s), \;\; t \geq 0, \;\; \mathcal{P} - q.s..$$

Proof. Apply Tanaka's formula under each $\mathbb{P} \in \mathcal{P}$ and proceed as in the proof of Theorem 6.4. □

Following exactly as in the previous results, we obtain a Girsanov theorem in this context as well.

Proposition 6.9 (Girsanov) *For $\mathcal{A}$ satisfying* (5.1)*, let ϕ be $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable and $\int_0^t |\phi_s|^2 ds < \infty$ for all $t \geq 0$, $\mathcal{P}$-quasi surely. Let*

$$Z_t := \exp\left(\int_0^t \phi_s dW_s - \frac{1}{2}\int_0^t |\phi_s|^2 ds\right) \quad \text{and} \quad \tilde{W}_t := W_t - \int_0^t \phi_s ds, \;\; t \geq 0,$$

where W is the $\mathcal{P}$-Brownian motion of (5.11)*. Suppose that for each $\mathbb{P} \in \mathcal{P}$, $\mathbb{E}^{\mathbb{P}}[Z_T] = 1$ for some $T \geq 0$. On $\hat{\mathcal{F}}_T$ we define the probability measure $\mathbb{Q}^{\mathbb{P}}$ by $d\mathbb{Q}^{\mathbb{P}} = Z_T d\mathbb{P}$. Then,*

$$\mathbb{Q}^{\mathbb{P}} \circ \tilde{W}^{-1} = \mathbb{P} \circ W^{-1} \quad \text{for every} \quad \mathbb{P} \in \mathcal{P},$$

i.e. $\tilde{W}$ is a $\mathbb{Q}^{\mathbb{P}}$-Brownian motion on $[0, T,]$ for every $\mathbb{P} \in \mathcal{P}$.

We finally discuss stochastic differential equations in this framework. Set $\mathbf{Q}^m := \{(t, \mathbf{x}) : t \geq 0, \mathbf{x} \in C[0,t]^m\}$. Let b, σ be two functions from $\Omega \times \mathbf{Q}^m$ to $\mathbb{R}^m$ and $\mathcal{M}_{m,d}(\mathbb{R})$, respectively. Here, $\mathcal{M}_{m,d}(\mathbb{R})$ is the space of $m \times d$ matrices with real entries. We are interested in the problem of solving the following stochastic differential equation *simultaneously under all* $\mathbb{P} \in \mathcal{P}$,

$$X_t = X_0 + \int_0^t b_s(\underline{X}_s)ds + \int_0^t \sigma_s(\underline{X}_s)dB_s, \quad t \geq 0, \quad \mathcal{P} - \text{q.s.}, \tag{6.2}$$

where $\underline{X}_t := (X_s, s \leq t)$.

Proposition 6.10 *Let $\mathcal{A}$ be satisfying* (5.1)*, and assume that, for every $\mathbb{P} \in \mathcal{P}$ and $\tau \in \mathcal{T}$, the equation* (6.2) *has a unique $\mathbb{F}^{\mathbb{P}}$-progressively measurable strong solution on interval* $[0, \tau]$*. Then there is a $\mathcal{P}$-quasi surely aggregated solution to* (6.2)*.*

Proof. For each $\mathbb{P} \in \mathcal{A}$, there is a $\mathbb{P}$-solution $X^{\mathbb{P}}$ on $[0, \infty)$, which we may consider in its $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable version by Lemma 2.4. The uniqueness on each $[0, \tau]$,$\tau \in \mathcal{T}$ implies that the family $\{X^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ satisfies the consistency condition of Theorem 5.1. □

7 An application

As an application of our theory, we consider the problem of super-hedging contingent claims under volatility uncertainty, which was studied by Denis and Martini [5]. In contrast with their approach, our framework allows to obtain the existence of the optimal hedging strategy. However, this is achieved at the price of restricting the non-dominated family of probability measures.

We also mention a related recent paper by Fernholz and Karatzas [8] whose existence results are obtained in the Markov case with a continuity assumption on the corresponding value function.

Let $\mathcal{A}$ be a separable class of diffusion coefficients generated by $\mathcal{A}_0$, and $\mathcal{P} := \{\mathbb{P}^a : a \in \mathcal{A}\}$ be the corresponding family of measures. We consider a fixed time horizon, say $T = 1$. Clearly all the results in previous sections can be extended to this setting, after some obvious modifications. Fix a nonnegative $\hat{\mathcal{F}}_1-$measurable real-valued random variable ξ. The superhedging cost of ξ is defined by

$$v(\xi) \quad := \quad \inf \left\{ x : \; x + \int_0^1 H_s dB_s \geq \xi, \quad \mathcal{P}\text{-q.s. for some} \quad H \in \mathcal{H} \right\},$$

where the stochastic integral $\int_0^\cdot H_s dB_s$ is defined in the sense of Theorem 6.4 and $H \in \mathbb{H}^0$ belongs to $\mathcal{H}$ if and only if

$$\int_0^1 H_t^T \hat{a}_t H_t dt < \infty \; \mathcal{P}\text{-q.s.} \quad \text{and} \quad \int_0^\cdot H_s dB_s \text{ is a } \mathcal{P}\text{-q.s. supermartingale.}$$

We shall provide a dual formulation of the problem $v(\xi)$ in terms of the following dynamic optimization problem,

$$V_{\hat{\tau}}^{\mathbb{P}^a} \quad := \quad \underset{b\in\mathcal{A}(\hat{\tau},a)}{\text{ess sup}}^{\mathbb{P}^a}\, \mathbb{E}^{\mathbb{P}^b}[\xi|\hat{\mathcal{F}}_{\hat{\tau}}], \quad \mathbb{P}^a\text{-a.s.}, \quad a\in\mathcal{A},\ \hat{\tau}\in\hat{\mathcal{T}}, \tag{7.1}$$

where

$$\mathcal{A}(\hat{\tau},a) := \{b\in\mathcal{A} : \theta^{a,b} > \hat{\tau} \ \text{ or } \ \theta^{a,b} = \hat{\tau} = 1\}.$$

Theorem 7.1 *Let $\mathcal{A}$ be a separable class of diffusion coefficients generated by $\mathcal{A}_0 \subset \mathcal{A}_{MRP}$. Assume that the family of random variables $\{V_{\hat{\tau}}^{\mathbb{P}}, \hat{\tau}\in\hat{\mathcal{T}}\}$ is uniformly integrable under all $\mathbb{P}\in\mathcal{P}$. Then*

$$v(\xi) = V(\xi) \ := \ \sup_{a\in\mathcal{A}} \|V_0^{\mathbb{P}^a}\|_{\mathbb{L}^\infty(\mathbb{P}^a)}. \tag{7.2}$$

Moreover, if $v(\xi)<\infty$, then there exists $H\in\mathcal{H}$ such that $v(\xi)+\int_0^1 H_s dB_s \ge \xi$, $\mathcal{P}$-q.s..

To prove the theorem, we need the following (partial) dynamic programming principle.

