The problem of super-replication under constraints

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Abstract

These notes present an overview of the problem of super-replication under portfolio constraints. We start by examining the duality approach and its limitations. We then concentrate on the direct approach in the Markov case which allows to handle general large investor problems and gamma constraints. In the context of the Black and Scholes model, the main result from the practical view-point is the so-called *face-lifting* phenomenon of the payoff function.

Key words: Super-replication, duality, dynamic programming, Hamilton-Jacobi-Bellman equation, viscosity solutions.

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1 Introduction

There is a large literature on the problem of super-replication in finance, i.e. the minimal initial capital which allows to hedge some given contingent claim at some terminal time T. Put in a stochastic control terms, the value function of the super-replication problem is the minimal initial data of some controlled process (the wealth process) which allows to hit some given target at time T. This stochastic control problem does not fit in the class of *standard* problems as presented in the usual textbooks, see e.g. [13]. This may explain the important attraction that this problem had on mathematicians.

In its simplest form, this problem is an alternative formulation of the Black and Scholes theory in terms of a stochastic control problem. The Black and Scholes solution appears naturally as a (degenerate) dual formulation of the super-replication problem. However, real financial markets are subject to constraints. The most popular example is the case of incomplete markets which was studied by Harrisson and Kreps (1979), and developed further by ElKaroui and Quenez (1995) in the diffusion case. The effect of the no short-selling constraint has been studied by Jouini and Kallal (1995). The dual formulation in the general convex constraints framework has been obtained by Cvitanić and Karatzas (1993) in the diffusion case and further extended by Föllmer and Kramkov (1997) to the general semimartingale case.

In the general constrained portfolio case, the above-mentioned dual formulation does not close the problem : except the complete market case, it provides an alternative stochastic control problem. The good news is that this problem is formulated in standard form. But there is still some specific complications since the controls are valued in unbounded sets. We are then in the context of singular control problems which typically exhibit a *jump* in the terminal condition. The main point is to characterize precisely this *face-lifting* phenomenon.

However, the duality approach has not been successful to solve more general super-replication problems. Namely, the dual formulation of the general large investor problem is still open. The same comment prevails for the super-replication problem under *gamma* constraints, i.e. constraints on the unbounded variation part of the portfolio. We provide a treatment of these problems which avoids the passage from the dual formulation. The key-point is an original dynamic programming principle stated directly on the initial formulation of the super-replication problem.

Further implications of this new dynamic programming principle are reported in our accompanying paper [22]. These notes are organized as follows.

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2 Problem formulation

2.1 The financial market

Given a finite time horizon $T > 0$, we shall consider throughout these notes a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a standard Brownian motion $B = \{(B_1(t), \ldots, B_d(t)), 0 \leq t \leq T\}$ valued in $I\!R^d$, and generating the ($P-$augmentation of the) filtration $I\!F$. We denote by ℓ the Lebesgue measure on $[0, T]$.

The financial market consists of a non-risky asset S^0 normalized to unity, i.e. $S^0 \equiv 1$, and d risky assets with price process $S = (S^1, \ldots, S^d)$ whose dynamics is defined by a stochastic differential equation. More specifically, given a vector process μ valued in $I\!R^n$, and a matrix-valued process σ valued in $I\!R^{n\times n}$, the price process S^i is defined as the unique strong solution of the stochastic differential equation :

$$S^i(0) \ = \ s^i \ , \qquad dS^i(t) \ = \ S^i(t) \left[b^i(t)dt + \sum_{j=1}^{d} \sigma^{ij}(t)dB^j(t) \right] \ ; \qquad (2.1)$$

here b and σ are assumed to be bounded $I\!F-$adapted processes.

Remark 2.1 The normalization of the non-risky asset to unity is, as usual, obtained by discounting, i.e. taking the non-risky asset as a *numéraire*.

In the financial literature, σ is known as the *volatility* process. We assume it to be invertible so that the *risk premium* process

$$\lambda_0(t) \quad := \quad \sigma(t)^{-1}b(t) \ , \qquad 0 \le t \le T \ ,$$

is well-defined. Throughout these notes, we shall make use of the process

$$Z_0(t) \quad = \quad \mathcal{E}\left(-\int_0^t \lambda_0(r)'dB(r)\right) \quad := \quad \exp\left(-\int_0^t \lambda_0(r)'dB(r) - \frac{1}{2}\int_0^t |\lambda_0(r)|^2\right) \ ,$$

where prime denotes transposition.

Standing Assumption. The volatility process σ satisfies :

$$E\left[\exp \frac{1}{2}\int_0^T |\sigma'\sigma|^{-1}\right] \ < \ \infty \quad \text{and} \quad \sup_{[0,T]} |\sigma'\sigma|^{-1} \ < \ \infty \quad P-\text{a.s.}$$

Since b is bounded, this condition ensures that the process λ_0 satisfies the Novikov condition $E[\exp \int_0^T |\lambda_0|^2/2] < \infty$, and we have $E[Z_0(T)] = 1$. The process Z_0 is then a martingale, and induces the probability measure P_0 defined by :

$$P_0(A) \ := \ E\left[Z_0(t)\mathbf{1}_A\right] \quad \text{for all} \quad A \in \mathcal{F}(t) \ , \quad 0 \le t \le T \ .$$

Clearly P_0 is equivalent to the original probability measure P. By Girsanov's Theorem, the process

$$B_0(t) \quad := \quad B(t) + \int_0^t \lambda_0(t)dt \ , \quad 0 \le t \le T \ ,$$

is a standard Brownian motion under P_0.

2.2 Portfolio and wealth process

Let $W(t)$ denote the wealth at time t of some investor on the financial market. We assume that the investor allocates continuously his wealth between the non-risky asset and the risky assets. We shall denote by $\pi^i(t)$ the proportion of wealth invested in the $i-th$ risky asset. This means that

$$\pi^i(t)W(t) \qquad \text{is the amount invested at time } t \text{ in the } i-th \text{ risky asset.}$$

The remaining proportion of wealth $1 - \sum_{i=1}^d \pi^i(t)$ is invested in the non-risky asset.

The *self-financing condition* states that the variation of the wealth process is only affected by the variation of the price process. Under this condition, the wealth process satisfies :

$$
\begin{aligned}
dW(t) &= W(t) \sum_{i=1}^d \pi^i(t) \frac{dS^i(t)}{S^i(t)} \\
&= W(t)\pi(t)'[b(t)dt + \sigma(t)dB(t)] \;=\; W(t)\pi(t)'\sigma(t)dB_0(t) \;. \qquad (2.2)
\end{aligned}
$$

Hence, the investment strategy π should be restricted so that the above stochastic differential equation has a well-defined solution. Also $\pi(t)$ should be based on the information available at time t. This motivates the following definition.

Definition 2.1 *An investment strategy is an $\mathbb{F}-$adapted process π valued in $\mathbb{R}^d$ and satisfying $\int_0^T |\sigma'\pi|^2(t)dt < \infty$ $P-$a.s.*

We shall denote by $\mathcal{A}$ the set of all investment strategies.

Clearly, given an initial capital $w \geq 0$ together with an investment strategy π, the stochastic differential equation (2.2) has a unique solution

$$
W_w^\pi(t) \;:=\; w\mathcal{E}\left(\int_0^t \pi(r)'\sigma(r)dB_0(r)\right) \;, \quad 0 \leq t \leq T \;.
$$

We then have the following trivial, but very important, observation :

$$
W_w^\pi \quad \text{is a } P_0-\text{supermartingale} \;, \qquad (2.3)
$$

as a non-negative local martingale under P_0.

2.3 Problem formulation

Let K be a closed convex subset of $\mathbb{R}^d$ containing the origin, and define the set of constrained strategies :

$$
\mathcal{A}_K \;:=\; \{\pi \in \mathcal{A} \;:\; \pi \in K \;\ell \otimes P - \text{a.s.}\} \;.
$$

The set K represents some constraints on the investment strategies.

Example 2.1 *Incomplete market* : taking $K = \{x \in \mathbb{R}^d \;:\; x^i = 0\}$, for some integer $1 \leq i \leq d$, means that trading on the $i-$th risky asset is forbidden.

Example 2.2 *No short-selling constraint* : taking $K = \{x \in I\!R^d \ : \ x^i \geq 0\}$, for some integer $1 \leq i \leq d$, means that the financial market does not allow to sell short the $i-$th asset.

Example 2.3 *No borrowing constraint* : taking $K = \{x \in I\!R^d \ : \ x^1 + \ldots + x^d \leq 1\}$ means that the financial market does not allow to sell short the non-risky asset or, in other word, borrowing from the bank is not available.

Now, let G be a non-negative $\mathcal{F}(T)-$ measurable random variable. The chief goal of these notes is to study the following stochastic control problem

$$V(0) \ := \ \inf\left\{w \in I\!R \ : \ W_w^\pi(T) \ \geq \ G \ P - \text{a.s. for some } \pi \in \mathcal{A}_K\right\} \ . \quad (2.4)$$

The random variable G is called a *contingent claim* in the financial mathematics literature, or a *derivative asset* in the financial engineering world. Loosely speaking, this is a contract between two counterparts stipulating that the seller has to pay G at time T to the buyer. Therefore, $V(0)$ is the minimal initial capital which allows the seller to face without risk the payment G at time T, by means of some clever investment strategy on the financial market.

We conclude this section by summarizing the main results which will be presented in these notes.

1. We start by proving that existence holds for the problem $V(0)$ under very mild conditions, i.e. there exists a constrained investment strategy $\pi \in \mathcal{A}_K$ such that $W^\pi_{V(0)}(T) \geq G \ P-$a.s. We say that π is an optimal *hedging* strategy for the contingent claim G.

The existence of an optimal hedging strategy will be obtained by means of some representation result which is now known as the *optional decomposition theorem* (in the framework of these notes, we can even call it a *predictable decomposition theorem*). As a by-product of this existence result, we will obtain a general dual formulation of the control problem $V(0)$. This will be developed in section 3.

2. In section 4, we seek for more information on the optimal hedging strategy by focusing on the Markov case. The main result is a characterization of $V(0)$ by means of a nonlinear partial differential equation (PDE) with appropriate terminal condition. In some cases, we will be able to solve explicitly the problem. The solution is typically of the *face lifting* type, a desirable property from the viewpoint of the practioners.

The derivation of the above-mentioned PDE is obtained from the dual formulation of $V(0)$ by classical arguments.

3. Section 6 develops the important observation that the same PDE can be obtained by working directly on the original formulation of the problem $V(0)$. Further developments of this idea will be reported in the accompanying paper [22]. In particular, such a direct treatment of the problem allows to solve some super-replication problems for which the dual formulation is not available.

4. The final section of this paper is devoted to the problem of super-replication under Gamma constraints, for which no dual formulation is available in the literature. The solution is again of the *face lifting* type.

3 Existence of optimal hedging strategies and dual formulation

In this section, we concentrate on the duality approach to the problem of super-replication under portfolio constraints $V(0)$. Our main objective is to convince the reader that the presence of constraints does not affect the general methodology of the proof : the main ingredient is a stochastic representation theorem. We therefore start by recalling the problem solution in the unconstrained case. This corresponds to the so-called *complete market* framework. In the general constrained case, the proof relies on the same arguments except that : we need to use a more advanced stochastic representation result, namely *the optional decomposition theorem.*

Remark 3.1 *local martingale representation theorem.*

(i) Theorem. Let Y be a local $P-$local martingale. Then there exits an $\mathbb{R}^d-$valued process ϕ such that

$$Y(t) \;=\; Y(0) + \int_0^t \phi(r)' dB(r) \;\; 0 \le t \le T \quad \text{and} \quad \int_0^T |\phi|^2 \;<\; \infty \quad P-\text{a.s.}$$

(see e.g. Dellacherie and Meyer VIII 62).

(ii) We shall frequently need to apply the above theorem to a $Q-$local martingale Y, for some equivalent probability measure Q defined by the density $(dQ/dP) = Z(T)$ $:= \mathcal{E}\left(-\int_0^T \lambda(r)' dB(r)\right)$, with Brownian motion $B^Q := B + \int_0^\cdot \lambda(r) dr$. To do this, we first apply the local martingale representation theorem to the$P-$local martingale

ZY. The result is $ZY = Y(0) + \int_0^\cdot \phi dB$ for some adapted process ϕ with $\int_0^T |\phi|^2 < \infty$. Applying Itô's lemma, one can easily check that we have :

$$Y(t) \;=\; Y(0) + \int_0^t \psi(r)' dB^Q(r) \quad 0 \le t \le T \quad \text{where} \quad \psi \;:=\; Z^{-1}\phi + \lambda Y \;.$$

Since Z and Y are continuous processes on the compact interval $[0,T]$, it is immediately checked that $\int_0^T |\psi|^2 < \infty$ $Q-$a.s.

3.1 Complete market : the unconstrained Black-Scholes world

In this paragraph, we consider the unconstrained case $K = \mathbb{R}^d$. The following result shows that $V(0)$ is obtained by the same rule than in the celebrated Black-Scholes model, which was first developed in the case of constant coefficients μ and σ.

Theorem 3.1 *Assume that $G > 0$ $P-$a.s. Then :*

(i) $V(0) = E_0[G]$

(ii) *if $E_0[G] < \infty$, then $W^\pi_{V(0)}(T) = G$ $P-$a.s. for some $\pi \in \mathcal{A}$.*

Proof. 1. Set $F := \{w \in \mathbb{R} : W^\pi_w(T) \ge G$ for some $\pi \in \mathcal{A}\}$. From the P_0-supermartingale property of the wealth process (2.3), it follows that $w \ge E_0[G]$ for all $w \in F$. This proves that $V(0) \ge E_0[G]$. Observe that this concludes the proof of (i) in the case $E_0[G] = +\infty$.

2. We then concentrate on the case $E_0[G] < \infty$. Define

$$Y(t) \;:=\; E_0[G|\mathcal{F}(t)] \quad \text{for} \quad 0 \le t \le T \;.$$

Apply the local martingale representation theorem to the P_0-martingale Y, see Remark 3.1. This provides

$$Y(t) \;=\; Y(0) + \int_0^t \psi(r)' dB_0(r) \quad \text{for some process } \psi \text{ with } \quad \int_0^T |\psi|^2 < \infty \;.$$

Now set $\pi := (Y\sigma')^{-1}\psi$. Since Y is a positive continuous process, it follows from Standing Assumption that $\pi \in \mathcal{A}$, and $Y = Y(0)\mathcal{E}\left(\int_0^\cdot \pi(r)^* \sigma(r) dB_0(r)\right) = W^\pi_{Y(0)}$. The statement of the theorem follows from the observation that $Y(T) = G$. $\square$

Remark 3.2 Statement (ii) in the above theorem implies that existence holds for the control problem $V(0)$, i.e. there exists an optimal trading strategy. But it provides a further information, namely that the optimal hedging strategy allows to *attain* the contingent claim G. Hence, in the unconstrained setting, all (positive) contingent claims are attainable. This is the reason for calling this financial market *complete*.