Lemma 7.2 *Let $\mathcal{A}$ be satisfying* (5.1)*, and assume $V(\xi)<\infty$. Then, for any $\hat{\tau}_1,\hat{\tau}_2\in\hat{\mathcal{T}}$ with $\hat{\tau}_1\le\hat{\tau}_2$,*

$$V_{\hat{\tau}_1}^{\mathbb{P}^a} \ge \mathbb{E}^{\mathbb{P}^b}\big[V_{\hat{\tau}_2}^{\mathbb{P}^b}|\hat{\mathcal{F}}_{\hat{\tau}_1}\big], \mathbb{P}^a\textit{-almost surely for all } a\in\mathcal{A} \textit{ and } b\in\mathcal{A}(a,\hat{\tau}_1).$$

Proof. By the definition of essential supremum, see e.g. Neveu [12] (Proposition VI-1-1), there exist a sequence $\{b_j, j\ge 1\}\subset\mathcal{A}(b,\hat{\tau}_2)$ such that $V_{\hat{\tau}_2}^{\mathbb{P}^b} = \sup_{j\ge1}\mathbb{E}^{\mathbb{P}^{b_j}}[\xi|\hat{\mathcal{F}}_{\hat{\tau}_2}]$, $\mathbb{P}^b$-almost surely. For $n\ge1$, denote $V_{\hat{\tau}_2}^{b,n} := \sup_{1\le j\le n}\mathbb{E}^{\mathbb{P}^{b_j}}[\xi|\hat{\mathcal{F}}_{\hat{\tau}_2}]$. Then $V_{\hat{\tau}_2}^{b,n}\uparrow V_{\hat{\tau}_2}^{\mathbb{P}^b}$, $\mathbb{P}^b$-almost surely as $n\to\infty$. By the monotone convergence theorem, we also have $\mathbb{E}^{\mathbb{P}^b}[V_{\hat{\tau}_2}^{b,n}|\hat{\mathcal{F}}_{\hat{\tau}_1}]\uparrow \mathbb{E}^{\mathbb{P}^b}[V_{\hat{\tau}_2}^{\mathbb{P}^b}|\hat{\mathcal{F}}_{\hat{\tau}_1}]$, $\mathbb{P}^b$-almost surely, as $n\to\infty$. Since $b\in\mathcal{A}(a,\hat{\tau}_1)$, $\mathbb{P}^b=\mathbb{P}^a$ on $\hat{\mathcal{F}}_{\hat{\tau}_1}$. Then $\mathbb{E}^{\mathbb{P}^b}[V_{\hat{\tau}_2}^{b,n}|\hat{\mathcal{F}}_{\hat{\tau}_1}]\uparrow \mathbb{E}^{\mathbb{P}^b}[V_{\hat{\tau}_2}^{\mathbb{P}^b}|\hat{\mathcal{F}}_{\hat{\tau}_1}]$, $\mathbb{P}^a$-almost surely, as $n\to\infty$. Thus it suffices to show that

$$V_{\hat{\tau}_1}^{\mathbb{P}^a} \ge \mathbb{E}^{\mathbb{P}^b}[V_{\hat{\tau}_2}^{b,n}|\hat{\mathcal{F}}_{\hat{\tau}_1}], \qquad \mathbb{P}^a\text{-almost surely for all } n\ge 1. \tag{7.3}$$

Fix n and define

$$\theta_n^b := \min_{1\le j\le n}\theta^{b,b_j}.$$

By Lemma 5.2, θ^{b,b_j} are $\mathbb{F}$-stopping times taking only countably many values, then so is θ_n^b. Moreover, since $b_j\in\mathcal{A}(b,\hat{\tau}_2)$, we have either $\theta_n^b>\hat{\tau}_2$ or $\theta_n^b=\hat{\tau}_2=1$. Following exactly the same arguments as in the proof of (5.4), we arrive at

$$\hat{\mathcal{F}}_{\hat{\tau}_2}\subset\left(\mathcal{F}_{\theta_n^b}\vee\mathcal{N}^{\mathbb{P}^b}(\mathcal{F}_1)\right).$$

Since $\mathbb{P}^{b_j} = \mathbb{P}^b$ on $\hat{\mathcal{F}}_{\hat{\tau}_2}$, without loss of generality we may assume the random variables $\mathbb{E}^{\mathbb{P}^{b_j}}[\xi|\hat{\mathcal{F}}_{\hat{\tau}_2}]$ and $V^{b,n}_{\hat{\tau}_2}$ are $\mathcal{F}_{\theta^b_n}$-measurable. Set $A_j := \{\mathbb{E}^{\mathbb{P}^{b_j}}[\xi|\hat{\mathcal{F}}_{\hat{\tau}_2}] = V^{b,n}_{\hat{\tau}_2}\}$ and $\tilde{A}_1 := A_1$, $\tilde{A}_j := A_j \setminus \cup_{i<j} A_i$, $2 \le j \le n$. Then $\tilde{A}_1, \cdots, \tilde{A}_n$ are $\mathcal{F}_{\theta^b_n}$-measurable and form a partition of Ω. Now set

$$\tilde{b}(t) := b(t)\mathbf{1}_{[0,\hat{\tau}_2)}(t) + \sum_{j=1}^{n} b_j(t)\mathbf{1}_{\tilde{A}_j}\mathbf{1}_{[\hat{\tau}_2,1]}(t).$$

We claim that $\tilde{b} \in \mathcal{A}$. Equivalently, we need to show that $\tilde{b}$ takes the form (4.12). We know that b and b_j have the form

$$b(t) = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} b_i^{0,m}\mathbf{1}_{E_i^{0,m}}\mathbf{1}_{[\tau^0_m,\tau^0_{m+1})} \quad \text{and} \quad b_j(t) = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} b_i^{j,m}\mathbf{1}_{E_i^{j,m}}\mathbf{1}_{[\tau^j_m,\tau^j_{m+1})}$$

with the stopping times and sets as before. Since $b_j(t) = b(t)$ for $t \le \theta^b_n$ and $j = 1, \cdots, n$,

$$\begin{aligned}
\tilde{b}(t) &= b(t)\mathbf{1}_{[0,\theta^b_n)} + \sum_{j=1}^{n}\mathbf{1}_{\tilde{A}_j} b_j(t)\mathbf{1}_{[\theta^b_n,1]}(t) \\
&= \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} b_i^{0,m}\mathbf{1}_{E_i^{0,m}\cap\{\tau^0_m<\theta^b_n\}}\mathbf{1}_{[\tau^0_m\wedge\theta^b_n,\tau^0_{m+1}\wedge\theta^b_n)} \\
&\quad + \sum_{j=1}^{n}\sum_{m=0}^{\infty}\sum_{i=1}^{\infty} b_i^{j,m}\mathbf{1}_{E_i^{j,m}\cap\tilde{A}_j\cap\{\tau^j_{m+1}>\theta^b_n\}}\mathbf{1}_{[\tau^j_m\vee\theta^b_n,\tau^j_{m+1}\vee\theta^b_n)}.
\end{aligned}$$

By Definition 4.8, it is clear that $\tau^0_m \wedge \theta^b_n$ and $\tau^j_m \vee \theta^b_n$ are $\mathbb{F}$-stopping times and take only countably many values, for all $m \ge 0$ and $1 \le j \le n$. For $m \ge 0$ and $1 \le j \le n$, one can easily see that $E_i^{0,m} \cap \{\tau^0_m < \theta^b_n\}$ is $\mathcal{F}_{\tau^0_m\wedge\theta^b_n}$-measurable and that $E_i^{j,m} \cap \tilde{A}_j \cap \{\tau^j_{m+1} > \theta^b_n\}$ is $\mathcal{F}_{\tau^j_m\vee\theta^b_n}$-measurable. By ordering the stopping times $\tau^0_m \wedge \theta^b_n$ and $\tau^j_m \vee \theta^b_n$ we prove our claim that $\tilde{b} \in \mathcal{A}$.

It is now clear that $\tilde{b} \in \mathcal{A}(b, \hat{\tau}_2) \subset \mathcal{A}(a, \hat{\tau}_1)$. Thus,

$$\begin{aligned}
V^{\mathbb{P}^a}_{\hat{\tau}_1} &\ge \mathbb{E}^{\mathbb{P}^{\tilde{b}}}[\xi|\hat{\mathcal{F}}_{\hat{\tau}_1}] = \mathbb{E}^{\mathbb{P}^{\tilde{b}}}\Big[\mathbb{E}^{\mathbb{P}^{\tilde{b}}}[\xi|\hat{\mathcal{F}}_{\hat{\tau}_2}]\Big|\hat{\mathcal{F}}_{\hat{\tau}_1}\Big] \\
&= \mathbb{E}^{\mathbb{P}^{\tilde{b}}}\Big[\sum_{j=1}^{n}\mathbb{E}^{\mathbb{P}^{\tilde{b}}}[\xi\mathbf{1}_{\tilde{A}_j}|\hat{\mathcal{F}}_{\hat{\tau}_2}]\Big|\hat{\mathcal{F}}_{\hat{\tau}_1}\Big] \\
&= \mathbb{E}^{\mathbb{P}^{\tilde{b}}}\Big[\sum_{j=1}^{n}\mathbb{E}^{\mathbb{P}^{b_j}}[\xi\mathbf{1}_{\tilde{A}_j}|\hat{\mathcal{F}}_{\hat{\tau}_2}]\Big|\hat{\mathcal{F}}_{\hat{\tau}_1}\Big] \\
&= \mathbb{E}^{\mathbb{P}^{\tilde{b}}}\Big[\sum_{j=1}^{n}V^{b,n}_{\hat{\tau}_2}\mathbf{1}_{\tilde{A}_j}\Big|\hat{\mathcal{F}}_{\hat{\tau}_1}\Big] = \mathbb{E}^{\mathbb{P}^{\tilde{b}}}[V^{b,n}_{\hat{\tau}_2}\big|\hat{\mathcal{F}}_{\hat{\tau}_1}], \quad \mathbb{P}^a\text{-almost surely.}
\end{aligned}$$

Finally, since $\mathbb{P}^{\tilde{b}} = \mathbb{P}^b$ on $\hat{\mathcal{F}}_{\hat{\tau}_2}$ and $\mathbb{P}^b = \mathbb{P}^a$ on $\hat{\mathcal{F}}_{\hat{\tau}_1}$, we prove (7.3) and hence the lemma. □

Proof of Theorem 7.1. We first prove that $v(\xi) \geq V(\xi)$. If $v(\xi) = \infty$, then the inequality is obvious. If $v(\xi) < \infty$, there are $x \in \mathbb{R}$ and $H \in \mathcal{H}$ such that the process $X_t := x + \int_0^t H_s dB_s$ satisfies $X_1 \geq \xi$, $\mathcal{P}-$quasi surely. Notice that the process X is a $\mathbb{P}^b$-supermartingale for every $b \in \mathcal{A}$. Hence

$$x = X_0 \geq \mathbb{E}^{\mathbb{P}^b}[X_1|\hat{\mathcal{F}}_0] \geq \mathbb{E}^{\mathbb{P}^b}[\xi|\hat{\mathcal{F}}_0], \quad \mathbb{P}^b - \text{a.s.} \quad \forall\, b \in \mathcal{A}.$$

By Lemma 5.2, we know that $\mathbb{P}^a = \mathbb{P}^b$ on $\hat{\mathcal{F}}_0$ whenever $a \in \mathcal{A}$ and $b \in \mathcal{A}(0, a)$. Therefore,

$$x \geq \mathbb{E}^{\mathbb{P}^b}[\xi|\hat{\mathcal{F}}_0], \quad \mathbb{P}^a\text{-a.s.}.$$

The definition of $V^{\mathbb{P}^a}$ and the above inequality imply that $x \geq V_0^{\mathbb{P}^a}$, $\mathbb{P}^a$-almost surely. This implies that $x \geq \|V_0^{\mathbb{P}^a}\|_{\mathbb{L}^\infty(\mathbb{P}^a)}$ for all $a \in \mathcal{A}$. Therefore, $x \geq V(\xi)$. Since this holds for any initial data x that is super-replicating ξ, we conclude that $v(\xi) \geq V(\xi)$.

We next prove the opposite inequality. Again, we may assume that $V(\xi) < \infty$. Then $\xi \in \hat{\mathbb{L}}^1$. For each $\mathbb{P} \in \mathcal{P}$, by Lemma 7.2 the family $\{V_{\hat{\tau}}^{\mathbb{P}}, \hat{\tau} \in \hat{\mathcal{T}}\}$ satisfies the (partial) dynamic programming principle. Then following standard arguments (see e.g. [7] Appendix A2), we construct from this family a càdlàg $(\hat{\mathbb{F}}^{\mathcal{P}}, \mathbb{P})$-supermartingale $\hat{V}^{\mathbb{P}}$ defined by,

$$\hat{V}_t^{\mathbb{P}} := \overline{\lim_{\mathbb{Q} \ni r \downarrow t}} V_r^{\mathbb{P}}, \quad t \in [0,1]. \tag{7.4}$$

Also for each $\hat{\tau} \in \hat{\mathcal{T}}$, it is clear that the family $\{V_{\hat{\tau}}^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ satisfies the consistency condition (5.8). Then it follows immediately from (7.4) that $\{\hat{V}_t^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ satisfies the consistency condition (5.8) for all $t \in [0,1]$. Since $\mathbb{P}$-almost surely $\hat{V}^{\mathbb{P}}$ is càdlàg, the family of processes $\{\hat{V}^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ also satisfy the consistency condition (5.2). We then conclude from Theorem 5.1 that there exists a unique aggregating process $\hat{V}$.

Note that $\hat{V}$ is a $\mathcal{P}$-quasi sure supermartingale. Then it follows from the Doob-Meyer decomposition of Proposition 6.6 that there exist a $\mathcal{P}$-quasi sure local martingale M and a $\mathcal{P}$-quasi sure increasing process K such that $M_0 = K_0 = 0$ and $\hat{V}_t = \hat{V}_0 + M_t - K_t$, $t \in [0,1)$, $\mathcal{P}$-quasi surely. Using the uniform integrability hypothesis of this theorem, we conclude that the previous decomposition holds on $[0,1]$ and the process M is a $\mathcal{P}$-quasi sure martingale on $[0,1]$.

In view of the martingale representation Theorem 6.5, there exists an $\hat{\mathbb{F}}^{\mathcal{P}}$-progressively measurable process H such that $\int_0^1 H_t^T \hat{a}_t H_t dt < \infty$ and $\hat{V}_t = \hat{V}_0 + \int_0^t H_s dB_s - K_t$, $t \geq 0$, $\mathcal{P}$-quasi surely. Notice that $\hat{V}_1 = \xi$ and $K_1 \geq K_0 = 0$. Hence $\hat{V}_0 + \int_0^1 H_s dB_s \geq \xi$, $\mathcal{P}$-quasi surely. Moreover, by the definition of $V(\xi)$, it is clear that $V(\xi) \geq \hat{V}_0$, $\mathcal{P}$-quasi surely. Thus $V(\xi) + \int_0^1 H_s dB_s \geq \xi$, $\mathcal{P}$-quasi surely.