Remark 3.3 The proof of Theorem 3.1 suggests that the optimal hedging strategy π is such that the P_0- martingale Y has the stochastic representation $Y = E[G] + \int_0^{\cdot} Y\pi'\sigma dB_0$. In the Markov case, we have $Y(t) = v(t, S(t))$. Assuming that v is smooth, it follows from an easy application of Itô's lemma that

$$\Delta^i(t) \;:=\; \frac{\pi^i(t)W^{\pi}_{V(0)}(t)}{S^i(t)} \;=\; \frac{\partial v}{\partial s^i}(t, S(t))\;.$$

We now focus on the positivity condition in the statement of Theorem 3.1, which rules out the main example of contingent claims, namely European call options $[S^i(T) - K]^+$, and European put options $[K - S^i(T)]^+$. Indeed, since the portfolio process is defined in terms of proportion of wealth, the implied wealth process is strictly positive. Then, it is clear that such contingent claims can not be attained, in the sense of Remark 3.2, and there is no hope for Claim (ii) of Theorem 3.1 to hold in this context. However, we have the following easy consequence.

Corollary 3.1 *Let G be a non-negative contingent claim. Then*

(i) *For all $\varepsilon > 0$, there exists an investment strategy $\pi_\varepsilon \in \mathcal{A}$ such that $W^{\pi_\varepsilon}_{V(0)}(T) = G + \varepsilon$.*

(ii) $V(0) = E_0[G]$.

Proof. Statement (i) follows from the application of Theorem 3.1 to the contingent claim $G+\varepsilon$. Now let $V_\varepsilon(0)$ denote the value of the super-replication problem for the contingent claim $G + \varepsilon$. Clearly, $V(0) \le V_\varepsilon(0) = E_0[G + \varepsilon]$, and therefore $V(0) \le E_0[G]$ by sending ε to zero. The reverse inequality holds since Part 1 of the proof of Theorem 3.1 does not require the positivity of G. $\square$

Remark 3.4 In the Markov setting of Remark 3.3 above, and assuming that v is smooth, the approximate optimal hedging strategy of Corollary 3.1 (i) is given by

$$\Delta^i_\varepsilon(t) \;:=\; \frac{\pi^i_\varepsilon(t)W^{\pi}_{V_\varepsilon(0)}(t)}{S^i(t)} \;=\; \frac{\partial}{\partial s^i}\left\{v(t, S(t)) + \varepsilon\right\} \;=\; \frac{\partial v}{\partial s^i}(t, S(t))\;;$$

observe that $\Delta := \Delta_\varepsilon$ is independent of ε.

Example 3.1 *The Black and Scholes formula* : consider a financial market with a single risky asset $d = 1$, and let μ and σ be constant coefficients, so that the P_0-distribution of $\ln[S(T)/S(t)]$, conditionally on $\mathcal{F}(t)$, is gaussian with mean $-\sigma^2(T-t)/2$ and variance $\sigma^2(T-t)$. As a contingent claim, we consider the example of a European call option, i.e. $G = [S(T) - K]^+$ for some exercise price $K > 0$. Then, one can compute directly that :

$$
\begin{aligned}
V(t) \;=\; v(t,S(t)) \quad \text{where} \quad v(t,s) &:= sF(d(t,s)) - KF\left(d(t,s) - \sigma\sqrt{T-t}\right) ,\\
d(t,s) &:= (\sigma\sqrt{T-t})^{-1}\ln(K^{-1}s) + \frac{1}{2}\sigma\sqrt{T-t} ,
\end{aligned}
$$

and $F(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-u^2/2}du$ is the cumulative function of the gaussian distribution. According to Remark 3.3, the optimal hedging strategy in terms of number of shares is given by :

$$\Delta(t) \;=\; F(d(t,S(t))) \; .$$

3.2 Optional decomposition theorem

We now turn to the general constrained case. The key-point in the proof of Theorem 3.1 was the representation of the P_0-martingale Y as a stochastic integral with respect to B_0; the integrand in this representation was then identified to the investment strategy. In the constrained case, the investment strategy needs to be valued in the closed convex set K, which is not guaranteed by the representation theorem. We then need to use a *more advanced* representation theorem. The results of this section were first obtained by ElKaroui and Quenez (1995) for the incomplete market case, and further extended by Cvitanić and Karatzas (1993).

We first need to introduce some notations. Let

$$\delta(y) \;:=\; \sup_{x\in K} x'y$$

be the support function of the closed convex set K. Since K contains the origin, δ is non-negative. We shall denote by

$$\tilde{K} \;:=\; \mathrm{dom}(K) \;=\; \{y \in I\!R^d \;:\; \delta(y) < \infty\}$$

the effective domain of δ. For later use, observe that $\tilde{K}$ is a closed convex cone of $I\!R^d$. Recall also that, since K is closed and convex, we have the following classical result from convex analysis (see e.g. Rockafellar 1970) :

$$x \in K \quad \text{if and only if} \quad \delta(y) - x'y \geq 0 \quad \text{for all } y \in \tilde{K} , \tag{3.1}$$

$$x \in \mathrm{ri}(K) \quad \text{if and only if} \quad x \in K \quad \text{and} \quad \inf_{y \in \tilde{K}_1} (\delta(y) - x'y) > 0 , \tag{3.2}$$

where

$$\tilde{K}_1 := \tilde{K} \cap \{y \in \mathbb{R}^d : |y| = 1 \text{ and } \delta(y) + \delta(-y) \neq 0\} .$$

We next denote by $\mathcal{D}$ the collection of all bounded adapted processes valued in $\tilde{K}$. For each $\nu \in \mathcal{D}$, we set

$$\beta_\nu(t) := \exp\left(-\int_0^t \delta(\nu(r))dr\right) , \quad 0 \leq t \leq T ,$$

and we introduce the Doléans-Dade exponential

$$Z_\nu(t) := \mathcal{E}\left(-\int_0^t \lambda_\nu(r)'dB(r)\right) \quad \text{where} \quad \lambda_\nu := \sigma^{-1}(b - \nu) = \lambda_0 - \sigma^{-1}\nu .$$

Since b and ν are bounded, λ_ν inherits the Novikov condition $E\left[\exp\left(\frac{1}{2}\int_0^T |\lambda_\nu|^2\right)\right] < \infty$ from Standing Assumption. We then introduce the family of probability measures

$$P_\nu(A) := E\left[Z_\nu(t)\mathbf{1}_A\right] \quad \text{for all} \quad A \in \mathcal{F}(t) , \quad 0 \leq t \leq T .$$

Clearly P_ν is equivalent to the original probability measure P. By Girsanov Theorem, the process

$$B_\nu(t) := B(t) + \int_0^t \lambda_\nu(r)dr = B_0(t) - \int_0^t \sigma(r)^{-1}\nu(r)dr , \quad 0 \leq t \leq T , \tag{3.3}$$

is a standard Brownian motion under P_ν.

Remark 3.5 The reason for introducing these objects is that the important property (2.3) extends to the family $\mathcal{D}$:

$$\beta_\nu W_w^\pi \quad \text{is a } P_\nu\text{−supermartingale for all} \quad \nu \in \mathcal{D},\ \pi \in \mathcal{A}_K , \tag{3.4}$$

and $w > 0$. Indeed, by Itô's lemma together with (3.3),

$$d(W_w^\pi \beta_\nu) = W_w^\pi \beta_\nu \left[-(\delta(\nu) - \pi'\nu)dt + \pi'\sigma dB_\nu\right] .$$

In view of (3.1), this shows that $W_w^\pi \beta_\nu$ is a non-negative local P_ν−supermartingale, which provides (3.4).

Theorem 3.2 *Let Y be an $\mathbb{F}-$ adapted positive càdlàg process. Assume that*

$$\text{the process } \beta_\nu Y \text{ is a } P_\nu-\text{supermartingale for all } \nu \in \mathcal{D}.$$

Then, there exists a predictable non-decreasing process C, with $C(0) = 0$, and a constrained portfolio $\pi \in \mathcal{A}_K$ such that $Y = W^\pi_{Y(0)} - C$.

Proof. We start by applying the Doob (unique) decomposition theorem (see e.g. Dellacherie and Meyer VII 12) to the P_0-supermartingale $Y\beta_0 = Y$, together with the local martingle representation theorem, under the probability measure P_0. This implies the existence of an adapted process ψ_0 and a non-decreasing predictable process C_0 satisfying $C_0(0) = 0$, $\int_0^T |\psi_0|^2 < \infty$, and :

$$Y(t) \;=\; Y(0) + \int_0^t \psi_0(r)dB_0(r) - C_0(t)\;, \tag{3.5}$$

see Remark 3.1. Observe that

$$M_0 \;:=\; Y(0) + \int_0^\cdot \psi_0 dB_0 \;=\; Y + C_0 \;\geq\; Y \;>\; 0\,. \tag{3.6}$$

We then define

$$\pi_0 \;:=\; M_0^{-1}(\sigma')^{-1}\psi_0\,.$$

From Standing Assumption together with the continuity of M_0 on $[0,T]$ and the fact that $\int_0^T |\psi_0|^2 < \infty$, it follows that $\pi_0 \in \mathcal{A}$. Then $M_0 = W^\pi_{Y(0)}$ and by (3.6),

$$Y \;=\; W^{\pi_0}_{Y(0)} - C_0\,.$$

In order to conclude the proof, it remains to show that the process π is valued in K.

2. By Itô's lemma together with (3.3), it follows that :

$$d(Y\beta_\nu) \;=\; M_0\beta_\nu\pi_0'\sigma dB_\nu - \beta_\nu\left[(Y\delta(\nu) - M_0\pi_0'\nu)dt + dC_0\right]\,.$$

This provides the unique decomposition of the $P_\nu-$supermartingale $Y\beta_\nu = M_\nu + C_\nu$, with

$$M_\nu \;:=\; Y(0) + \int_0^\cdot M_0\beta_\nu\pi_0'\sigma dB_\nu \quad\text{and}\quad C^\nu \;:=\; \int_0^\cdot \beta_\nu\left[(Y\delta(\nu) - M_0\pi_0'\nu)dt + dC_0\right]\,,$$

into a $P_\nu-$local martingale M_ν and a predictable non-decreasing process C_ν starting from the origin. We conclude from this that :

$$\begin{aligned} 0 \;\leq\; \int_0^t \beta_\nu^{-1}dC_\nu \;&=\; C_0(t) + \int_0^t \left(Y\delta(\nu) - M_0\pi_0'\nu\right)(r)dr \\ &\leq\; C_0(t) + \int_0^t M_0\left(\delta(\nu) - \pi_0'\nu\right)(r)dr \quad\text{for all } \nu \in \mathcal{D}\,, \end{aligned} \tag{3.7}$$

where the last inequality follows from (3.6) and the non-negativity of the support function δ.

3. Now fix some $\nu \in \mathcal{D}$, and define the set $F_\nu := \{(t,\omega) : [\pi_0'\nu + \delta(\nu)](t,\omega) < 0\}$. Consider the process

$$\nu^{(n)} \;=\; \nu 1_{F_y^c} + n\nu 1_{F_y} \;, \quad n \in I\!N \;.$$

Clearly, since $\tilde{K}$ is a cone, we have $\nu^{(n)} \in \mathcal{D}$ for all $n \in I\!N$. Writing (3.7) with $\nu^{(n)}$, we see that, whenever $\ell \otimes P[F_\nu] > 0$, the left hand-side term converges to $-\infty$ as $n \to \infty$, a contradiction. Hence $\ell \otimes P[F_\nu] = 0$ for all $\nu \in \mathcal{D}$. From (3.1), this proves that $\pi \in K$ $\ell \otimes P-$a.s. $\square$

3.3 Dual formulation

Let $\mathcal{T}$ be the collection of all stopping times valued in $[0,T]$, and define the family of random variables :

$$Y_\tau \;:=\; \mathrm{esssup}_{\nu\in\mathcal{D}}\, E_\nu\left[G\gamma_\nu(\tau,T)|\mathcal{F}(\tau)\right] \;;\;\; \tau \;\in\; \mathcal{T} \quad \text{where} \quad \gamma_\nu(\tau,T) \;:=\; \frac{\beta_\nu(T)}{\beta_\nu(\tau)} \;,$$

and $E_\nu[\cdot]$ denotes the conditional expectation operator under P_ν. The purpose of this section is to prove that $V(0) = Y_0$, and that existence holds for the control problem $V(0)$. As a by-product, we will also see that existence for the control problem Y_0 holds only in very specific situations. These results are stated precisely in Theorem 3.3. As a main ingredient, their proof requires the following (classical) dynamic programming principle.

Lemma 3.1 (Dynamic Programming). *Let $\tau \le \theta$ be two stopping times in $\mathcal{T}$. Then :*

$$Y_\tau \;=\; \mathrm{esssup}_{\nu\in\mathcal{D}}\, E_\nu\left[Y_\theta\gamma_\nu(\tau,\theta)|\mathcal{F}(\tau)\right] \;.$$

Proof. 1. Conditioning by $\mathcal{F}(\theta)$, we see that

$$Y_\tau \;\le\; \mathrm{esssup}_{\nu\in\mathcal{D}}\, E_\nu\left[\gamma_\nu(\tau,\theta)E_\nu[G\gamma_\nu(\theta,T)|\mathcal{F}(\theta)]\right] \;\le\; \mathrm{esssup}_{\nu\in\mathcal{D}}\, E_\nu\left[\gamma_\nu(\tau,\theta)Y_\theta\right] \;.$$

2. To see that the reverse inequality holds, fix any $\mu \in \mathcal{D}$, and let $\mathcal{D}_{\tau,\theta}(\mu)$ be the subset of $\mathcal{D}$ whose elements coincide with μ on the stochastic interval $[\![\tau,\mu]\!]$. Let $(\nu_k)_k$ be a maximizing sequence of Y_θ, i.e.

$$Y_\theta \;=\; \lim_{k\to\infty} J_{\nu_k}(\theta) \quad \text{where} \quad J_\nu(\theta) \;:=\; E_\nu[G\gamma_\nu(\theta,T)|\mathcal{F}(\theta)] \;;$$

the existence of such a sequence follows from the fact that the family $\{J_\nu(\theta), \nu \in \mathcal{D}\}$ is directed upward. Also, since $J_\nu(\theta)$ depends on ν only through its realization on the stochastic interval $[\theta, T]$, we can assume that $\nu_k \in \mathcal{D}_{\tau,\theta}(\mu)$. We now compute that

$$Y_\tau \;\geq\; E_{\nu_k}\left[G\gamma_{\nu_k}(\tau,T)|\mathcal{F}(\tau)\right] \;=\; E_\mu\left[\gamma_\mu(\tau,\theta)J_{\nu_k}(\theta)|\mathcal{F}(\tau)\right] \;,$$

which implies that $Y_\tau \geq E_\mu\left[\gamma_\mu(\tau,\theta)Y_\theta|\mathcal{F}(\tau)\right]$ by Fatou's lemma. □

Now, observe that we may take the stopping times τ in the definition of the family $\{Y_\tau,\, \tau \in \mathcal{T}\}$ to be deterministic and thereby obtain a non-negative adapted process $\{Y(t),\, 0 \leq t \leq T\}$. A natural question is whether this process is consistent with the family $\{Y_\tau,\, \tau \in \mathcal{T}\}$ in the sense that $Y_\tau(\omega) = Y(\tau(\omega))$ for a.e. $\omega \in \Omega$. For general control problems, this is a delicate issue. However, in our context, it follows from the above dynamic programming principle that the family $\{Y_\tau,\, \tau \in \mathcal{T}\}$ satisfies a *supermartingale* property :

$$E\left[\beta_\nu(\theta)Y_\theta|\mathcal{F}(\tau)\right] \;\leq\; \beta_\nu(\tau)Y_\tau \quad \text{for all } \tau,\theta \in \mathcal{T} \text{ with } \quad \tau \;\leq\; \theta \;.$$

By a classical argument, this allows to extract a process Y out of this family, which satisfies the supermartingale property in the usual sense. We only state precisely this technical point, and send the interested reader to Karatzas and Shreve (1999) Appendix D or Cvitanić and Karatzas (1993), Proposition 6.3.