Finally, since ξ is nonnegative, $\hat{V} \geq 0$. Therefore,

$$V(\xi) + \int_0^t H_s dB_s \geq \hat{V}_0 + \int_0^t H_s dB_s \geq \hat{V}_t \geq 0, \qquad \mathcal{P} - q.s..$$

This implies that $H \in \mathcal{H}$, and thus $V(\xi) \geq v(\xi)$. $\square$

Remark 7.3 Denis and Martini [5] require

$$\underline{a} \leq a \leq \overline{a} \quad \text{for all} \quad a \in \mathcal{A}, \tag{7.5}$$

for some given constant matrices $\underline{a} \leq \overline{a}$ in $\mathbb{S}_d^{>0}$. We do not impose this constraint. In other words, we may allow $\underline{a} = 0$ and $\overline{a} = \infty$. Such a relaxation is important in problems of static hedging in finance, see e.g. [2] and the references therein. However, we still require that each $a \in \mathcal{A}$ takes values in $\mathbb{S}_d^{>0}$. $\square$

We shall introduce the set $\mathcal{A}_S \subset \mathcal{A}_{\text{MRP}}$ induced from strong formulation in Section 8. When $\mathcal{A}_0 \subset \mathcal{A}_S$, we have the following additional interesting properties.

Remark 7.4 If each $\mathbb{P} \in \mathcal{P}$ satisfies the Blumenthal zero-one law (e.g. if $\mathcal{A}_0 \subset \mathcal{A}_S$ by Lemma 8.2 below), then $V_0^{\mathbb{P}^a}$ is a constant for all $a \in \mathcal{A}$, and thus (7.2) becomes

$$v(\xi) = V(\xi) := \sup_{a \in \mathcal{A}} V_0^{\mathbb{P}^a}.$$

Remark 7.5 In general, the value $V(\xi)$ depends on $\mathcal{A}$, then so does $v(\xi)$. However, when ξ is uniformly continuous in ω under the uniform norm, we show in [16] that

$$\sup_{\mathbb{P} \in \overline{\mathcal{P}}_S} \mathbb{E}^{\mathbb{P}}[\xi] = \inf \left\{ x : \ x + \int_0^1 H_s dB_s \geq \xi, \quad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \overline{\mathcal{P}}_S, \quad \text{for some } H \in \overline{\mathcal{H}} \right\}, \tag{7.6}$$

and the optimal superhedging strategy H exists, where $\overline{\mathcal{H}}$ is the space of $\mathbb{F}$-progressively measurable H such that, for all $\mathbb{P} \in \overline{\mathcal{P}}_S$, $\int_0^1 H_t^T \hat{a}_t H_t dt < \infty$, $\mathbb{P}$-almost surely and $\int_0^\cdot H_s dB_s$ is a $\mathbb{P}$-supermartingale. Moreover, if $\mathcal{A} \subset \mathcal{A}_S$ is dense in some sense, then

$$V(\xi) = v(\xi) = \text{the } \overline{\mathcal{P}}_S\text{-superhedging cost in (7.6)}.$$

In particular, all functions are independent of the choice of $\mathcal{A}$. This issue is discussed in details in our accompanying paper [16] (Theorem 5.3 and Proposition 5.4), where we establish a duality result for a more general setting called the second order target problem. However, the set-up in [16] is more general and this independence can be proved by the above arguments under suitable assumptions. $\square$

8 Mutually singular measures induced by strong formulation

We recall the set $\overline{\mathcal{P}}_S$ introduced in the Introduction as

$$\overline{\mathcal{P}}_S := \left\{ \mathbb{P}_S^\alpha : \alpha \in \overline{\mathcal{A}} \right\} \quad \text{where} \quad \mathbb{P}_S^\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1}, \tag{8.7}$$

and X^α is given in (1.1). Clearly $\overline{\mathcal{P}}_S \subset \overline{\mathcal{P}}_W$. Although we do not use it in the present paper, this class is important both in theory and in applications. We remark that Denis-Martini [5] and our paper [15] consider the class $\overline{\mathcal{P}}_W$ while Denis-Hu-Peng [6] and our paper [17] consider the class $\overline{\mathcal{P}}_S$, up to some technical restriction of the diffusion coefficients.

We start the analysis of this set by noting that

$$\alpha \text{ is the quadratic variation density of } X^\alpha \text{ and } dB_s = \alpha_s^{-1/2} dX_s^\alpha, \text{ under } \mathbb{P}_0. \tag{8.8}$$

Since B under $\mathbb{P}_S^\alpha$ has the same distribution as X^α under $\mathbb{P}_0$, it is clear that

$$\text{the } \mathbb{P}_S^\alpha\text{-distribution of } (B, \hat{a}, W^{\mathbb{P}_S^\alpha}) \text{ is equal to the } \mathbb{P}_0\text{-distribution of } (X^\alpha, \alpha, B). \tag{8.9}$$

In particular, this implies that

$$\begin{gathered} \hat{a}(X^\alpha) = \alpha(B), \quad \mathbb{P}_0\text{-a.s.}, \quad \hat{a}(B) = \alpha(W^{\mathbb{P}_S^\alpha}), \quad \mathbb{P}_S^\alpha\text{-a.s.}, \\ \text{and for any } a \in \overline{\mathcal{A}}_W(\mathbb{P}_S^\alpha), \ X^\alpha \text{ is a strong solution to SDE (4.4) with coefficient } a. \end{gathered} \tag{8.10}$$

Moreover we have the following characterization of $\overline{\mathcal{P}}_S$ in terms of the filtrations.

Lemma 8.1 $\overline{\mathcal{P}}_S = \left\{ \mathbb{P} \in \overline{\mathcal{P}}_W : \overline{\mathbb{F}^{W^{\mathbb{P}}}}^{\mathbb{P}} = \overline{\mathbb{F}}^{\mathbb{P}} \right\}$.

Proof. By (8.8), α and B are $\overline{\mathbb{F}^{X^\alpha}}^{\mathbb{P}_0}$-progressively measurable. Since $\mathbb{F}$ is generated by B, we conclude that $\mathbb{F} \subset \overline{\mathbb{F}^{X^\alpha}}^{\mathbb{P}_0}$. By completing the filtration we next obtain that $\overline{\mathbb{F}}^{\mathbb{P}_0} \subset \overline{\mathbb{F}^{X^\alpha}}^{\mathbb{P}_0}$. Moreover, for any $\alpha \in \overline{\mathcal{A}}$, it is clear that $\mathbb{F}^{X^\alpha} \subset \overline{\mathbb{F}}^{\mathbb{P}_0}$. Thus, $\overline{\mathbb{F}^{X^\alpha}}^{\mathbb{P}_0} = \overline{\mathbb{F}}^{\mathbb{P}_0}$. Now, we invoke (8.9) and conclude $\overline{\mathbb{F}^{W^{\mathbb{P}}}}^{\mathbb{P}} = \overline{\mathbb{F}}^{\mathbb{P}}$ for any $\mathbb{P} = \mathbb{P}_S^\alpha \in \overline{\mathcal{P}}_S$.