Corollary 3.2 *There exists a càdlàg process $Y = \{Y(t),\, 0 \leq t \leq T\}$, consistent with the family $\{Y_\tau,\, \tau \in \mathcal{T}\}$, and such that $Y\beta_\nu$ is a $P_\nu-$supermartingale for all $\nu \in \mathcal{D}$.*

We are now able for the main result of this section.

Theorem 3.3 *Assume that $G > 0$ $P-$a.s. Then :*

(i) $V(0) = Y(0)$,

(ii) *if $Y(0) < \infty$, existence holds for the problem $V(0)$, i.e. $W^\pi_{V(0)}(T) \geq G$ $P-$a.s. for some $\pi \in \mathcal{A}_K$,*

(iii) *existence holds for the problem $Y(0)$ if and only if*

$$W^{\hat\pi}_{V(0)}(T) \;=\; G \quad \textit{and} \quad \beta_{\hat\nu}W^{\hat\pi}_{V(0)} \textit{ is a } P_{\hat\nu}-\textit{martingale}$$

for some pair $(\hat\pi, \hat\nu) \in \mathcal{A}_K \times \mathcal{D}$.

Proof. 1. We concentrate on the proof of $Y(0) \geq V(0)$ as the reverse inequality is a direct consequence of (3.4). The process Y, extracted from the family $\{Y_\tau, \tau \in \mathcal{T}\}$ in Corollary 3.2, satisfies Condition (i) of the optional decomposition theorem 3.2. Then $Y = W^\pi_{Y(0)} - C$ for some constrained portfolio $\pi \in \mathcal{A}_K$, and some predictable non-decreasing process C with $C(0) = 0$. In particular, $W^\pi_{Y(0)}(T) \geq Y(T) = G$. This proves that $Y(0) \geq V(0)$, completing the proof of (i) and (ii).

2. It remains to prove (iii). Suppose that $W^{\hat\pi}_{V(0)}(T) = G$ and $\beta_{\hat\nu} W^{\hat\pi}_{V(0)}$ is a $P_{\hat\nu}-$martingale for some pair $(\hat\pi, \hat\nu) \in \mathcal{A}_K \times \mathcal{D}$. Then, by the first part of this proof, $Y(0) = V(0) = E_{\hat\nu}\left[W^{\hat\pi}_{V(0)}(T)\beta_{\hat\nu}(T)\right] = E_{\hat\nu}\left[G\beta_{\hat\nu}(T)\right]$, i.e. $\hat\nu$ is a solution of $Y(0)$.

Conversely, assume that $Y(0) = E_{\hat\nu}[G\beta_{\hat\nu}(T)]$ for some $\hat\nu \in \mathcal{D}$. Let $\hat\pi$ be the solution of $V(0)$, whose existence is established in the first part of this proof. By definition $W^{\hat\pi}_{V(0)}(T) - G \geq 0$. Since $\beta_{\hat\nu} W^{\hat\pi}_{V(0)}$ is a $P_{\hat\nu}-$super-martingale, it follows that $E_{\hat\nu}\left[\beta_{\hat\nu}(W^{\hat\pi}_{V(0)}(T) - G)\right] \leq 0$. This proves that $W^{\hat\pi}_{V(0)}(T) - G = 0$ $P-$a.s. We finally see that the $P_{\hat\nu}-$super-martingale $\beta_{\hat\nu} W^{\hat\pi}_{V(0)}$ has constant $P_{\hat\nu}-$expectation :

$$\begin{aligned} Y(0) &\geq E_{\hat\nu}\left[\beta_{\hat\nu}(t) W^{\hat\pi}_{V(0)}(t)\right] \\ &\geq E_{\hat\nu}\left[E_{\hat\nu}\left(\beta_{\hat\nu}(T) W^{\hat\pi}_{V(0)}(T) \middle| \mathcal{F}(t)\right)\right] \;=\; E_{\hat\nu}\left[\beta_{\hat\nu}(T) G\right] \;=\; Y(0) \;, \end{aligned}$$

and therefore $\beta_{\hat\nu} W^{\hat\pi}_{V(0)}$ is a $P_{\hat\nu}-$martingale. □

3.4 Extensions

The results of this section can be extended in many directions.

3.4.1 Simple Large investor models

Suppose that the drift coefficient of the price process depends on the investor's strategy, so that the price dynamics are influenced by the action of the investor. More precisely, the price dynamics ar defined by :

$$S^i(0) \;=\; s^i \;, \qquad dS^i(t) \;=\; S^i(t)\left[b^i(t, S(t), \pi(t))dt + \sum_{j=1}^d \sigma^{ij}(t) dB^j(t)\right] \;, \quad (3.8)$$

where $b(t, s, \cdot)$ is convex and non-negative. Define the Legendre-Fenchel transform :

$$\tilde b(t, s, y) \;:=\; \sup_{x \in K}\left(b^i(t, \omega, x) + x'y\right) \quad \text{for } \; y \in I\!R^d \;,$$

together with the associated effective domain :

$$\tilde K(t, s) \;:=\; \left\{\, y \in I\!R^d \;:\; \tilde b(t, s, y) < \infty \,\right\} \;.$$

Next, we introduce the set $\mathcal{D}$ consisting of all bounded adapted processes ν satisfying $\nu(t) \in \tilde{K}(t, S(t))$ $\ell \otimes P-$a.s.

For any process $\nu \in \mathcal{D}$, define the equivalent probability measure P_ν by the density process

$$Z_\nu(t) \;\; := \;\; \mathcal{E}\left(\int_0^t [\sigma(t)^{-1}\nu(t)]' dW(t)\right) \; ,$$

and the discount process :

$$\beta_\nu(t) \;\; := \;\; \exp\left(-\int_0^t \tilde{b}(r, S(r), \nu(r)) dr\right) \; .$$

Then, the results of the previous section can be extended to this context by defining the dynamic version of the dual problem

$$Y_\tau \;:=\; \mathrm{esssup}_{\nu\in\mathcal{D}} E_\nu \left[G\gamma_\nu(\tau,T) \middle| \mathcal{F}(\tau)\right] \;;\;\; \tau \in \mathcal{T} \quad \text{where} \quad \gamma_\nu(\tau,T) \;:=\; \frac{\beta_\nu(T)}{\beta_\nu(\tau)} \; .$$

Important observation : in the above *simple* large investor model, only the drift coefficient in the dynamics of S is influenced by the portfolio π. Unfortunately, the analysis developed in the previous paragraphs does not extend to the *general* large investor problem, where the volatility process is influenced by the action of the investor. This is due to the fact that there is no way to get rid of the dependence of σ on π by proceeding to some equivalent change of measure : it is well-known that the measures induced by diffusions with different diffusion coefficients are singular. In section 6, the general large investor problem will be solved by an alternative technique.

Remark 3.6 From an economic viewpoint, one may argue that the dynamics of the price process should be influenced by the variation of the portfolio holdings of the investor $d[\pi W]$. This is related to the problem of hedging under *gamma* constraints. The last section of these notes presents some preliminary results in this direction.

3.4.2 Semimartingale price processes

All the results of the previous sections extend to the case where the price process S is defined as a semimartingale valued in $(0, \infty)^d$. Let us state the generalization of Theorem 3.3; we refer the interested reader to Föllmer and Kramkov (1997) for a deep analysis.

the wealth process is defined by

$$W_w^\pi(t) \ := \ \mathcal{E}\left(\int_0^t \pi(r)'\mathrm{diag}[S(r)]^{-1}dS(r)\right) \ ,$$

where diag$[s]$ denotes the diagonal matrix with diagonal components s^i, and the set of admissible portfolio $\mathcal{A}_K$ is the collection of all predictable processes π valued in K for which the above stochastic integral is well-defined.

Let $\mathcal{P}$ be the collection of all probability measures $Q \sim P$ such that :

$$W_1^\pi e^{-C} \quad \text{is a } Q\text{−local supermartingale} \quad \text{for all } \pi \in \mathcal{A} \ , \tag{3.9}$$

for some increasing predictable process C (depending on Q). An increasing predictable process C_Q is called an *upper variation process under* Q if it satisfies (3.9) and is minimal with respect to this property. Such a process is shown to exist.

Then, the results of the previous sections can be extended to this context by defining the dynamic version of the dual problem

$$Y_\tau \ := \ \mathrm{esssup}_{Q\in\mathcal{P}} E_Q\left[G\gamma_Q(\tau,T)|\,\mathcal{F}(\tau)\right] \ ; \ \ \tau \ \in \ \mathcal{T} \ , \qquad \gamma_Q(\tau,T) \ := \ e^{C_Q(\tau)-C_Q(T)} \ ,$$

where E_Q denotes the expectation operator under Q.

The main ingredient for this result is an extension of the optional decomposition result of Theorem 3.2. Observe that the non-decreasing process C in this more general framework is optional, and not necessarily predictable. We recall that

- the *optional tribe* is generated by $\{\mathcal{F}_t\}$−adapted processes with cd̀-làg trajectories;

- a process X is *optional* if the map $(t,\omega) \longmapsto X(t,\omega)$ is measurable with respect to the optional tribe;

- the *predictable tribe* is generated by $\{\mathcal{F}_{t-}\}$−adapted processes with left-continuous trajectories;

- a process X is *predictable* if the map $(t,\omega) \longmapsto X(t,\omega)$ is measurable with respect to the predictable tribe.

4 HJB equation from the dual problem

4.1 Dynamic programming equation

In order to characterize further the solution of the super-replication problem, we now focus on the Markov case :

$$b(t) \;=\; b(t,S(t)) \quad \text{and} \quad \sigma(t) \;=\; \sigma(t,S(t))\;,$$

where b and σ are now vector and matrix valued functions defined on $[0,T]\times I\!R^d$. In order to guarantee existence of a strong solution to the SDE defining S, we assume that b and σ are Lipschitz functions in the s variable, uniformly in t. We also consider the special case of European contingent claim :

$$G \;=\; g\left(S(T)\right)\;,$$

where g is a map from $[0,\infty)^d$ into $I\!R_+$. We first extend the definition of the dual problem in order to allow for a moving time origin :

$$v(t,s) \;:=\; \sup_{\nu\in\mathcal{D}} E\left[g\left(S_{t,s}(T)\right)\gamma_\nu(t,T)\right] \;=\; \sup_{\nu\in\mathcal{D}} E\left[g\left(S_{t,s}(T)\right)e^{-\int_t^T\delta(\nu(r))dr}\right]\;.$$

Here, $S_{t,s}$ denotes the unique strong solution of (2.1) with initial data $S_{t,s}(t)=s$. Notice that, although the control process ν is path dependent, the value function of the above control problem $v(t,s)$ is a function of the current values of the state variables. This is a consequence of the important results of Haussman (1985) and ElKaroui, Nguyen and Jeanblanc (1986).

Let v_* and v^* be respectively the lower and upper semi-continuous envelopes of v :

$$v_*(t,s) \;:=\; \liminf_{(t',s')\to(t,s)} v(t',s') \quad \text{and} \quad v^*(t,s) \;:=\; \limsup_{(t',s')\to(t,s)} v(t',s')\;.$$

We shall frequently appeal to the infinitesimal generator of the process S under P_0 :

$$\mathcal{L}\varphi \;:=\; \varphi_t+\frac{1}{2}\mathrm{Trace}\left[\mathrm{diag}[s]\sigma\sigma'\mathrm{diag}[s]D^2\varphi\right]\;,$$

where $\mathrm{diag}[s]$ denotes the diagonal matrix with diagonal components s^i, the t subscript denotes the partial derivative with respect to the t variable, $D\varphi$ is the gradient vector with respect to s, and $D^2\varphi$ is the Hessian matrix with respect to s. We will also make use of the first order differential operator :

$$H^y\varphi \;:=\; \delta(y)\varphi - y'\mathrm{diag}[s]D\varphi \quad \text{for} \quad y\;\in\;\tilde{K}\;.$$

An important role will be played by the following *face-lifted* payoff function

$$\hat{g}(s) \;:=\; \sup_{y \in \tilde{K}} \; g(se^y)e^{-\delta(y)} \quad \text{for} \quad s \in I\!R^d_+ \;, \tag{4.1}$$

where se^y is the $I\!R^d$ vector with components $s^i e^{y^i}$. We finally recall from (3.2) the notation $\tilde{K}_1 := \tilde{K} \cap \{|y| = 1 \text{ and } \delta(y) + \delta(-y) \neq 0\}$, and we denote by

$$x_K \quad \text{the orthogonal projection of } x \text{ on} \quad \text{vect}(K) \;,$$

where vect(K) is the vector space generated by K. The main result of this section is the following.

Theorem 4.1 *Let σ be bounded and suppose that v is locally bounded. Assume further that g is lower semi-continuous, $\hat{g}$ is upper semi-continuous with linear growth. Then :*

(i) *For all $(t,x) \in [0,T) \times I\!R^d$ and $y \in \tilde{K}$, the function $r \longmapsto \delta(y)r - \ln\left(v(t, e^{x+ry})\right)$ is non decreasing; in particular, $v(t,e^x) = v(t, e^{x_K})$ for all $(t,x) \in [0,T) \times I\!R^d$,*

(ii) *v is a (discontinuous) viscosity solution of*

$$\min\left\{ -\mathcal{L}v \;,\; \inf_{y \in \tilde{K}_1} \; H^y v \right\} \;=\; 0 \;\; on \; [0,T) \times (0,\infty)^d \;, \qquad v(T,\cdot) \;=\; \hat{g} \;. \tag{4.2}$$

The proof of the above statement is a direct consequence of Propositions 4.1, 4.2, 4.4, 4.5, and Corollary 4.1, reported in the following paragraphs.