Conversely, suppose $\mathbb{P} \in \overline{\mathcal{P}}_W$ be such that $\overline{\mathbb{F}^{W^{\mathbb{P}}}}^{\mathbb{P}} = \overline{\mathbb{F}}^{\mathbb{P}}$. Then $B = \beta(W_\cdot^{\mathbb{P}})$ for some measurable mapping $\beta : \mathbf{Q} \to \mathbb{S}_d^{>0}$. Set $\alpha := \beta(B.)$, we conclude that $\mathbb{P} = \mathbb{P}_S^\alpha$. □

The following result shows that the measures $\mathbb{P} \in \overline{\mathcal{P}}_S$ satisfy MRP and the Blumental zero-one law.

Lemma 8.2 *$\overline{\mathcal{P}}_S \subset \overline{\mathcal{P}}_{\mathit{MRP}}$ and every $\mathbb{P} \in \overline{\mathcal{P}}_S$ satisfies the Blumenthal zero-one law.*

Proof. Fix $\mathbb{P} \in \overline{\mathcal{P}}_S$. We first show that $\mathbb{P} \in \overline{\mathcal{P}}_{\mathrm{MRP}}$. Indeed, for any $(\overline{\mathbb{F}}^{\mathbb{P}}, \mathbb{P})$-local martingale M, Lemma 8.1 implies that M is a $(\overline{\mathbb{F}^{W^{\mathbb{P}}}}^{\mathbb{P}}, \mathbb{P})$-local martingale. Recall that $W^{\mathbb{P}}$ is a $\mathbb{P}$ Brownian motion. Hence, we now can use the standard martingale representation theorem. Therefore, there exists a unique $\overline{\mathbb{F}^{W^{\mathbb{P}}}}^{\mathbb{P}}$-progressively measurable process $\tilde{H}$ such that

$$\int_0^t |\tilde{H}_s|^2 ds < \infty \quad \text{and} \quad M_t = M_0 + \int_0^t \tilde{H}_s dW_s^{\mathbb{P}}, \quad t \geq 0, \quad \mathbb{P}\text{-a.s.}.$$

Since $\hat{a} > 0$, $dW^{\mathbb{P}} = \hat{a}^{-1/2} dB$. So one can check directly that the process $H := \hat{a}^{-1/2} \tilde{H}$ satisfies all the requirements.

We next prove the Blumenthal zero-one law. For this purpose fix $E \in \mathcal{F}_{0+}$. By Lemma 8.1, $E \in \overline{\mathcal{F}_0^{W^{\mathbb{P}}}}^{\mathbb{P}}$. Again we recall that $W^{\mathbb{P}}$ is a $\mathbb{P}$ Brownian motion and use the standard Blumenthal zero-one law for the Brownian motion. Hence $\mathbb{P}(E) \in \{0, 1\}$. $\square$

We now define analogously the following spaces of measures and diffusion processes.

$$\mathcal{P}_S := \overline{\mathcal{P}}_S \cap \mathcal{P}_W, \qquad \mathcal{A}_S := \{a \in \mathcal{A}_W : \mathbb{P}^a \in \mathcal{P}_S\}. \tag{8.11}$$

Then it is clear that

$$\mathcal{P}_S \subset \mathcal{P}_{\text{MRP}} \subset \mathcal{P}_W \quad \text{and} \quad \mathcal{A}_S \subset \mathcal{A}_{\text{MRP}} \subset \mathcal{A}_W.$$

The conclusion $\mathcal{P}_S \subset \mathcal{P}_W$ is strict, see Barlow [1]. We remark that one can easily check that the diffusion process a in Examples 4.4 and 4.5 and the generating class $\mathcal{A}_0$ in Examples 4.9, 4.10, and 4.14 are all in $\mathcal{A}_S$.

Our final result extends Proposition 4.11.

Proposition 8.3 *Let $\mathcal{A}$ be a separable class of diffusion coefficients generated by $\mathcal{A}_0$. If $\mathcal{A}_0 \subset \mathcal{A}_S$, then $\mathcal{A} \subset \mathcal{A}_S$.*

Proof. Let a be given in the form (4.12) and, by Proposition 4.11, $\mathbb{P}$ be the unique weak solution to SDE (4.4) on $[0, \infty)$ with coefficient a and initial condition $\mathbb{P}(B_0 = 0) = 1$. By Lemma 8.1 and its proof, it suffices to show that a is $\overline{\mathbb{F}^{W^{\mathbb{P}}}}^{\mathbb{P}}$-adapted. Recall (4.12). We prove by induction on n that

$$a_t \mathbf{1}_{\{t<\tau_n\}} \text{ is } \overline{\mathcal{F}_{t\wedge\tau_n}^{W^{\mathbb{P}}}}^{\mathbb{P}} - \text{measurable for all } t \geq 0. \tag{8.12}$$

Since $\tau_0 = 0$, a_0 is $\mathcal{F}_0$-measurable, and $\mathbb{P}(B_0 = 0) = 1$, (8.12) holds when $n = 0$. Assume (8.12) holds true for n. Now we consider $n + 1$. Note that

$$a_t \mathbf{1}_{\{t<\tau_{n+1}\}} = a_t \mathbf{1}_{\{t<\tau_n\}} + a_t \mathbf{1}_{\{\tau_n \leq t<\tau_{n+1}\}}.$$

By the induction assumption it suffices to show that

$$a_t \mathbf{1}_{\{t<\tau_{n+1}\}} \text{ is } \overline{\mathcal{F}_{\tau_n \vee t\wedge\tau_{n+1}}^{W^{\mathbb{P}}}}^{\mathbb{P}} - \text{measurable for all } t \geq 0. \tag{8.13}$$

Apply Lemma 4.12, we have $a_t = \sum_{m\geq 1} a_m(t)\mathbf{1}_{E_m}$ for $t < \tau_{n+1}$, where $a_m \in \mathcal{A}_0$ and $\{E_m, m \geq 1\} \subset \mathcal{F}_{\tau_n}$ form a partition of Ω. Let $\mathbb{P}^m$ denote the unique weak solution to SDE (4.4) on $[0, \infty)$ with coefficient a_m and initial condition $\mathbb{P}^m(B_0 = 0) = 1$. Then by Lemma 5.2 we have, for each $m \geq 1$,

$$\mathbb{P}(E \cap E_m) = \mathbb{P}^m(E \cap E_m), \quad \forall E \in \mathcal{F}_{\tau_{n+1}}. \tag{8.14}$$

Morover, by (4.2) it is clear that

$$W_t^{\mathbb{P}} = W_t^{\mathbb{P}^m}, \quad 0 \le t \le \tau_{n+1} \quad , \mathbb{P} - \text{a.s. on} \quad E_m \quad (\text{and} \quad \mathbb{P}^m - \text{a.s. on} \quad E_m). \tag{8.15}$$

Now since $a_m \in \mathcal{A}_0 \subset \mathcal{A}_S$, we know $a_m(t)\mathbf{1}_{\{t<\tau_{n+1}\}}$ is $\overline{\mathcal{F}^{W^{\mathbb{P}^m}}_{t\wedge\tau_{n+1}}}^{\mathbb{P}^m}$ $-$measurable. This, together with the fact that $E_m \in \mathcal{F}_{\tau_n}$, implies that $a_m(t)\mathbf{1}_{\{t<\tau_{n+1}\}}\mathbf{1}_{E_m}$ is $\overline{\mathcal{F}^{W^{\mathbb{P}^m}}_{\tau_n\vee t\wedge\tau_{n+1}}}^{\mathbb{P}^m}$ $-$measurable. By (8.14), (8.15) and that $a_t = a_m(t)$ for $t < \tau_{n+1}$ on E_m, we see that $a_t\mathbf{1}_{\{t<\tau_{n+1}\}}\mathbf{1}_{E_m}$ is $\overline{\mathcal{F}^{W^{\mathbb{P}}}_{\tau_n\vee t\wedge\tau_{n+1}}}^{\mathbb{P}}$ $-$measurable. Since m is arbitrary, we get

$$a_t\mathbf{1}_{\{t<\tau_{n+1}\}} = \sum_{m\ge 1} a_t\mathbf{1}_{\{t<\tau_{n+1}\}}\mathbf{1}_{E_m}$$

is $\overline{\mathcal{F}^{W^{\mathbb{P}}}_{\tau_n\vee t\wedge\tau_{n+1}}}^{\mathbb{P}}$ $-$measurable. This proves (8.13), and hence the proposition. □

9 Appendix

In this Appendix we provide a few more examples concerning weak solutions of (4.4) and complete the remaining technical proofs.