We conclude this paragraph by recalling the notion of discontinuous viscosity solutions. The interested reader can find an overview of this theory in [5] or [13]. Given a real-valued function u, we shall denote by u_* and u^* its lower and the upper semi-continuous envelopes. We also denote by $\S_n$ the set of all $n \times n$ symmetric matrices with real coefficients.

Definition 4.1 *Let F be a map from $I\!R^n \times I\!R^n \times \mathcal{S}_n$ into $I\!R$, $\mathcal{O}$ an open domain of $I\!R^n$, and consider the non-linear PDE*

$$F\left(x, u(x), Du(x), D^2u(x)\right) \;=\; 0 \quad for \quad x \;\in\; \mathcal{O} \;. \tag{4.3}$$

Assume further that $F(x,r,p,A)$ is non-increasing in A (in the sense of symmetric matrices). Let $u : \mathcal{O} \longrightarrow I\!R$ be a locally bounded function.

(i) We say that u is a (discontinuous) viscosity supersolution of (4.3) *if for any $x_0 \in \mathcal{O}$ and $\varphi \in C^2(\mathcal{O})$ satisfying*

$$(u_* - \varphi)(x_0) \;=\; \min_{x\in\mathcal{O}} (u_* - \varphi) \;,$$

we have

$$F^*\Big(x_0, u_*(x_0), D\varphi(x_0), D^2\varphi(x_0)\Big) \;\geq\; 0 \;.$$

(ii) We say that u is a (discontinuous) viscosity subsolution of (4.3) *if for any $x_0 \in \mathcal{O}$ and $\varphi \in C^2(\mathcal{O})$ satisfying*

$$(u^* - \varphi)(x_0) \;=\; \max_{x\in\mathcal{O}} (u^* - \varphi) \;,$$

we have

$$F_*\Big(x_0, u^*(x_0), D\varphi(x_0), D^2\varphi(x_0)\Big) \;\leq\; 0 \;.$$

(iii) We say that u is a (discontinuous) viscosity solution of (4.3) *if it satisfies the above requirements* (i) *and* (ii)*.*

In these notes the map F defining the PDE (4.3) will always be continuous. In the accompanying paper [22], the case where F is not continuous will be frequently met.

4.2 Super-solution property

We first start by proving that v_* is a viscosity super-solution of the HJB equation (4.2).

Proposition 4.1 *Suppose that the value function v is locally bounded. Then v is a (discontinuous) viscosity super-solution of the PDE*

$$\min\left\{-\mathcal{L}v_* \;,\; \inf_{y\in\tilde{K}} \; H^y v_*\right\}(t,s) \;\geq\; 0 \;. \tag{4.4}$$

Proof. Let (t, s) be fixed in $[0, T) \times I\!R^d_+$, and consider a smooth test function φ, mapping $[0, T] \times I\!R^d_+$ into $I\!R$, and satisfying

$$0 \;=\; (v_* - \varphi)(t, s) \;=\; \min_{[0,T]\times I\!R^d_+} (v_* - \varphi) \;. \tag{4.5}$$

Let $(t_n, s_n)_{n\geq 0}$ be a sequence of elements of $[0,T)\times I\!R^d_+$ satisfying

$$v(t_n, s_n) \longrightarrow v_*(t,s) \quad \text{as} \quad n\to\infty \; .$$

In view of (4.5), this implies that :

$$b_n \;:=\; v(t_n,s_n) - \varphi(t_n,s_n) \longrightarrow 0 \quad \text{as} \quad n\to\infty \; . \tag{4.6}$$

The starting point of this proof is the dynamic programming principle of Lemma 3.1 together with the fact that $v \geq v_*$, which provide :

$$\begin{aligned}
\varphi(t_n,s_n) + b_n \;=\; v(t_n,s_n) \;&\geq\; E_\nu\left[v\left(\theta_n, S_{t_n,s_n}(\theta)\right)\gamma_\nu(t_n,\theta_n)\right] \qquad (4.7)\\
&\geq\; E_\nu\left[v_*\left(\theta_n, S_{t_n,s_n}(\theta)\right)\gamma_\nu(t_n,\theta_n)\right] \;\text{ for all } \nu\in\mathcal{D}\;,
\end{aligned}$$

where $\theta_n > t_n$ is an arbitrary stopping time, to be fixed later on. In view of (4.5) we have $v_* \geq \varphi$, and therefore

$$\begin{aligned}
0 \;&\leq\; b_n + E_\nu\left[\varphi(t_n,s_n)\gamma_\nu(t_n,t_n) - \varphi\left(\theta_n, S_{t_n,s_n}(\theta_n)\right)\gamma_\nu(t_n,\theta_n)\right]\\
&=\; b_n - E_\nu\left[\int_{t_n}^{\theta_n}\gamma_\nu(t_n,r)\left(\mathcal{L}^{\nu(r)}\varphi - H^{\nu(r)}\varphi\right)\left(r, S_{t_n,s_n}(r)\right)dr\right.\\
&\qquad\qquad \left. - \int_{t_n}^{\theta_n}\gamma_\nu(t_n,r)\left(\mathrm{diag}[s]D\varphi\right)\left(r,S_{t_n,s_n}(r)\right)' dB_\nu(r)\right] \;, (4.8)
\end{aligned}$$

by Itô's lemma. Now, fix some large constant M, and define stopping times

$$\theta_n \;:=\; h_n \wedge \inf\left\{t\geq t_n \;:\; \sum_{i=1}^d \left|\ln\left(S^i_{t_n,s_n}(t)/s^i_n\right)\right| \geq M\right\}\;,$$

where

$$h_n \;:=\; |\beta_n|^{1/2}\mathbf{1}_{\{\beta_n\neq 0\}} + n^{-1}\mathbf{1}_{\{\beta_n=0\}}\;.$$

We now take the process ν to be constant $\nu(r)=y$ for some $y\in\tilde{K}$, divide (4.8) by h_n, and send n to infinity using the dominated convergence theorem. The result is

$$-\mathcal{L}\varphi(t,s) + H^y\varphi(t,s) \;\geq\; 0 \quad \text{for all} \quad y\in\tilde{K}\;. \tag{4.9}$$

For $y=0$, this provides $-\mathcal{L}\varphi(t,s)\geq 0$. It remain to prove that $H^y\varphi(t,s)\geq 0$ for all $y\in\tilde{K}$. This is a direct consequence of (4.9) together with the fact that $\tilde{K}$ is a cone, and H^y is positively homogeneous in y. $\Box$

Remark 4.1 The above proof can be simplified by observing that the value function v inherits the lower semi-continuity of the payoff function g, assumed in Theorem 4.1. We did not include this simplification in order to highlight the main point of the proof where the lower semi-continuity of g is needed.

We are now in a position to prove statement (i) of Theorem 4.1.

Corollary 4.1 *Suppose that the value function v is locally bounded. For fixed $(t,x) \in [0,T) \times I\!R^d$ and $y \in \tilde{K}$, consider the function*

$$h_y \ : \ r \ \longmapsto \ \delta(y)r - \ln\left(v_*(t, e^{x+ry})\right) \ .$$

Then

(i) *h_y is non-decreasing;*

(ii) *$v_*(t, e^x) = v_*(t, e^{x_K})$, where $x_K := proj_{\mathrm{vect}(K)}(x)$ is the orthogonal projection of x on the vector space generated by K;*

(iii) *assuming further that g is lower semi-continuous, we have $v(t, e^x) = v(t, e^{x_K})$.*

Proof. (i) From Proposition 4.1, function v_* is a viscosity supersolution of

$$H^y v_* \ = \ \delta(y)v - y'\mathrm{diag}[s]Dv_* \ \geq \ 0 \quad \text{for all} \quad y \in \tilde{K} \ .$$

Consider the change of variable $w(t,x) := \ln\left[v_*(t, e^{x^1}, \ldots, e^{x^d})\right] \exp\left(-\delta(y)t\right)$. Then w is a viscosity supersolution of the equation $-w_x \geq 0$, and therefore w is non-increasing. This completes the proof.

(ii) We first observe that $\mathrm{vect}(K)^{\perp} \subset \tilde{K}$ and $\delta(y) = 0$ for all $y \in \mathrm{vect}(K)^{\perp}$. Now, since $\hat{y} := x - x_K \in \mathrm{vect}(K)^{\perp}$, it follows from (i) that

$$\begin{aligned} -\ln v_*(t, e^{x_K}) \ = \ h_{\hat{y}}(1) \ \geq \ h_{\hat{y}}(0) \ &= \ -\ln v_*(t, e^x) \\ &= \ h_{-\hat{y}}(0) \ \geq \ h_{-\hat{y}}(-1) \ = \ -\ln v_*(t, e^{x_K}) \ . \end{aligned}$$

Now, assuming further that g is lower semi-continuous, it follows from a simple application of Fatou's lemma that v is lower semicontinuous, and (iii) is just a restatement of (ii). □

4.3 Subsolution property

The purpose of this paragraph is to prove that the value function v is a (discontinuous) viscosity subsolution of the HJB equation (4.4). Since it has been shown in Proposition 4.1 that v is a supersolution of this PDE, this will prove that v is a (discontinuous) viscosity solution of the HJB equation (4.4).

Proposition 4.2 *Assume that the payoff function g is lower semi-continuous, and the value function v is locally bounded. Then, v is a (discontinuous) viscosity sub-solution of the PDE*

$$\min\left\{-\mathcal{L}v^* \ , \ \inf_{y\in\tilde{K}_1} H^y v^*\right\} \ \leq \ 0 \ . \tag{4.10}$$

Proof. Let $\varphi : [0,T]\times I\!R^d \longrightarrow I\!R$ be a smooth function and $(t_0,s_0)\in[0,T)\times I\!R^d$ be such that :

$$0 \ = \ (v^*-\varphi)(t_0,s_0) \ > \ (v^*-\varphi)(t,s) \quad \text{for all} \quad (t,s)\neq(t_0,s_0) \ , \tag{4.11}$$

i.e. (t_0,s_0) is a strict global maximizer of $(v^*-\varphi)$ on $[0,T)\times I\!R^d$. Since $v^*>0$ and $v^*(t,e^x)=v^*(t,e^{x_K})$ by Corollary 4.1, we may assume that $\varphi>0$ and $\varphi(t,e^x)=\varphi(t,e^{x_K})$ as well. Observe that this is equivalent to

$$\mathrm{diag}[s](D\varphi/\varphi)(t,s) \ \in \ \mathrm{vect}(K) \quad \text{for all} \quad (t,s)\in[0,T)\times(0,\infty)^d \ . \tag{4.12}$$

In order to prove the required result we shall assume that :

$$-\mathcal{L}\varphi(t_0,s_0) \ > \ 0 \quad \text{and} \quad \inf_{y\in\tilde{K}_1} H^y\varphi(t_0,s_0) \ > \ 0 \ . \tag{4.13}$$

In view of (4.12), the second inequality in (4.13) is equivalent to

$$\mathrm{diag}[s_0]\left(D\varphi/\varphi\right)(t_0,s_0) \ \in \ \mathrm{ri}(K) \ . \tag{4.14}$$

Our final goal will be to end up with a contradiction of the dynamic programming principle of Lemma 3.1.

1. By smoothness of the test function φ, it follows from (4.13)-(4.14) that one can find some parameter $\delta>0$ such that

$$-\mathcal{L}\varphi(t,s) \ \geq \ 0 \ \text{ and } \ \mathrm{diag}[s]\left(D\varphi/\varphi\right)(t,s) \ \in \ K \tag{4.15}$$

$$\text{for all} \ (t,s) \ \in \ D \ := \ (t_0-\delta,t_0+\delta)\times(s_0e^{\delta B}) \ , \tag{4.16}$$

where B is the unit closed ball of $I\!R^1$ and $s_0e^{\delta B}$ is the collection of all points ξ in $I\!R^d$ such that $|\ln(\xi^i/s_0^i)|<\delta$ for $i=1,\ldots,d$. We also set

$$\max_{\partial D}\left(\frac{v^*}{\varphi}\right) \ =: \ e^{-2\beta} \ < \ 1 \ , \tag{4.17}$$

where strict inequality follows from (4.11).

Next, let $(t_n, s_n)_n$ be a sequence in D such that :

$$(t_n, s_n) \longrightarrow (t, s) \quad \text{and} \quad v(t_n, s_n) \longrightarrow v^*(t, s) . \tag{4.18}$$

Using the fact that $v \le v^*$ together with the smoothness of φ, we may assume that the sequence $(t_n, s_n)_n$ satisfies the additional requirement

$$v(t_n, s_n) \;\ge\; e^{-\beta}\varphi(t_n, s_n) . \tag{4.19}$$

For ease of notation, we denote $S_n(\cdot) := S_{t_n,s_n}(\cdot)$, and we introduce the stopping times

$$\theta_n \;:=\; \inf\{r \ge t_n \;:\; (r, S_n(r)) \notin D\} .$$

Observe that, by continuity of the process S_n,

$$(\theta_n, S_n(\theta_n)) \in \partial D \quad \text{so that} \quad v^*\left(\theta_n, S_n(\theta_n)\right) \;\le\; e^{-2\beta}\varphi\left(\theta_n, S_n(\theta_n)\right) , \tag{4.20}$$

as a consequence of (4.17).

2. Let ν be any control process in $\mathcal{D}$. Since $v \le v^*$, it follows from (4.19) and (4.20) that :

$$\begin{aligned} v\left(\theta_n, S_n(\theta_n)\right)\gamma_\nu(t_n, \theta_n) - v(t_n, s_n) \;\le\;& e^{-2\beta}\left(1 - e^{\beta}\right)\varphi(t_n, s_n) \\ & + e^{-2\beta}\left(\varphi\left(\theta_n, S_n(\theta_n)\right)\gamma_\nu(t_n, \theta_n) - \varphi(t_n, s_n)\right) . \end{aligned}$$

We now apply Itô's lemma, and use the definition of θ_n together with (4.15). The result is :

$$\begin{aligned} v\left(\theta_n, S_n(\theta_n)\right)\gamma_\nu(t_n, \theta_n) - v(t_n, s_n) \;=\;& e^{-2\beta}\left(1 - e^{\beta}\right)\varphi(t_n, s_n) \\ & + e^{-2\beta}\int_{t_n}^{\theta_n} \gamma_\nu(t_n, r)(-H^{\nu(r)} + \mathcal{L})\varphi(r, S_n(r))dr \\ & + e^{-2\beta}\int_{t_n}^{\theta_n} \gamma_\nu(t_n, r)(D\varphi'\mathrm{diag}[s]\sigma)(r, S_n(r))dB_\nu(r) \\ \le\;& e^{-2\beta}\left(1 - e^{\beta}\right)\varphi(t_n, s_n) \\ & + e^{-2\beta}\int_{t_n}^{\theta_n} (D\varphi'\mathrm{diag}[s]\sigma)(r, S_n(r))dB_\nu(r) . \end{aligned}$$

We finally take expected values under P_ν, and use the arbitrariness of $\nu \in \mathcal{D}$ to see that :

$$\begin{aligned} v(t_n, s_n) \;\le\;& \sup_{\nu\in\mathcal{D}} E_\nu\left[v\left(\theta_n, S_n(\theta_n)\right)\gamma_\nu(t_n, \theta_n)|\mathcal{F}(t_n)\right] + e^{-2\beta}\left(1 - e^{\beta}\right)\varphi(t_n, s_n) \\ <\;& \sup_{\nu\in\mathcal{D}} E_\nu\left[v\left(\theta_n, S_n(\theta_n)\right)\gamma_\nu(t_n, \theta_n)|\mathcal{F}(t_n)\right] , \end{aligned}$$

which is in contradiction with the dynamic programming principle of Lemma 3.1.
□

4.4 Terminal condition

From the definition of the value function v, we have :

$$v(T, s) \ = \ g(s) \quad \text{for all} \quad s \in \mathbb{R}_+^d \ .$$

However, we are facing a singular stochastic control problem, as the controls ν are valued in an unbounded set. Typically this situation induces a *jump* in the terminal condition so that we only have :

$$v_*(T, s) \ \geq \ v(T, s) \ = \ g(s) \ .$$

The main difficulty is then to characterize the terminal condition of interest $v_*(T, \cdot)$. The purpose of this section is to prove that $v_*(T, \cdot)$ is related to the function $\hat{g}$ defined in 4.1.

We first start by deriving the PDE satisfied by $v_*(T, \cdot)$, as inherited from Proposition 4.1.

Proposition 4.3 *Suppose that g is lower semi-continuous and v is locally bounded. Then $v_*(T, \cdot)$ is a viscosity super-solution of*

$$\min \left\{ v_* - g \ , \ \inf_{y \in \tilde{K}} \ H^y v_* \right\} (T, \cdot) \ .$$

Proof. 1. We first check that $v_*(T, \cdot) \geq g$. Let $(t_n, s_n)_n$ be a sequence of $[0, T) \times (0, \infty)^d$ converging to (T, s), and satisfying $v(t_n, s_n) \longrightarrow v_*(T, s)$. Since $\delta(0) = 0$, it follows from the definition of v that

$$v(t_n, s_n) \ \geq \ E_0 \left[g \left(S_{t_n, s_n}(T) \right) \right] \ .$$

Since $g \geq 0$, we may apply Fatou's lemma, and derive the required inequality using the lower semi-continuity condition on g, together with the continuity of $S_{t,s}(T)$ in (t, s).

2. It remains to prove that $v_*(T, \cdot)$ is a viscosity super-solution of

$$H^y v_*(T, \cdot) \ \geq \ 0 \quad \text{for all} \quad y \in \tilde{K} \ . \tag{4.21}$$

Let $f \leq v_*(T, \cdot)$ be a C^2 function satisfying, for some $s_0 \in (0, \infty)^d$,

$$0 \ = \ (v_*(T, \cdot) - f)(s_0) \ \ = \ \ \min_{\mathbb{R}_+^d} \left(v_*(T, \cdot) - f \right) \ .$$

Since v_* is lower semi-continuous, we have $v_*(T, s_0) = \liminf_{(t,s)\to(T,s_0)} v_*(t, s)$, and

$$v_*(T_n, s_n) \longrightarrow v_*(T, s_0) \quad \text{for some sequence} \quad (T_n, s_n) \longrightarrow (T, s_0) .$$

Define

$$\varphi_n(t, s) \ := \ f(s) - \frac{1}{2}|s - s_0|^2 + \frac{T - t}{T - T_n} ,$$

let $\bar{B} = \{s \in I\!R^d_+ \ : \ \sum_i |\ln(s^i/s^i_0)| \le 1\}$, and choose $(\bar{t}_n, \bar{s}_n)$ such that :

$$(v_* - \varphi_n)(\bar{t}_n, \bar{s}_n) \ = \ \min_{[T_n,T]\times\bar{B}} (v_* - \varphi_n) .$$

We shall prove the following claims :

$$\bar{t}_n < T \text{ for large n} , \tag{4.22}$$

$$\bar{s}_n \ \longrightarrow \ s_0 \ \text{ along some subsequence, and } \ v_*(\bar{t}_n, \bar{s}_n) \ \longrightarrow \ v_*(T, s_0) . \tag{4.23}$$

Admitting this, wee see that, for sufficiently large n, $(\bar{t}_n, \bar{s}_n)$ is a local minimizer of the difference $(v_* - \varphi_n)$. Then, the viscosity super-solution property, established in Proposition 4.1, holds at $(\bar{t}_n, \bar{s}_n)$, implying that $H^y v_*(\bar{t}_n, \bar{s}_n) \ge 0$, i.e.

$$\delta(y) v_*(\bar{t}_n, \bar{s}_n) - y' \mathrm{diag}[s] \left(Df(\bar{s}_n) - (\bar{s}_n - s_0)\right) \ \ge \ 0 \quad \text{for all} \quad y \in \tilde{K}_1 ,$$

by definition of φ_n in terms of f. In view of (4.23), this provides the required inequality (4.21).

Proof of (4.22) : Observe that for all $s \in \bar{B}$,

$$(v_* - \varphi_n)(T, s) \ = \ v_*(T, s) - f(s) + \frac{1}{2}|s - s_0|^2 \ \ \ge \ \ v_*(T, s) - f(s) \ \ge \ 0 .$$

Then, the required result follows from the fact that :

$$\begin{aligned} \lim_{n\to\infty} (v_* - \varphi_n)(T_n, s_n) \ &= \ \lim_{n\to\infty} \left\{ v_*(T_n, s_n) - f(s_n) + \frac{1}{2}|s_n - s_0|^2 - \frac{1}{T - T_n} \right\} \\ &= \ -\infty . \end{aligned}$$

Proof of (4.23) : Since $(\bar{s}_n)_n$ is valued in the compact subset $\bar{B}$, we have $\bar{s}_n \longrightarrow \bar{s}$ along some subsequence, for some $\bar{s} \in \bar{B}$. We now use respectively the following facts : s_0 minimizes the difference $v_*(T, \cdot) - f$, v_* is lower semi-continuous, $s_n \longrightarrow s_0$, $\bar{t}_n \ge T_n$, and $(\bar{t}_n, \bar{s}_n)$ minimizes the difference $v_* - \varphi_n$ on $[T_n, T] \times \bar{B}$. The result is :

$$0 \ \le \ (v_*(T, \cdot) - f)(\bar{s}) - (v_*(T, \cdot) - f)(s_0)$$

$$
\begin{aligned}
&\le \liminf_{n\to\infty} \left\{(v_* - \varphi_n)(\bar t_n, \bar s_n) - \frac{1}{2}|\bar s_n - s_0|^2 \right. \\
&\qquad\qquad \left. -(v_* - \varphi_n)(T_n, s_n) + \frac{1}{2}|s_n - s_0|^2 - \frac{\bar t_n - T_n}{T - T_n}\right\} \\
&\le -\frac{1}{2}|\bar s - s_0|^2 + \liminf_{n\to\infty} \ \{(v_* - \varphi_n)(\bar t_n, \bar s_n) - (v_* - \varphi_n)(T_n, s_n)\} \\
&\le -\frac{1}{2}|\bar s - s_0|^2 + \limsup_{n\to\infty} \ \{(v_* - \varphi_n)(\bar t_n, \bar s_n) - (v_* - \varphi_n)(T_n, s_n)\} \\
&\le -\frac{1}{2}|\bar s - s_0|^2 \ \le \ 0 \ ,
\end{aligned}
$$

so that all above inequalities hold with equality, and (4.23) follows. □

We are now able to characterize a precise bound on the terminal condition of the singular stochastic control problem $v(t,s)$.

Proposition 4.4 *Suppose that g is lower semi-continuous and v is locally bounded. Then $v_*(T,\cdot) \ge \hat g$.*

Proof. We first change variables by setting $F(x) := \ln[v_*(T, e^x)]$, with $e^x := (e^{x^1}, \ldots, e^{x^d})'$. From Proposition 4.3, it follows that F satisfies :

$$
\delta(y) - y'DF \ \ge \ 0 \quad \text{for all} \quad y \in \tilde K \ .
$$

Introducing the lower semi-continuous function $h(t) := F(x + ty) - \delta(y)t$, for fixed $x \in I\!R^d$ and $y \in \tilde K$, we see that h is a viscosity super-solution of the equation $-h' \ge 0$. By classical result in the theory of viscosity solutions, we conclude that h is non-increasing. In particular $h(0) \ge h(1)$, i.e. $F(x) \ge F(x+y) - \delta(y)$ for all $x \in I\!R^d$ and $y \in \tilde K$, and

$$
\begin{aligned}
F(x) \ &\ge \ \sup_{y\in\tilde K} \ \{F(x+y) - \delta(y)\} \\
&\ge \ \sup_{y\in\tilde K} \ \ln\left\{g\left(e^{x+y}\right) e^{-\delta(y)}\right\} \quad \text{for all} \quad x \in I\!R^d \ .
\end{aligned}
$$

This provides the required result by simply turning back to the initial variables. □

We now intend to provide the reverse inequality of Proposition 4.4. In order to simplify the presentation, we shall provide an easy proof which requires strong assumptions.

Proposition 4.5 *Let σ be a bounded function, and $\hat g$ be an upper semi-continuous function with linear growth. Suppose that v is locally bounded. Then $v^*(T,\cdot) \le \hat g$.*

Proof. Suppose to the contrary that $V^*(T,s) - \hat{g}(s) =: 2\eta > 0$ for some $s \in (0,\infty)^d$. Let (T_n, s_n) be a sequence in $[0,T] \times (0,\infty)^d$ satisfying :

$$(T_n, s_n) \longrightarrow (T, s)\ , \qquad V(T_n, s_n) \longrightarrow V^*(T,s)$$

and

$$V(T_n, s_n) \ > \ \hat{g}(s) + \eta \quad \text{for all} \quad n \geq 1\ .$$

From the (dual) definition of V, this shows the existence of a sequence $(\nu^n)_n$ in $\mathcal{D}$ such that :

$$E^0_{T_n,s_n}\left[g\left(S_T^{(n)} e^{\int_{T_n}^T \nu_r^n dr}\right) e^{-\int_{T_n}^T \delta(\nu_r^n) dr}\right] \ > \ \hat{g}(s) + \eta \quad \text{for all} \quad n \geq 1\ , \tag{4.24}$$

where

$$S_T^{(n)} \ := \ s_n \mathcal{E}\left(\int_{T_n}^T \sigma(t, S_t^{\nu^n}) dW_t\right)\ .$$

We now use the sublinearity of δ to see that :

$$\begin{aligned}
&E^0_{T_n,s_n}\left[g\left(S_T^{(n)} e^{\int_{T_n}^T \nu_r^n dr}\right) e^{-\int_{T_n}^T \delta(\nu_r^n) dr}\right] \\
\leq\ &E^0_{T_n,s_n}\left[g\left(S_T^{(n)} e^{\int_{T_n}^T \nu_r^n dr}\right) e^{-\delta(\int_{T_n}^T \nu_r^n dr)}\right] \\
&\leq\ E^0_{T_n,s_n}\left[\hat{g}\left(S_T^{(n)}\right)\right]\ ,
\end{aligned}$$

where we also used the definition of $\hat{g}$ together with the fact that $\tilde{K}$ is a closed convex cone of $\mathbb{R}^d$. Plugging this inequality in (4.24), we see that

$$\hat{g}(s) + \eta \ \leq \ E^0_{t_n,s_n}\left[\hat{g}\left(S_T^{(n)}\right)\right]\ . \tag{4.25}$$

By easy computation, it follows from the linear growth condition on $\hat{g}$ that

$$E^0\left|\hat{g}(S_T^{(n)})\right|^2 \ \leq \ Const\ \left(1 + e^{(T-t)\|\sigma\|^2_\infty}\right)\ .$$

This shows that the sequence $\left\{\hat{g}(S_T^{(n)}),\ n \geq 1\right\}$ is bounded in $L^2(P^0)$, and is therefore uniformly integrable. We can therefore pass to the limit in (4.25) by means of the dominated convergence theorem. The required contradiction follows from the upper semi-continuity of $\hat{g}$ together with the a.s. continuity of S_T in the initial data (t,s). $\square$

5 Applications

5.1 The Black-Scholes model with portfolio constraints

In this paragraph we report an explicit solution of the super-replication problem under portfolio constraints in the context of the Black-Scholes model. This result was obtained by Broadie, Cvitanić and Soner (1997).

Proposition 5.1 *Let $d = 1$, $\sigma(t,s) = \sigma > 0$, and consider a lower semi-continuous payoff function $g \; : I\!\!R_+ \longrightarrow I\!\!R$. Assume that the face-lifted payoff function $\hat{g}$ is upper semi-continuous and has linear growth. Then :*

$$v(t,s) \;\; = \;\; E_0\left[\hat{g}\left(S_{t,s}(T)\right)\right] \;,$$

i.e. $v(t,s)$ is the unconstrained Black-and Scholes price of the face-lifted contingent claim $\hat{g}\left(S_{t,s}(T)\right)$.

Proof. By a direct application of Theorem 4.1, the value function $v(t,s)$ solves the PDE (4.2). When σ is constant, it follows from the maximum principle that the PDE (4.2) reduces to :

$$-\mathcal{L}v \; = \; 0 \;\; \text{on} \;\; [0,T) \times (0,\infty)^d \;, \qquad v(T,\cdot) \; = \; \hat{g} \;,$$

and the required result follows from the Feynman-Kac representation formula. $\Box$

5.2 The uncertain volatility model

In this paragraph, we study the simplest incomplete market model. The number of risky assets is now $d = 2$. We consider the case $K = I\!\!R \times \{0\}$ so that the second risky asset is not tradable. In order to satisfy the conditions of Theorem 4.1, we assume that the contingent claim is defined by $G = g(S^1(T))$, where the payoff function $g : I\!\!R_+ \longrightarrow I\!\!R_+$ is continuous and has polynomial growth. We finally introduce the notations :

$$\overline{\sigma}(t,s_1) \; := \; \sup_{s_2>0} \, [\sigma_{11}^2 + \sigma_{12}^2](t,s_1,s_2) \quad \text{and} \quad \underline{\sigma}(t,s_1) \; := \; \inf_{s_2>0} \, [\sigma_{11}^2 + \sigma_{12}^2](t,s_1,s_2)$$

We report the following result from Cvitanić, Pham and Touzi (1999) which follows from Theorem 4.1. We leave its verification for the reader.