9.1 Examples

Example 9.1 (No weak solution) Let $a_0 = 1$, and for $t > 0$,

$$a_t := 1 + \mathbf{1}_E, \quad \text{where} \quad E := \Big\{ \overline{\lim_{h\downarrow 0}} \frac{B_h - B_0}{\sqrt{2h\ln\ln h^{-1}}} \neq 2 \Big\}.$$

Then $E \in \mathcal{F}_{0+}$. Assume $\mathbb{P}$ is a weak solution to (4.4). On E, $a = 2$, then $\overline{\lim}_{h\downarrow 0} \frac{B_h - B_0}{\sqrt{2h\ln\ln h^{-1}}} = 2$, $\mathbb{P}$-almost surely, thus $\mathbb{P}(E) = 0$. On E^c, $a = 1$, then $\overline{\lim}_{h\downarrow 0} \frac{B_h - B_0}{\sqrt{2h\ln\ln h^{-1}}} = 1$, $\mathbb{P}$-almost surely and thus $\mathbb{P}(E^c) = 0$. Hence there can not be any weak solutions. □

Example 9.2 (Martingale measure without Blumenthal 0-1 law) Let $\Omega' := \{1, 2\}$ and $\mathbb{P}'_0(1) = \mathbb{P}'_0(2) = \frac{1}{2}$. Let $\tilde{\Omega} := \Omega \times \Omega'$ and $\tilde{\mathbb{P}}_0$ the product of $\mathbb{P}_0$ and $\mathbb{P}'_0$. Define

$$\tilde{B}_t(\omega, 1) := \omega_t, \quad \tilde{B}_t(\omega, 2) := 2\omega_t.$$

Then $\tilde{\mathbb{P}} := \tilde{\mathbb{P}}_0 \circ (\tilde{B})^{-1}$ is in $\overline{\mathcal{P}}_W$. Denote

$$E := \left\{ \overline{\lim_{t\downarrow 0}} \frac{\tilde{B}_h - \tilde{B}_0}{\sqrt{2h\ln\ln h^{-1}}} = 1 \right\}.$$

Then $E \in \mathcal{F}^{\tilde{B}}_{0+}$, and $\tilde{\mathbb{P}}_0(E) = \mathbb{P}'_0(1) = \frac{1}{2}$. □

Example 9.3 (Martingale measure without MRP) Let $\tilde{\Omega} := (C[0,1])^2$, $(\tilde{W}, \tilde{W}')$ the canonical process, and $\tilde{\mathbb{P}}_0$ the Wiener measure so that $\tilde{W}$ and $\tilde{W}'$ are independent Brownian motions under $\tilde{\mathbb{P}}_0$. Let $\varphi : \mathbb{R} \to [0,1]$ be a measurable function, and

$$\tilde{B}_t := \int_0^t \tilde{\alpha}_s d\tilde{W}_s \quad \text{where} \quad \tilde{\alpha}_t := [1 + \varphi(\tilde{W}'_t)]^{\frac{1}{2}},\ t \geq 0,$$

This induces the following probability measure $\mathbb{P}$ on Ω with $d = 1$,

$$\mathbb{P} \ := \ \tilde{\mathbb{P}}_0 \circ \tilde{B}^{-1}.$$

Then $\mathbb{P}$ is a square integrable martingale measure with $d\langle B\rangle_t/dt \in [1,2]$, $\mathbb{P}$-almost surely.

We claim that B has no MRP under $\mathbb{P}$. Indeed, if B has MRP under $\mathbb{P}$, then so does $\tilde{B}$ under $\tilde{\mathbb{P}}_0$. Let $\tilde{\xi} := \mathbb{E}^{\tilde{\mathbb{P}}_0}[\tilde{W}'_1 | \mathcal{F}_1^{\tilde{B}}]$. Since $\tilde{\xi} \in \mathcal{F}_1^{\tilde{B}}$ and is obviously $\tilde{\mathbb{P}}_0$-square integrable, then there exists $\tilde{H}^a \in \mathcal{H}^2(\tilde{\mathbb{P}}_0, \mathbb{F}^{\tilde{B}})$ such that

$$\tilde{\xi} = \mathbb{E}^{\tilde{\mathbb{P}}_0}[\tilde{\xi}] + \int_0^1 \tilde{H}^a_t d\tilde{B}_t = \mathbb{E}^{\tilde{\mathbb{P}}_0}[\tilde{\xi}] + \int_0^1 \tilde{H}^a_t \tilde{\alpha}_t d\tilde{W}_t, \qquad \tilde{\mathbb{P}}_0 - a.s..$$

Since $\tilde{W}$ and $\tilde{W}'$ are independent under $\tilde{\mathbb{P}}_0$, we get $0 = \mathbb{E}^{\tilde{\mathbb{P}}_0}[\tilde{\xi}\tilde{W}'_1] = \mathbb{E}^{\tilde{\mathbb{P}}_0}[|\tilde{\xi}|^2]$. Then $\tilde{\xi} = 0$, $d\tilde{\mathbb{P}}_0$-almost surely, and thus

$$\mathbb{E}^{\tilde{\mathbb{P}}_0}[\tilde{W}'_1 |\tilde{B}_1|^2] \ = \ \mathbb{E}^{\tilde{\mathbb{P}}_0}[\tilde{\xi} |\tilde{B}_1|^2] \ = \ 0. \tag{9.1}$$

However, it follows from Itô's formula, together with the independence of W and W', that

$$\begin{aligned} \mathbb{E}^{\tilde{\mathbb{P}}_0}[\tilde{W}'_1 |\tilde{B}_1|^2] &= \mathbb{E}^{\tilde{\mathbb{P}}_0}\Big[\tilde{W}'_1 \int_0^1 2\tilde{B}_t \tilde{\alpha}_t d\tilde{W}_t\Big] + \mathbb{E}^{\tilde{\mathbb{P}}_0}\Big[\tilde{W}'_1 \int_0^1 \tilde{\alpha}_t^2 dt\Big] \\ &= \mathbb{E}^{\tilde{\mathbb{P}}_0}\Big[\int_0^1 \tilde{W}'_t\big(1 + \varphi(\tilde{W}'_t)\big) dt\Big] = \mathbb{E}^{\tilde{\mathbb{P}}_0}\Big\{\int_0^1 \tilde{W}'_t \varphi(\tilde{W}'_t) dt\Big\}, \end{aligned}$$

and we obtain a contradiction to (9.1) by observing that the latter expectation is non-zero for $\varphi(x) := \mathbf{1}_{\mathbb{R}_+}(x)$. □

We note that, however, we are not able to find a good example such that $a \in \mathcal{A}_W$ (so that (4.4) has unique weak solution) but B has no MRP under $\mathbb{P}^a$ (and consequently (4.4) has no strong solution).