Proposition 5.2 (i) *Assume that $\overline{\sigma} < \infty$ on $[0,T] \times I\!R_+$. Then $v(t,s) = v(t,s_1)$ solves the Black-Scholes-Barrenblatt equation*

$$-v_t - \frac{1}{2}\left[\overline{\sigma}^2 v^+_{s_1 s_1} - \underline{\sigma}^2 v^-_{s_1 s_1}\right] = 0, \text{ on } [0,T) \times (0,\infty), \qquad v(T,s_1) = g(s_1) \text{ for } s_1 > 0.$$

(ii) *Assume that $\overline{\sigma} = \infty$ and*

$$\text{either } g \text{ is convex} \quad \text{or} \quad \underline{\sigma} = 0 .$$

Then $v(t,s) = g^{conc}(s_1)$, where g^{conc} is the concave envelope of g.

6 HJB equation from the primal problem for the general large investor problem

In this section, we present an original dynamic programming principle stated directly on the initial formulation of the super-replication problem. We then prove that the HJB equation of Theorem 4.1 can be obtained by means of this dynamic programming principle. Hence, if one is only interested in the derivation of the HJB equation, then the dual formulation is not needed any more.

An interesting feature of the analysis of this section is that it can be extended to problems where the dual formulation is not available. A first example is the large investor problem with feedback of the investor's portfolio on the price process through the drift and the volatility coefficients; see section 3.4.1. This problem is treated in the present section and does not present any particular difficulty. Another example is the problem of super-replication under gamma constraints which will be discussed in the last section of these notes. In an accompanying paper, the analysis of this section will be further extended to cover front propagation problems related to the theory of differential geometry.

6.1 Dynamic programming principle

In this section, the price process S is defined by the controlled stochastic differential equation :

$$\begin{aligned} S^\pi_{t,s}(t) &= s \\ dS^\pi_{t,s}(r) &= \operatorname{diag}[S^\pi_{t,s}](r)\left[b(r, S^\pi_{t,s}(r), \pi(r))dr + \sigma(r, S^\pi_{t,s}(r), \pi(r))dB(r)\right] \ , \end{aligned}$$

where $b : [0,T] \times I\!R^d \times K \longrightarrow I\!R^d$ and $\sigma : [0,T] \times I\!R^d \times K \longrightarrow I\!R^{d\times d}$ are bounded functions, Lipschitz in the s variable uniformly in (t,π).

The wealth process is defined by

$$\begin{aligned} W^{\pi}_{t,s,w}(t) &= w \\ dW^{\pi}_{t,s,w}(r) &= W^{\pi}_{t,s,w}(r)\pi(r)' \left[b(r, S^{\pi}_{t,s}(r), \pi(r))dr + \sigma(r, S^{\pi}_{t,s}(r), \pi(r))dB(r)\right] . \end{aligned}$$

Given a European contingent claim $G = g(S_{t,s}(T)) \geq 0$, the super-replication problem is defined by

$$v(t,s) \;:=\; \inf \left\{ w > 0 \;:\; W^{\pi}_{t,s,w}(T) \;\geq\; g\left(S^{\pi}_{t,s}(T)\right) \; P-\text{a.s. for some } \pi \in \mathcal{A}_K \right\} .$$

The main result of this section is the following.

Lemma 6.1 *Let $(t,s) \in [0,T) \times (0,\infty)^d$ be fixed, and consider an arbitrary stopping time θ valued in $[t,T]$. Then,*

$$v(t,s) \;=\; \inf \left\{ w > 0 \;:\; W^{\pi}_{t,s,w}(\theta) \;\geq\; v\left(\theta, S^{\pi}_{t,s}(\theta)\right) \; P-\textit{a.s. for some } \pi \in \mathcal{A}_K \right\} .$$

In order to derive the HJB equation by means of the above dynamic programming principle, we shall write it in the following equivalent form :

DP1 Let $(t,s) \in [0,T) \times (0,\infty)^d$, $(w,\pi) \in (0,\infty) \times \mathcal{A}_K$ be such that $W^{\pi}_{t,s,w}(T) \geq g\left(S^{\pi}_{t,s}(T)\right)$ $P-$a.s. Then

$$W^{\pi}_{t,s,w}(\theta) \;\geq\; v\left(\theta, S^{\pi}_{t,s}(\theta)\right) \qquad P-\text{a.s.}$$

DP2 Let $(t,s) \in [0,T) \times (0,\infty)^d$, and set $\hat{w} := v(t,s)$. then for all stopping time θ valued in $[t,T]$,

$$P\left[W^{\pi}_{t,s,\hat{w}-\eta}(\theta) \;>\; v\left(\theta, S^{\pi}_{t,s}(\theta)\right)\right] \;<\; 1 \quad \text{for all} \quad \eta > 0 \text{ and } \pi \in \mathcal{A}_K .$$

Idea of the proof. The proof of DP1 follows from the trivial observation that

$$S^{\pi}_{t,s}(T) \;=\; S^{\pi}_{\theta, S^{\pi}_{t,s}(\theta)}(T) \quad \text{and} \quad W^{\pi}_{t,s,w}(T) \;=\; W^{\pi}_{\theta, S_{t,s}(\theta), W_{t,s,w}(\theta)}(T) .$$

The proof of DP2 is more technical, we only outline the main idea. Let θ be some stopping time valued in $[t,T]$, and suppose that

$$\hat{W} \;:=\; W^{\pi}_{t,s,\hat{w}-\eta}(\theta) \;>\; v\left(\theta, S^{\pi}_{t,s}(\theta)\right) \quad P-\text{a.s. for some} \quad \eta > 0 \text{ and } \pi \in \mathcal{A}_K .$$

Set $\hat{S} := S^{\pi}_{t,s}(\theta)$. By definition of the super-replication problem $v\left(\theta, \hat{S}\right)$ starting at time θ, there exists an admissible portfolio $\hat{\pi} \in \mathcal{A}_K$ such that

$$W^{\hat{\pi}}_{\theta,\hat{S},\hat{W}}(T) \;\; \geq \;\; g\left(S^{\hat{\pi}}_{\theta,\hat{S}}(T)\right) \quad P-\text{a.s.}$$

This is the delicate place of this proof, as one has to appeal to a measurable selection argument in order to define the portfolio $\hat{\pi}$. In order to complete the proof, it suffices to define the admissible portfolio $\tilde{\pi} := \pi \mathbf{1}_{[t,\theta)} + \hat{\pi}\mathbf{1}_{[\![\theta,T]\!]}$, and to observe that :

$$W^{\tilde{\pi}}_{t,s,\hat{w}-\eta}(T) \;\; = \;\; W^{\hat{\pi}}_{\theta,\hat{S},\hat{W}}(T) \;\; \geq \;\; g\left(S^{\hat{\pi}}_{\theta,\hat{S}}(T)\right) \;\; = \;\; g\left(S^{\tilde{\pi}}_{t,s}(T)\right) \quad P-\text{a.s.}$$

which is in contradiction with the definition of $\hat{w}$. □

6.2 Supersolution property from DP1

In this paragraph, we prove the following result.

Proposition 6.1 *Suppose that the value function v is locally bounded, and the constraint set K is compact. Then v is a discontinuous viscosity supersolution of the PDE*

$$\min\left\{ -\hat{\mathcal{L}}v_* \;,\; \inf_{y\in\tilde{K}} \; H^y v_* \right\}(t,s) \;\; \geq \;\; 0$$

where

$$\begin{aligned} \hat{\mathcal{L}}u \;\; &:= \;\; u_t(t,s) + \frac{1}{2}\text{Tr}\left[\text{diag}[s]\sigma\sigma'(t,s,\hat{\alpha})\text{diag}[s]D^2u(t,s)\right] \\ \hat{\alpha} \;\; &:= \;\; \text{diag}[s]\frac{Du}{u}(t,s) \;. \end{aligned}$$

The compactness condition on the constraints set K is assumed in order to simplify the proof. In the case of a *small* investor model (b and σ do not depend on π), the following argument is an alternative proof of Proposition 4.1 which uses part DP1 of the above direct dynamic programming principle of Lemma 6.1, instead of the classical dynamic programming principle of Lemma 3.1 stated on the dual formulation of the problem. Recall that such a dual formulation is not available in the context of a general large investor model with σ depending on π.

Proof. Let (t,s) be fixed in $[0,T) \times \mathbb{R}^d_+$, and consider a smooth test function φ, mapping $[0,T] \times \mathbb{R}^d_+$ into $(0,\infty)$, and satisfying

$$0 \;\; = \;\; (v_* - \varphi)(t,s) \;\; = \;\; \min_{[0,T]\times\mathbb{R}^d_+} \; (v_* - \varphi) \;. \tag{6.1}$$

Let $(t_n, s_n)_{n\geq 0}$ be a sequence of elements of $[0,T)\times I\!R^d_+$ satisfying

$$v(t_n,s_n) \longrightarrow v_*(t,s) \quad \text{as} \quad n\to\infty\ .$$

We now set

$$w_n \;:=\; v(t_n,s_n)+\max\{n^{-1}\ ,\ 2|v(t_n,s_n)-\varphi(t_n,s_n)|\} \;\text{ for all } n\geq 0\ ,$$

and observe that

$$c_n \;:=\; \ln w_n-\ln\varphi(t_n,s_n) \;>\; 0 \quad\text{and}\quad c_n \;\longrightarrow\; 0 \text{ as } n\to\infty\ , \tag{6.2}$$

in view of (6.1). Since $w_n > v(t_n,s_n)$, it follows from DP1 that :

$$\ln\left\{v\left(\theta_n, S^{\pi_n}_{t_n,s_n}(\theta_n)\right)\right\} \;\leq\; \ln\left\{W^{\pi_n}_{t_n,s_n,w_n}(\theta_n)\right\} \quad\text{for some}\quad \pi_n\in\mathcal{A}_K\ ,$$

where, for each $n\geq 0$, θ_n is an arbitrary stopping time valued in $[t_n,T]$, to be chosen later on. In the rest of this proof, we simply denote $(S_n,W_n) := \left(S^{\pi_n}_{t_n,s_n}, W^{\pi_n}_{t_n,s_n,w_n}\right)$.

By definition of the test function φ, we have $v\geq v_*\geq\varphi$, and therefore :

$$\begin{aligned} 0 \;\leq\;& c_n+\ln\varphi(t_n,s_n)-\ln\varphi\left(\theta_n,S_n(\theta_n)\right)\\ &+\int_{t_n}^{\theta_n}\left[\pi_n' b-\frac{1}{2}\left|\sigma'\pi_n\right|^2\right](r,S_n(r),\pi_n(r))\,dr\\ &+\int_{t_n}^{\theta_n}\left(\pi_n'\sigma\right)(r,S_n(r),\pi_n(r))\,dB(r)\ . \end{aligned}$$

Applying Itô's lemma to the smooth test function $\ln\varphi$, we see that :

$$\begin{aligned} 0 \;\leq\;& c_n+\int_{t_n}^{\theta_n}(\pi_n-h_n)'\sigma\left(r,S_n(r),\pi_n(r)\right)dB(r)\\ &-\int_{t_n}^{\theta_n}\left[\frac{\mathcal{L}^{\pi_n(r)}\varphi}{\varphi}-(\pi_n-h_n)'b+\frac{|\sigma' h_n|^2-|\sigma'\pi_n|^2}{2}\right](r,S_n(r),\pi_n(r))\,dr\ , \end{aligned} \tag{6.3}$$

where we introduced the additional notations

$$h_n(r) \;:=\; \mathrm{diag}[S_n(r)]\frac{D\varphi}{\varphi}\;(r,S_n(r))\ ,$$

and

$$\mathcal{L}^\alpha\varphi(t,s) \;:=\; \varphi_t(t,s)+\frac{1}{2}\mathrm{Tr}\left\{\mathrm{diag}[s]\sigma\sigma'(t,s,\alpha)\mathrm{diag}[s]D^2\varphi(t,s)\right\}\ .$$

2. Given a positive integer k, we define the equivalent probability measure P^k_n by :

$$\left.\frac{dP^k_n}{dP}\right|_{\mathcal{F}(t_n)} \;:=\; \mathcal{E}\left(\int_{t_n}^{T}\sigma^{-1}\left[-b+k\frac{\pi_n-h_n}{|\pi_n-h_n|}1_{\{\pi_n-h_n\neq 0\}}\right](r,S_n(r))\,dB(r)\right)\ ;$$

recall that σ^{-1} and b are bounded functions. Taking expected values in (6.3) under P_n^k, conditionally on $\mathcal{F}(t_n)$, we see that :

$$0 \;\le\; c_n - E_k \int_{t_n}^{\theta_n} \left[\frac{\mathcal{L}^{\pi_n(r)}\varphi}{\varphi} + k|\pi_n - h_n| + \frac{|\sigma' h_n|^2 - |\sigma'\pi_n|^2}{2} \right] (r, S_n(r), \pi_n(r))\, dr \;,$$

where E_k denote the conditional expectation operator under P_n^k. We now devide by $\sqrt{c_n}$ and send n to infinity to get :

$$0 \;\le\; \liminf_n E_k \int_{t_n}^{\theta_n} \left[\frac{-\mathcal{L}^{\pi_n(r)}\varphi}{\varphi} - k|\pi_n - h_n| - \frac{|\sigma' h_n|^2 - |\sigma'\pi_n|^2}{2} \right] (r, S_n(r), \pi_n(r))\, dr \;, \tag{6.4}$$

3. We now fix the stopping times :

$$\theta_n \;:=\; \sqrt{c_n} \wedge \inf \left\{ r \ge t_n \;:\; S_n(r) \notin s_n e^{\gamma B} \right\} \;,$$

for some constant γ, where B is the unit closed ball of $\mathbb{R}^1$ and $s_n e^{\delta B}$ is the collection of all points ξ in $\mathbb{R}^d$ such that $|\ln(\xi^i/s_n^i)| \le \gamma$. Then, by dominated convergence, it follows from (6.4) that :

$$0 \;\le\; \inf_{\alpha \in K} \; -\frac{\mathcal{L}^{\alpha}\varphi}{\varphi} - k \left| \alpha - \mathrm{diag}[s] \frac{D\varphi}{\varphi}(t,s) \right| + \frac{1}{2} \left[\left| \sigma' \mathrm{diag}[s] \frac{D\varphi}{\varphi}(t,s) \right|^2 - |\sigma'\alpha|^2 \right] \;.$$

Since K is compact, the above minimum is attained at some $\alpha_k \in K$, and $\alpha_k \longrightarrow \hat{\alpha} \in K$ along some subsequence. Sending k to infinity, we see that one must have

$$\hat{\alpha} \;=\; \mathrm{diag}[s] \frac{D\varphi}{\varphi}(t,s) \;\in\; K \quad \text{and} \quad -\mathcal{L}^{\hat{\alpha}}\varphi \;\ge\; 0 \;.$$

□

6.3 Subsolution property from DP2

We now show that part DP2 of the dynamic programming principle stated in Lemma 6.1 allows to prove an extension of the subsolution property of Proposition 4.2 in the general large investor model. To simplify the presentation, we shall only work out the proof in the case where K has non-empty interior; the general case can be addressed by the same argument as in section 4.3, i.e. by first deducing, from the supersolution property, that $v(t, e^x)$ depends only on the projection x_K of x on $\mathrm{aff}(K) = \mathrm{vect}(K)$.