9.2 Some technical proofs

Proof of Lemma 2.4. The uniqueness is obvious. We now prove the existence.
(i) Assume X is càdlàg, $\mathbb{P}$-almost surely. Let $E_0 := \{\omega : X_\cdot(\omega) \text{ is not càdlàg}\}$. For each $r \in \mathbb{Q} \cap (0, \infty)$, there exists $\tilde{X}_r \in \mathcal{F}_r^+$ such that $E_r := \{\tilde{X}_r \neq X_r\} \in \mathcal{N}^{\mathbb{P}}(\mathcal{F}_\infty)$. Let $E := E_0 \cup (\cup_r E_r)$. Then $\mathbb{P}(E) = 0$. For integers $n \geq 1$ $k \geq 0$, set $t^n_k := k/n$, and define

$$X^n_t := \tilde{X}_{t^n_{k+1}} \text{ for } t \in \big(t^n_k, t^n_{k+1}\big], \quad \text{and} \quad \tilde{X} := (\overline{\lim_{n\to\infty}} X^n)\mathbf{1}_{\{\overline{\lim}_{n\to\infty} X^n \in \mathbb{R}\}}.$$

Then for any $t \in (t_k^n, t_{k+1}^n]$, $X_t^n \in \mathcal{F}^+_{t_{k+1}^n}$ and $X^n|_{[0,t]} \in \mathcal{B}([0,t]) \times \mathcal{F}^+_{t_{k+1}^n}$. Since $\mathbb{F}^+$ is right continuous, we get $\tilde{X}_t \in \mathcal{F}_t^+$ and $\tilde{X}|_{[0,t]} \in \mathcal{B}([0,t]) \times \mathcal{F}_t^+$. That is, $\tilde{X} \in \mathbb{F}^+$. Moreover, for any $\omega \notin E$ and $n \geq 1$, if $t \in (t_k^n, t_{k+1}^n]$, we get

$$\lim_{n\to\infty} X_t^n(\omega) = \lim_{n\to\infty} \tilde{X}_{t_{k+1}^n}(\omega) = \lim_{n\to\infty} X_{t_{k+1}^n}(\omega) = X_t(\omega).$$

So $\{\omega :$ there exists $t \geq 0$ such that $\tilde{X}_t(\omega) \neq X_t(\omega)\} \subset E$. Then, $\tilde{X}$ is $\mathbb{P}$-indistinguishable from X and thus $\tilde{X}$ also has càdlàg paths, $\mathbb{P}$-almost surely.

(ii) Assume X is $\overline{\mathbb{F}}^{\mathbb{P}}$-progressively measurable and is bounded. Let $Y_t := \int_0^t X_s ds$. Then Y is continuous. By (i), there exists $\mathbb{F}^+$-progressively measurable continuous process $\tilde{Y}$ such that $\tilde{Y}$ and Y are $\mathbb{P}$-indistinguishable. Let $E_0 := \{$there exists $t \geq 0$ such that $\tilde{Y}_t \neq Y_t\}$, then $\mathbb{P}(E_0) = 0$ and $\tilde{Y}.(\omega)$ is continuous for each $\omega \notin E_0$. Define,

$$X_t^n := n[\tilde{Y}_t - \tilde{Y}_{t-\frac{1}{n}}]; \quad \tilde{X} := (\overline{\lim_{n\to\infty}} X^n)\mathbf{1}_{\{\overline{\lim}_{n\to\infty} X^n \in \mathbb{R}\}} \quad \text{for} \quad n \geq 1.$$

As in (i), we see $\tilde{X} \in \mathbb{F}^+$. Moreover, for each $\omega \notin E_0$, $X_t^n(\omega) = n\int_{t-\frac{1}{n}}^t X_s(\omega)ds$. Then $\tilde{X}.(\omega) = X.(\omega), dt$-almost surely. Therefore, $\tilde{X} = X$, $\mathbb{P}$-almost surely.

(iii) For general $\overline{\mathbb{F}}^{\mathbb{P}}$-progressively measurable X, let $X_t^m := (-m) \vee (X \wedge m)$, for any $m \geq 1$. By (ii), X^m has an $\mathbb{F}^+$-adapted modification $\tilde{X}^m$. Then obviously the following process $\tilde{X}$ satisfies all the requirements: $\tilde{X} := (\overline{\lim}_{m\to\infty} \tilde{X}^m)\mathbf{1}_{\{\overline{\lim}_{m\to\infty} \tilde{X}^m \in \mathbb{R}\}}$. □

To prove Example 4.5, we need a simple lemma.

Lemma 9.4 *Let τ be an $\mathbb{F}$-stopping time and X is an $\mathbb{F}$-progressively measurable process. Then $\tau(X.)$ is also an $\mathbb{F}-$stopping time.*

Moreover, if Y is $\mathbb{F}$-progressively measurable and $Y_t = X_t$ for all $t \leq \tau(X.)$, then $\tau(Y.) = \tau(X.)$.

Proof. Since τ is an $\mathbb{F}$-stopping time, we have $\{\tau(X.) \leq t\} \in \mathcal{F}_t^X$ for all $t \geq 0$. Moreover, since X is $\mathbb{F}$-progressively measurable, we know $\mathcal{F}_t^X \subset \mathcal{F}_t^B$. Then $\{\tau(X.) \leq t\} \in \mathcal{F}_t^B$ and thus $\tau(X.)$ is an $\mathbb{F}-$stopping time.

Now assume $Y_t = X_t$ for all $t \leq \tau(X.)$. For any $t \geq 0$, on $\{\tau(X.) = t\}$, we have $Y_s = X_s$ for all $s \leq t$. Since $\{\tau(X.) = t\} \in \mathcal{F}_t^X$ and by definition $\mathcal{F}_t^X = \sigma(X_s, s \leq t\}$, then $\tau(Y.) = t$ on the event $\{\tau(X.) = t\}$. Therefore, $\tau(Y.) = \tau(X.)$. □

Proof of Example 4.5. Without loss of generality we prove only that (4.4) on $\mathbb{R}_+$ with $X_0 = 0$ has a unique strong solution. In this case the stochastic differential equation becomes

$$dX_t = \sum_{n=0}^{\infty} a_n(X.)\mathbf{1}_{[\tau_n(X.),\tau_{n+1}(X.))} dB_t, \quad t \geq 0, \quad \mathbb{P}_0 - a.s..$$

We prove the result by induction on n. Let X^0 be the solution to SDE:

$$X_t^0 = \int_0^t a_0^{1/2}(X_\cdot^0)dB_s, \quad t \geq 0 \quad, \mathbb{P}_0 - \text{almost surely}$$

Note that a_0 is a constant, thus $X_t^0 = a_0^{\frac{1}{2}} B_t$ and is unique. Denote $\tilde{\tau}_0 := 0$ and $\tilde{\tau}_1 := \tau_1(X_\cdot^0)$. By Lemma 9.4, $\tilde{\tau}_1$ is an $\mathbb{F}-$stopping time. Now let $X_t^1 := X_t^0$ for $t \leq \tilde{\tau}_1$, and

$$X_t^1 = X_{\tilde{\tau}_1}^0 + \int_{\tilde{\tau}_1}^t a_1^{1/2}(X_\cdot^1)dB_s, \quad t \geq \tilde{\tau}_1, \quad \mathbb{P}_0 - \text{a.s.}$$