Proposition 6.2 *Suppose that the value function v is locally bounded, and the constraints set K has non-empty interior. Then, v is a discontinuous viscosity subsolution of the PDE*

$$\min\left\{ -\hat{\mathcal{L}}v^* \ , \ \inf_{y\in\tilde{K}_1} \ H^y v^* \right\}(t,s) \ \geq \ 0$$

Proof. 1. In order to simplify the presentation, we shall pass to the log-variables. Set $x := \ln w$, $X^\pi_{t,s,x} := \ln W^\pi_{t,s,w}$, and $u := \ln v$. By Itô's lemma, the controlled process X^π is given by :

$$\begin{aligned} X^\pi_{t,s,x}(\tau) &= x + \int_t^\tau \left(\pi' b - \frac{1}{2}|\sigma'\pi|^2\right)\left(r, S^\pi_{t,s}(r), \pi(r)\right) dr \\ &\quad + \int_t^\tau (\pi'\sigma)\left(r, S^\pi_{t,s}(r), \pi(r)\right) dB(r) \ . \end{aligned} \tag{6.5}$$

With this change of variable, Proposition 6.2 states that u^* satisfies on $[0,T) \times (0,\infty)^d$ the equation :

$$\min\left\{ -\mathcal{G}u^* \ , \ \inf_{y\in\tilde{K}_1} \ (\delta(y) - \text{diag}[s]Du^*(t,s)) \right\} \ \leq \ 0$$

in the viscosity sense, where

$$\begin{aligned} \mathcal{G}u^*(t,s) &:= u^*_t(t,s) + \frac{1}{2}\left|\sigma'(t,s,\hat{\alpha})\text{diag}[s]Du^*(t,s)\right|^2 \\ &\quad + \frac{1}{2}\text{Tr}\left[\text{diag}[s]\sigma\sigma'(t,s,\hat{\alpha})\text{diag}[s]D^2u^*(t,s)\right] \ , \\ \text{and} \ \hat{\alpha} &:= \text{diag}[s]Du^*(t,s) \ . \end{aligned}$$

2. We argue by contradiction. Let $(t_0,s_0) \in [0,T) \times (0,\infty)^d$ and a C^2 $\varphi(t,s)$ be such that :

$$0 \ = \ (w^* - \varphi)(t_0,s_0) \ > \ (w^* - \varphi)(t,s) \quad \text{for all} \quad (t,s) \neq (t_0,s_0) \ ,$$

and suppose that

$$\mathcal{G}\varphi(t,s) \ > \ 0 \quad \text{and} \quad \text{diag}[s_0]D\varphi(t_0,s_0) \ \in \ \text{int}(K) \ .$$

Set $\hat{\pi}(t,s) := \text{diag}[s]D\varphi(t,s)$. Let $0 < \alpha < T - t_0$ be an arbitrary scalar and define the neighborhood of (t_0,s_0) :

$$\mathcal{N} \ := \ \left\{(t,s) \in (t_0-\alpha, t_0+\alpha) \times s_0 e^{\alpha B} \ : \ \hat{\pi}(t,s) \in K \ \text{and} \ -\mathcal{G}\varphi(t,s) \geq 0\right\} \ ,$$

where B is again the unit closed ball of $I\!R^1$ and $s_0 e^{\alpha B}$ is the collection of all points $\xi \in I\!R^d$ such that $|\ln(\xi^i/s_0^i)| \leq \gamma$. Since (t_0, s_0) is a strict maximizer of $(u^* - \varphi)$, observe that :

$$-4\beta \;:=\; \max_{\partial\mathcal{N}} (u^* - \varphi) \;<\; 0\,.$$

3. Let (t_1, s_1) be some element in $\mathcal{N}$ such that :

$$x_1 \;:=\; u(t_1, s_1) \;\geq\; u^*(t_0, s_0) - \beta \;=\; \varphi(t_0, s_0) - \beta \;\geq\; \varphi(t_1, s_1) - 2\beta\,,$$

and consider the controlled process

$$(\hat{S}, \hat{X}) \;:=\; \left(S^{\hat{\pi}}_{t_1, s_1}, \ln W^{\hat{\pi}}_{t_1, s_1, w_1 e^{-\beta}}\right) \text{ with feedback control } \hat{\pi}(r) \;:=\; \hat{\pi}\left(r, \hat{S}(r)\right)\,,$$

which is well-defined at least up to the stopping time

$$\theta \;:=\; \inf\left\{r > t_0 \;:\; \left(r, \hat{S}(r)\right) \not\in \mathcal{N}\right\}\,.$$

Observe that $u \leq u^* \leq \varphi - 3\beta$ on $\partial\mathcal{N}$ by continuity of the process $\hat{S}$. We then compute that :

$$\begin{aligned} x_1 - \beta - u\left(\theta, \hat{S}(\theta)\right) \;&\geq\; u(t_1, s_1) - 3\beta - \varphi\left(\theta, \hat{S}(\theta)\right) \\ &\geq\; \beta + \varphi(t_1, s_1) - \varphi\left(\theta, \hat{S}(\theta)\right)\,. \end{aligned} \tag{6.6}$$

4. From the definition of $\hat{\pi}$, the diffusion term of the difference $d\hat{X}(r) - d\varphi(r, \hat{S}(r))$ vanishes up to the stopping time θ. It then follows from (6.6), Itô's lemma, and (6.5) that

$$\begin{aligned} \hat{X}(\theta) - u\left(\theta, \hat{S}(\theta)\right) \;&\geq\; \beta + \int_{t_1}^{\theta} d\hat{X}(r) - d\varphi(r, \hat{S}(r)) \\ &=\; \beta + \int_{t_1}^{\theta} \mathcal{G}\varphi(r, \hat{S}(r)) \\ &\geq\; \beta \;>\; 0 \qquad P-\text{a.s.} \end{aligned}$$

where the last inequality follows from the definition of the stopping time θ and the neighborhood $\mathcal{N}$. This proves that $W^{\hat{\pi}}_{t_1, s_1, w_1 e^{-\beta}} > v(\theta, S^{\hat{\pi}}_{t_1, s_1}(\theta)$ $P-$a.s., which is the required contradiction to DP2. □

7 Hedging under Gamma constraints

In this section, we focus on an alternative constraint on the portfolio π. Let $Y := \text{diag}[S(t)]^{-1}\pi(t)W(t)$ denote the vector of number of shares of the risky assets held at each time. By definition of the portfolio strategy, the investor has to adjust his strategy at each time t, by passing the number of shares from $Y(t)$ to $Y(t+dt)$. His demand in risky assets at time t is then given by "$dY(t)$".

In an equilibrium model, the price process of the risky assets would be pushed upward for a large demand of the investor. We therefore study the hedging problem with constrained portfolio adjustment. This problem turns out to present serious mathematical difficulties. The analysis of this section is reported from [19], and provides a solution of the problem in a very specific situation. We hope that this presentation will encourage some readers to attack some of the possible extensions.

7.1 Problem formulation

We consider a financial market which consists of one bank account, with constant price process $S^0(t) = 1$ for all $t \in [0,T]$, and one risky asset with price process evolving according to the Black-Scholes model :

$$S_{t,s}(u) \;:=\; s\mathcal{E}\left(\sigma(B(t) - B(u)\right), \quad t \leq u \leq T.$$

Here B is a standard Brownian motion in $I\!R$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$. We shall denote by $I\!F = \{\mathcal{F}(t),\, 0 \leq t \leq T\}$ the P-augmentation of the filtration generated by B.

Observe that there is no loss of generality in taking S as a martingale, as one can always reduce the model to this case by judicious change of measure (P_0 in the previous sections). On the other hand, the subsequent analysis can be easily extended to the case of a varying volatility coefficient.

We denote by $Y = \{Y(u),\, t \leq u \leq T\}$ the process of number of shares of risky asset S held by the agent during the time interval $[t,T]$. Then, by the self-financing condition, the wealth process induced by some initial capital w and portfolio strategy Y is given by :

$$W(u) \;=\; w + \int_t^u Y(r)dS_{t,s}(r), \quad t \leq u \leq T.$$

In order to introduce constraints on the variations of the hedging portfolio Y, we restrict Y to the class of continuous semimartingales with respect to the filtration $I\!F$. Since $I\!F$ is the Brownian filtration, we define the controlled portfolio strategy $Y^{\alpha,\gamma}_{t,s,y}$ by :

$$Y^{\alpha,\gamma}_{t,y}(u) \;=\; y + \int_t^u \alpha(r)dr + + \int_t^u \gamma(r)\sigma dB(r), \quad t \leq u \leq T, \tag{7.1}$$

where $y \in I\!R$ is the initial portfolio and the *control* pair (α, γ) takes values in

$$\mathcal{B}_t \ := \ \left(L^\infty([t,T] \times \Omega \, ; \, \ell \otimes P) \right)^2 .$$

Hence a *trading strategy* is defined by the triple (y, α, γ) with $y \in I\!R$ and $(\alpha, \gamma) \in \mathcal{B}_t$. The associated wealth process, denoted by $W^{\alpha,\gamma}_{t,w,s,y}$, is given by :

$$W^{\alpha,\gamma}_{t,w,s,y}(u) \ = \ w + \int_t^u Y^{\alpha,\gamma}_{t,y}(r) dS_{t,s}(r), \quad t \le u \le T . \tag{7.2}$$

We now formulate the Gamma constraint in the following way. Let Γ be a constant fixed throughout the paper. Given some initial capital $w > 0$, we define the set of *w-admissible* trading strategies by :

$$\mathcal{A}_{t,s}(w) \ := \ \left\{ (y, \alpha, \gamma) \in I\!R \times \mathcal{B}_t \ : \ \gamma(.) \le \Gamma \ \text{ and } \ W^{\alpha,\gamma}_{t,w,s,y}(.) \ge 0 \right\} .$$

As in the previous sections, We consider the super-replication problem of some European type contingent claim $g(S_{t,s}(T))$:

$$v(t,s) \ := \ \inf \left\{ w \ : \ W^{\alpha,\gamma}_{t,w,s,y}(T) \ge g(S_{t,s}(T)) \ \text{ a.s. for some } (y,\alpha,\gamma) \in \mathcal{A}_{t,s}(w) \right\} . \tag{7.3}$$

7.2 The main result

Our goal is to derive the following explicit solution : $v(t,s)$ is the (unconstrained) Black-Scholes price of some convenient *face-lifted* contingent claim $\hat{g}(S_{t,s}(T))$, where the function $\hat{g}$ is defined by

$$\hat{g}(s) \ := \ h^{conc}(s) + \Gamma s \ln s \quad \text{with} \quad h(s) \ := \ g(s) - \Gamma s \ln s \ ,$$

and h^{conc} denotes the concave envelope of h. Observe that this function can be computed easily. The reason for introducing this function is the following.

Lemma 7.1 *$\hat{g}$ is the smallest function satisfying the conditions*

$$\text{(i)} \ \ \hat{g} \ \ge \ g \ , \quad \textit{and} \quad \text{(ii)} \ \ s \longmapsto \hat{g}(s) - s \ln s \ \textit{ is concave.}$$

The proof of this easy result is omitted.

Theorem 7.1 *Let g be a non-negative lower semi-continuous mapping on $I\!R_+$. Assume further that*

$$s \ \longmapsto \ \hat{g}(s) - C \, s \ln s \quad \textit{is convex for some constant} \ \ C . \tag{7.4}$$

Then the value function (7.3) *is given by :*

$$v(t,s) \ = \ E\left[\hat{g}\left(S_{t,s}(T) \right) \right] \quad \textit{for all} \quad (t,s) \in [0,T) \times (0,\infty) .$$

7.3 Discussion

1. We first make some comments on the model. Formally, we expect the optimal hedging portfolio to satisfy

$$\hat{Y}(u) \quad = \quad v_s(u, S_{t,s}(u)) \ ,$$

where v is the minimal super-replication cost; see Section 3.1. Assuming enough regularity, it follows from Itô's lemma that

$$d\hat{Y}(u) \quad = \quad A(u)du + \sigma(u, S_{t,s}(u))S_{t,s}(u)v_{ss}(u, S_{t,s}(u))dB(u) \ ,$$

where $A(u)$ is given in terms of derivatives of v. Compare this equation with (7.1) to conclude that the associated *gamma* is

$$\hat{\gamma}(u) \quad = \quad S_{t,s}(u) \ v_{ss}(u, S_{t,s}(u)) \ .$$

Therefore the bound on the process $\hat{\gamma}$ translates to a bound on sv_{ss}. Notice that, by changing the definition of the process γ in (7.1), we may bound v_{ss} instead of sv_{ss}. However, we choose to study sv_{ss} because it is a dimensionless quantity, i.e., if all the parameters in the problem are increased by the same factor, sv_{ss} still remains unchanged.

2. Observe that we only require an upper bound on the control γ. The similar problem with a lower bound on γ is still open, and presents some specific difficulties. In particular, it seems that the control $\int_0^t \alpha(r)dr$ has to be relaxed to the class of bounded variation processes...

3. The extension of the analysis of this section to the multi-asset framework is also an open problem. The main point is the extension of Lemma 7.5 below.

4. Intuitively, we expect to obtain a similar type solution to the case of portfolio constraints. If the Black-Scholes solution happens to satisfy the gamma constraint, then it solves the problem with gamma constraint. In this case v satisfies the PDE $-\mathcal{L}v = 0$. Since the Black-Scholes solution does not satisfy the gamma constraint, in general, we expect that function v solves the variational inequality :

$$\min\left\{-\mathcal{L}v, \Gamma - sv_{ss}\right\} \quad = \quad 0 \ . \tag{7.5}$$

5. An important feature of the log-normal Black and Sholes model is that the variational inequality (7.5) reduces to the Black-Scholes PDE $-\mathcal{L}v = 0$ as long as the terminal condition satisfies the gamma constraint (in a weak sense). From Lemma 7.1, the *face-lifted* payoff function $\hat{g}$ is precisely the minimal function above g which satisfies the gamma constraint (in a weak sense). This explains the nature of the solution reported in Theorem 7.1, namely $v(t,s)$ is the Black-Scholes price of the contingent claim $\hat{g}\left(S_{t,s}(T)\right)$.