Note that $a_1 \in \mathcal{F}_{\tau_1}$, that is, for any $y \in \mathbb{R}$ and $t \geq 0$, $\{a_1(B_\cdot) \leq y, \tau_1(B_\cdot) \leq t\} \in \mathcal{F}_t$. Thus, for any $\mathbf{x}, \tilde{\mathbf{x}} \in C(\mathbb{R}_+, \mathbb{R}^d)$, if $\mathbf{x}_s = \tilde{\mathbf{x}}_s, 0 \leq s \leq t$, then $a_1(\mathbf{x})\mathbf{1}_{\{\tau_1(\mathbf{x}) \leq t\}} = a_1(\tilde{\mathbf{x}})\mathbf{1}_{\{\tau_1(\tilde{\mathbf{x}}) \leq t\}}$. In particular, noting that $\tau_1(X_\cdot^1) = \tau_1(X_\cdot^0) = \tilde{\tau}_1$, for each ω by choosing $t = \tilde{\tau}$ we obtain that $a_1(X_\cdot^1) = a_1(X_\cdot^0)$. Thus $X_t^1 = X_{\tilde{\tau}_1}^0 + a_1(X_\cdot^0)[B_t - B_{\tilde{\tau}_1}]$, $t \geq \tilde{\tau}_1$, and is unique. Now repeat the procedure for $n = 1, 2, \cdots$ we obtain the unique strong solution X in $[0, \tilde{\tau}_\infty)$, where $\tilde{\tau}_\infty := \lim_{n\to\infty} \tau_n(X_\cdot)$. Since a is bounded, it is obvious that $X_{\tilde{\tau}_\infty} := \lim_{t\uparrow\tilde{\tau}_\infty} X_t$ exists $\mathbb{P}_0$-almost surely. Then, by setting $X_t := X_{\tilde{\tau}_\infty}$ for $t \in (\tilde{\tau}_\infty, \infty)$ we complete the construction. □

Proof of Lemma 4.12. Let a be given as in (4.12) and $\tau \in \mathcal{T}$ be fixed. First, since $\{E_i^n, i \geq 1\}$ is a partition of Ω, then for any $n \geq 0$,

$$\left\{\cap_{j=0}^n E_{i_j}^j, \quad (i_j)_{0\leq j\leq n} \in \mathbb{N}^{n+1}\right\} \quad \text{also form a partition of } \Omega.$$

Next, assume τ_n takes values t_k^n (possibly including the value ∞), $k \geq 1$. Then $\{\{\tau_n = t_k^n\}, k \geq 1\}$ form a partition of Ω. Similarly we have, for any $n \geq 0$,

$$\left\{\cap_{j=0}^{n+1}\{\tau_j = t_{k_j}^j\}, \ (k_j)_{0\leq j\leq n+1} \in \mathbb{N}^{n+2}\right\} \quad \text{form a partition of} \quad \Omega.$$

These in turn form another partition of Ω given by,

$$\left\{\left[\cap_{j=0}^n \left(E_{i_j}^j \cap \{\tau_j = t_{k_j}^j\}\right)\right] \bigcap \{\tau_{n+1} = t_{k_{n+1}}^{n+1}\}, \quad (i_j, k_j)_{0\leq j\leq n} \in \mathbb{N}^{2(n+1)}, \ k_{n+1} \in \mathbb{N}\right\}. \tag{9.2}$$

Denote by $\mathcal{I}$ the family of all finite sequence of indexes $I := (i_j, k_j)_{0\leq j\leq n}$ for some n such that $0 = t_{k_0}^0 < \cdots < t_{k_n}^n < \infty$. Then $\mathcal{I}$ is countable. For each $I \in \mathcal{I}$, denote by $|I|$ the corresponding n, and define

$$\begin{aligned}
E_I &:= \left(\cap_{j=0}^{|I|}\left[E_{i_j}^j \cap \{\tau_j = t_{k_j}^j \leq \tau\}\right]\right) \bigcap \left(\{\tau_{|I|+1} > \tau\} \cup \{\tau_{|I|+1} = \tau = \infty\}\right), \\
\tilde{\tau} &:= \sum_{I\in\mathcal{I}} \tau_{|I|+1}\mathbf{1}_{E_I}, \quad \text{and} \quad a_I := \sum_{j=0}^{|I|-1} a_{i_j}^j \mathbf{1}_{[t_{k_j}^j, t_{k_{j+1}}^{j+1})} + a_{i_{|I|}}^{|I|}\mathbf{1}_{[t_{k_{|I|}}^{|I|}, \infty)}.
\end{aligned}$$

It is clear that E_I is $\mathcal{F}_\tau-$measurable. Then, in view of the concatenation property of $\mathcal{A}_0$, $a_I \in \mathcal{A}_0$. In light of (9.2), we see that $\{E_I, I \in \mathcal{I}\}$ are disjoint. Moreover, since $\tau_n = \infty$ for

n large enough, we know $\{E_I, I \in \mathcal{I}\}$ form a partition of Ω. Then $\tilde{\tau}$ is an $\mathbb{F}-$stopping time and either $\tilde{\tau} > \tau$ or $\tilde{\tau} = \tau = \infty$. We now show that

$$a_t = \sum_{I \in \mathcal{I}} a_I(t) \mathbf{1}_{E_I} \quad \text{for all} \quad t < \tilde{\tau}. \tag{9.3}$$

In fact, for each $I = (i_j, k_j)_{0 \le j \le n} \in \mathcal{I}$, $\omega \in E_I$, and $t < \tilde{\tau}(\omega)$, we have $\tau_j(\omega) = t^j_{k_j} \le \tau(\omega)$ for $j \le n$ and $\tau_{n+1}(\omega) = \tilde{\tau}(\omega) > t$. Let $j_0 = j_0(t,\omega) \le n$ be such that $\tau_{j_0}(\omega) \le t < \tau_{j_0+1}(\omega)$. Then $\mathbf{1}_{[\tau_{j_0}(\omega), \tau_{j_0+1}(\omega))}(t) = 1$ and $\mathbf{1}_{[\tau_j(\omega), \tau_{j+1}(\omega))}(t) = 0$ for $j \neq j_0$, and thus

$$a_t(\omega) \;=\; \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} a^j_i(t,\omega) \mathbf{1}_{E^j_i}(\omega) \mathbf{1}_{[\tau_j(\omega), \tau_{j+1}(\omega))}(t) = \sum_{i=1}^{\infty} a^{j_0}_i(t,\omega) \mathbf{1}_{E^{j_0}_i}(\omega) = a^{j_0}_{i_{j_0}}(t,\omega),$$

where the last equality is due to the fact that $\omega \in E_I \subset E^{j_0}_{i_{j_0}}$ and that $\{E^{j_0}_i, i \ge 1\}$ is a partition of Ω. On the other hand, by the definition of a_I, it is also straightforward to check that $a_I(t,\omega) = a^{j_0}_{i_{j_0}}(t,\omega)$. This proves (9.3). Now since $\mathcal{I}$ is countable, by numerating the elements of $\mathcal{I}$ we prove the lemma.

Finally, we should point out that, if $\tau = \tau_n$, then we can choose $\tilde{\tau} = \tau_{n+1}$. $\square$

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