6. One can easily check formally that the variational inequality (7.5) is the HJB equation associated to the stochastic control problem :

$$\tilde{v}(t,s) \;:=\; \sup_{\nu\in\mathcal{N}} E\left[g\left(S^{\nu}_{t,s}(T)\right) - \frac{1}{2}\int_t^T \nu(t)[S^{\nu}_{t,s}(r)]^2 dr\right] ,$$

where $\mathcal{N}$ is the set of all non-negative, bounded, and $I\!\!F-$ adapted processes, and :

$$S^{\nu}_{t,s}(u) \;:=\; \mathcal{E}\left(\int_t^u [\sigma^2+\nu(r)]^{1/2} dB(r)\right) , \qquad \text{for } t \le u \le T .$$

The above stochastic control problem is a candidate for some dual formulation of the problem $v(t,s)$ defined in (7.3). Observe, however, that the dual variables ν are acting on the diffusion coefficient of the controlled process S^{ν}, so that the change of measure techniques of Section 3 do not help to prove the duality connection between v and $\tilde{v}$.

A direct proof of some duality connection between v and $\tilde{v}$ is again an open problem. In order to obtain the PDE characterization (7.5) of v, we shall make use of the dynamic programming principle stated directly on the initial formulation of the problem v.

7.4 Proof of Theorem 7.1

We shall denote

$$\hat{v}(t,s) \;:=\; E\left[\hat{g}\left(S_{t,s}(T)\right)\right] .$$

It is easy to check that $\hat{v}$ is a smooth function satisfying

$$\mathcal{L}\hat{v} \;=\; 0 \text{ and } s\hat{v}_{ss} \;\le\; \Gamma \quad \text{on} \quad [0,T)\times(0,\infty) . \tag{7.6}$$

1. We start with the inequality $v \le \hat{v}$. For $t \le u \le T$, set

$$y \;:=\; \hat{v}_s(t,s) , \quad \alpha(u) \;:=\; \mathcal{L}\hat{v}_s(u,S(u)) , \quad \gamma(u) \;:=\; S_{t,s}(u)\hat{v}_{ss}(u,S(u)) ,$$

and we claim that

$$(\alpha, \gamma) \ \in \ \mathcal{B}_t \quad \text{and} \quad \gamma \leq \Gamma \, . \tag{7.7}$$

Before verifying this claim, let us complete the proof of the required inequality. Since $g \leq \hat{g}$, we have

$$\begin{aligned} g\left(S_{t,s}(T)\right) \ &\leq \ \hat{g}\left(S_{t,s}(T)\right) \ = \ \hat{v}\left(T, S_{t,s}(T)\right) \\ &= \ \hat{v}(t,s) + \int_t^T \mathcal{L}\hat{v}(u, S_{t,s}(u))du + \hat{v}_s(u, S_{t,s}(u))dS_{t,s}(u) \\ &= \ \hat{v}(t,s) + \int_t^T Y_{t,y}^{\alpha,\gamma}(u)dS_{t,s}(u) \ ; \end{aligned}$$

in the last step we applied Itô's formula to $\hat{v}_s$. Now, set $w := \hat{v}(t,s)$, and observe that $W_{t,w,s,y}^{\alpha,\gamma}(u) = \hat{v}(u, S_{t,s}(u)) \geq 0$ by non-negativity of the payoff function g. Hence $(y, \alpha, \gamma) \in \mathcal{A}_{t,s}(\hat{v}(t,s))$, and by the definition of the super-replication problem (7.3), we conclude that $v \leq \hat{v}$.

It remains to prove (7.7). The upper bound on γ follows from (7.6). As for the lower bound, it is obtained as a direct consequence of Condition (7.4). Using again (7.6) and the smoothness of $\hat{v}$, we see that $0 = (\mathcal{L}\hat{v})_s = \mathcal{L}\hat{v}_s + \sigma^2 s \hat{v}_s s$, so that $\alpha = -\sigma^2\gamma$ is also bounded.

2. The proof of the reverse inequality $v \geq \hat{v}$ requires much more effort. The main step is the following dynamic programming principle which correspond to DP1 in Section 6.1.

Lemma 7.2 (Dynamic programming.) *Let $w \in I\!R$, $(y, \alpha, \gamma) \in \mathcal{A}_{t,s}(w)$ be such that $W_{t,w,s,y}^{\alpha,\gamma}(T) \geq g\left(S_{t,s}(T)\right)$ $P-$a.s. Then*

$$W_{t,w,s,y}^{\alpha,\gamma}(\theta) \ \geq \ v\left(\theta, S_{t,s}(\theta)\right) \qquad P - a.s.$$

for all stopping time θ valued in $[t,T]$.

The obvious proof of this claim is similar to the first part of the proof of Lemma 6.1. We continue by by stating two lemmas whose proofs rely heavily on the above dynamic programming principle, and will be reported later. We denote as usual by v_* the lower semi-continuous envelope of v.

Lemma 7.3 *The function v_* is viscosity supersolution of the equation*

$$-\mathcal{L}v_* \ \geq \ 0 \quad on \quad [0,T) \times (0,\infty) \, .$$

Lemma 7.4 *The function $s \longmapsto v_*(t,s) - \Gamma s \ln s$ is concave for all $t \in [0,T]$.*

Given the above results, we now proceed to the proof of the remaining inequality $v \geq \hat{v}$.

2.a. Given a trading strategy in $\mathcal{A}_{t,s}(w)$, the associated wealth process is a non-negative local martingale, and therefore a supermartingale. From this, one easily proves that $v_*(T,s) \geq g(s)$. By Lemma 7.4, $v_*(T,\cdot)$ also satisfies requirement (ii) of Lemma 7.1, and therefore

$$v_*(T,\cdot) \quad \geq \quad \hat{g} \; .$$

In view of Lemma 7.3, v_* is a viscosity supersolution of the equation $-\mathcal{L}v_* = 0$ and $v_*(T,\cdot) = \hat{g}$. Since $\hat{v}$ is a viscosity solution of the same equation, it follows from the classical comparison theorem that $v_* \geq \hat{v}$. □

Proof of Lemma 7.3 We split the argument in several steps.

1. We first show that the problem can be reduced to the case where the controls (α,γ) are uniformly bounded. For $\varepsilon \in (0,1]$, set

$$\mathcal{A}^{\varepsilon}_{t,s}(w) \;\; := \;\; \left\{ \, (y,\alpha,\gamma) \in \mathcal{A}_{t,s}(w) \;\; : \;\; |\alpha(.)| + |\gamma(.)| \leq \varepsilon^{-1} \, \right\},$$

and

$$v^{\varepsilon}(t,s) \;\; = \;\; \inf \left\{ w \;\; : \;\; W^{\alpha,\gamma}_{t,w,s,y}(T) \geq g(S_{t,s}(T)) \;\, P-\text{a.s. for some } (y,\alpha,\gamma) \in \mathcal{A}^{\varepsilon}_{t,s}(w) \right\} \; .$$

Let v^{ε}_* be the lower semi-continuous envelope of v^{ε}. It is clear that v^{ε} also satisfies the dynamic programming equation of Lemma 7.2.

Since

$$v_*(t,s) \;\; = \;\; \liminf{}_* \, v^{\varepsilon}(t,s) \quad = \quad \liminf_{\varepsilon \to 0, (t',s') \to (t,s)} \;\; v^{\varepsilon}_*(t',s') \; ,$$

we shall prove that

$$-\mathcal{L}v^{\varepsilon} \quad \geq \quad 0 \quad \text{in the viscosity sense,} \tag{7.8}$$

and the statement of the lemma follows from the classical stability result in the theory of viscosity solutions [5].

2. We now derive the implications of the dynamic programming principle of Lemma 7.2 applied to v^{ε}. Let $\varphi \in C^{\infty}(I\!R^2)$ and $(t_0,s_0) \in (0,T) \times (0,\infty)$ satisfy

$$0 \;\; = \;\; (v^{\varepsilon}_* - \varphi)(t_0,s_0) \quad = \quad \min_{(0,T)\times(0,\infty)} (v^{\varepsilon}_* - \varphi) \; ;$$

in particular, we have $v^\varepsilon_* \geq \varphi$. Choose a sequence $(t_n, s_n) \to (t_0, s_0)$ so that $v^\varepsilon(t_n, s_n)$ converges to $v^\varepsilon_*(t_0, s_0)$. For each n, by the definition of v^ε and the dynamic programming, there are $w_n \in [v^\varepsilon(t_n, s_n), v^\varepsilon(t_n, s_n) + 1/n]$, hedging strategies $(y_n, \alpha_n, \gamma_n) \in \mathcal{A}^\varepsilon_{t_n,s_n}(w_n)$ satisfying

$$W^{\alpha_n,\gamma_n}_{t_n,w_n,s_n,y_n}(\theta_n) - v^\varepsilon\left(t_n + t, S_{t_n,s_n}(t_n+t)\right) \;\; \geq \;\; 0$$

for every stopping time θ_n valued in $[t_n, T]$. Since $v^\varepsilon \geq v^\varepsilon_* \geq \varphi$,

$$w_n + \int_{t_n}^{\theta_n} Y^{\alpha_n,\gamma_n}_{t_n,y_n}(u) dS_{t_n,s_n}(u) - \varphi\left(\theta_n, S_{t_n,s_n}(\theta_n)\right) \;\; \geq \;\; 0 \; .$$

Observe that

$$\beta_n \;:=\; w_n - \varphi(t_n, s_n) \;\; \longrightarrow \;\; 0 \;\text{ as } n \longrightarrow \infty \; .$$

By Itô's Lemma, this provides

$$M_n(\theta_n) \;\; \leq \;\; D_n(\theta_n) + \beta_n \; , \tag{7.9}$$

where

$$\begin{aligned} M_n(t) \;\; &:= \;\; \int_0^t \left[\varphi_s(t_n+u, S_{t_n,s_n}(t_n+u)) - Y^{\alpha_n,\gamma_n}_{t_n,y_n}(t_n+u)\right] dS_{t_n,s_n}(t_n+u) \\ D_n(t) \;\; &:= \;\; -\int_0^t \mathcal{L}\varphi(t_n+u, S_{t_n,s_n}(t_n+u))du \; . \end{aligned}$$

We now chose conveniently the stopping time θ_n. For some sufficiently large positive constant λ and arbitrary $h > 0$, define the stopping time

$$\theta_n \;\; := \;\; (t_n + h) \wedge \inf\left\{u > t_n \;\; : \;\; \left|\ln\left(S_{t_n,s_n}(u)/s_n\right)\right| \geq \lambda\right\} \; .$$

3. By the smoothness of $\mathcal{L}\varphi$, the integrand in the definition of M_n is bounded up to the stopping time θ_n and therefore, taking expectation in (7.9) provides :

$$-E\left[\int_0^{t\wedge\theta_n} \;\mathcal{L}\varphi(t_n+u, S_{t_n,s_n}(t_n+u))du\right] \;\; \geq \;\; -\beta_n \; ,$$

We now send n to infinity, divide by h and take the limit as $h \searrow 0$. The required result follows by dominated convergence. □

It remains to prove Lemma 7.4. The key-point is the following result. We refer the reader to [19] for the tedious proof.

Lemma 7.5 *Let $(\{a_n(u),\ u \geq 0\})_n$ and $(\{b_n(u),\ u \geq 0\})_n$ be two sequences of real-valued, progressively measurable processes that are uniformly bounded in n. Let (t_n, s_n) be a sequence in $[0,T) \times (0,\infty)$ converging to $(0,s)$ for some $s > 0$. Suppose that*

$$\begin{aligned} M_n(t \wedge \tau_n) &:= \int_{t_n}^{t_n + t\wedge\tau_n} \left(z_n + \int_{t_n}^{u} a_n(r)dr + \int_{t_n}^{u} b_n(r) dS_{t_n,s_n}(r) \right) dS_{t_n,s_n}(u) \\ &\leq \beta_n + Ct \wedge \tau_n \end{aligned}$$

for some real numbers $(z_n)_n$, $(\beta_n)_n$, and stopping times $(\theta_n)_n \geq t_n$. Assume further that, as n tends to zero,

$$\beta_n \longrightarrow 0 \quad \text{and} \quad t \wedge \theta_n \longrightarrow t \wedge \theta_0 \quad P - a.s.,$$

where θ_0 is a strictly positive stopping time. Then :

(i) $\lim_{n\to\infty} z_n = 0$.

(ii) $\lim_{t\searrow 0} \operatorname{ess\,inf}_{0\leq u\leq t} b(u) \leq 0$, *where b be a weak limit process of $(b_n)_n$.*

Proof of Lemma 7.4 We start exactly as in the previous proof by reducing the problem to the case of uniformly bounded controls, and writing the dynamic programming principle on the value function v^ε.

By a further application of Itô's lemma, we see that :

$$M_n(t) = \int_0^t \left(z_n + \int_0^u a_n(r)dr + \int_0^u b_n(r) dS_{t_n,s_n}(t_n + r) \right) dS_{t_n,s_n}(t_n + u) ,$$

where

$$\begin{aligned} z_n &:= \varphi_s(t_n, s_n) - y_n \\ a_n(r) &:= \mathcal{L}\varphi_s(t_n + r, S_{t_n,s_n}(t_n + r)) - \alpha_n(t_n + r) \\ b_n(r) &:= \varphi_{ss}(t_n + r, S_{t_n,s_n}(t_n + r)) - \frac{\gamma_n(t_n + r)}{S_{t_n,s_n}(t_n + r)} . \end{aligned}$$

Observe that the processes $a_n(. \wedge \theta_n)$ and $b_n(. \wedge \theta_n)$ are bounded uniformly in n since $\mathcal{L}\varphi_s$ and φ_{ss} are smooth functions. Also since $\mathcal{L}\varphi$ is bounded on the stochastic interval $[t_n, \theta_n]$, it follows from (7.9) that

$$M_n(\theta_n) \leq C\, t \wedge \theta_n + \beta_n$$

for some positive constant C. We now apply the results of Lemma 7.5 to the martingales M_n. The result is :

$$\lim_{n\to\infty} y_n = \varphi_s(t_0, y_0) \quad \text{and} \quad \liminf_{n\to\infty,\, t\searrow 0} b(t) \leq 0.$$

where b is a weak limit of the sequence (b_n). Recalling that $\gamma_n(t) \leq \Gamma$, this provides that :

$$-s\varphi_{ss}(t_0, s_0) + \Gamma \geq 0 \; .$$

Hence v_*^ε, and therefore v_*, is a viscosity supersolution of the equation $-s(v_*)_{ss}+\Gamma \geq 0$, and the required result follows by standard arguments in the theory of viscosity solutions. □

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