

Dual Formulation of Second Order Target Problems

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Abstract

We provide a new point of view for the formulation of second order stochastic target problems. The main new ingredient is to modify the reference probability so as to allow for different scales. Our main result is a dual formulation of this control problem as a supremum of the solutions of standard backward stochastic differential equations. In particular, in the Markov case the dual problem is immediately seen to correspond to a fully nonlinear PDE, thus avoiding the heavy technicalities in [19].

Key words: Stochastic target problem, Singular probability measures, Backward SDEs, Duality.

AMS 2000 subject classifications: 60H10, 60H30.

1 Introduction

This work is motivated by the recent extension of [6] of backward stochastic differential equations (BSDE) to the second order. Loosely speaking, while standard BSDEs as introduced by Pardoux and Peng [14] are connected to semilinear PDEs in the Markov case, second order BSDEs are connected to fully nonlinear PDEs. However, no existence theory is available for this extension, except in the Markov case with strong smoothness conditions on the solution. A uniqueness result in the Markov case is proved in [6] in a class defined by very technical conditions.

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In this paper, we consider the closely related problem of second order stochastic target as introduced in [19]. We believe that a better understanding of this problem is crucial to achieve some progress in our study of second order BSDEs of [6]. More precisely, let B be a Brownian motion under the probability measure \mathbb{P}_0 and $\{\mathcal{F}_t, t \geq 0\}$ the corresponding filtration. In this introduction, we specialize the discussion to the one-dimensional case for ease of presentation. For a continuous semimartingale Z , we denote by Γ the density of its covariation with B , and we define the process Y by:

$$Y_t := y - \int_0^t H_s(Y_s, Z_s, \Gamma_s) ds + \int_0^t Z_s \circ dB_s, \quad d\langle Z, B \rangle_t = \Gamma_t dt, \quad (1.1)$$

where \circ denoted the Fisk-Stratonovich stochastic integration, and $H : \Omega \times [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfies the required Lipschitz and measurability conditions for existence and uniqueness of Y . We shall denote such a solution by $Y^{y,Z}$. Given an \mathcal{F}_1 -measurable random variable ξ , [19] introduced the following second order stochastic target problem:

$$\mathcal{V}^0 := \inf \left\{ y : Y_1^{y,Z} \geq \xi \text{ } \mathbb{P}_0 - \text{a.s. for some } Z \in \mathcal{Z}^0 \right\}. \quad (1.2)$$

The definition of the set of admissible controls \mathcal{Z}^0 is crucial, and we shall not say more about it in the present context.

In this paper, we adopt a new point of view in order to provide a duality theory for this class of problems. The formulation of this paper differs from that of [19] at two places. First, we re-enforce the constraint $Y_1^{y,Z} \geq \xi$ in (1.2) by requiring that it should hold under various singular measures, and not only on the support of \mathbb{P}_0 . Second, the set of admissible controls is more natural and avoids the technical aspects of [19]. We believe that the duality result of this paper is an important step towards a better understanding of second order BSDEs.

This new point of view is motivated by the work of Denis and Martini [7] who introduced the important notion of quasi-sure stochastic analysis. We instead employ a direct approach by defining a dominating measure of a countable family of singular measures. The main drawback of this approach is that strong continuity assumptions are needed in order to have a robust theory (with respect to the choice of the dense family). However, two important advantages are stimulating our effort in investigating this approach. First, there is no need to redo from scratch the theory of stochastic integration. Second, and more importantly, in contrast with [7], a representation theorem is available in our framework.

For a special kind of nonlinearity (of the so-called Black-Scholes-Barrenblatt type), Peng [16] also provides an original approach in the Markov case, based on the beautiful notion of G -Brownian motion. However, similar to [7], the theory of stochastic integration needs to be revisited in the new context. In a recent paper by Denis, Hu and Peng [8], this approach is shown to be intimately related to the quasi-sure stochastic analysis.

We next provide an intuitive justification of our approach building on the corresponding PDE in the Markov case:

$$H_t(y, z, \gamma) = h(t, X_t, y, z, \gamma), \quad \xi = g(X_T), \quad (1.3)$$

where X is the solution of a Markov stochastic differential equation, and h, g are deterministic scalar functions. Let $\mathcal{V}^0(t, x)$ be defined as in (1.2) with time origin at t and $X_t = x$. Let $\gamma \mapsto h(t, x, y, z, \gamma)$ be non-decreasing. A highly technical set of admissible controls is introduced in [19] in order to establish the connection of this problem with the corresponding dynamic programming equation:

$$-\frac{\partial u}{\partial t} - h(t, x, u(t, x), Du(t, x), D^2u(t, x)) = 0, \quad u(1, x) = g(x). \quad (1.4)$$

We next assume that $\gamma \mapsto h(t, x, r, p, \gamma)$ is convex. Then

$$h(t, x, r, p, \gamma) = \sup_{a \geq 0} \left\{ \frac{1}{2} a \gamma - f(t, x, r, p, a) \right\}, \quad (1.5)$$

where f is the (partial) convex conjugate of h with respect to γ . Let $\text{dom}(f)$ be the domain of f as a function of a . By the maximum principle, we expect that, for every $a \in \text{dom}(f)$, the solution $u \geq u^a$, where u^a is defined as the solution of the semilinear PDE:

$$-\frac{\partial u}{\partial t} - \frac{1}{2} a D^2 u(t, x) + f(t, x, u(t, x), Du(t, x), a) = 0, \quad u(1, x) = g(x). \quad (1.6)$$

By standard results, u^a can be represented by means of the solution of the backward stochastic differential equation

$$Y_t^a = g(X_T^a) - \int_t^T f(s, X_s^a, Y_s^a, Z_s^a, a_s) ds - \int_t^T Z_s^a a_s^{1/2} dB_s, \quad X_s^a = x + \int_t^s a_r^{1/2} dB_r.$$

It is then natural to consider the problem

$$V_t := \sup_{a \in \bar{\mathcal{A}}^f} Y_t^a \quad (1.7)$$

where $\bar{\mathcal{A}}^f$ is the collection of all processes with values in $\text{dom}(f)$. The optimization problem (1.7) corresponds to the dual formulation of the second order target problem in the Markov case. Such a duality relation was suggested in the specific example of [18], and can be proved rigorously by showing that $v(t, x) := V_t$ is a viscosity solution of the fully nonlinear PDE (1.4) which would imply that $v = \mathcal{V}^0$ by uniqueness. Of course, such an argument requires some technical conditions at least to guarantee that comparison of viscosity supersolutions and subsolutions holds true for the PDE (1.4).

The main object of this paper is to provide a purely probabilistic proof of this duality result. Moreover, our duality result does not require to restrict the problem to the Markov

framework. The paper is organized as follows. Section 2 introduces a convenient setup for simultaneous stochastic integration and stochastic analysis under singular measures. In Section 3, we introduce the second order target problem \mathcal{V} , together with two successive relaxations $\bar{\mathcal{V}}$ and $\hat{\mathcal{V}}$, from which a dual formulation v appears. In this general setting, our main duality result of Section 4 is that $\hat{\mathcal{V}} = v$, but we could only show that $\mathcal{V} \geq \bar{\mathcal{V}} \geq \hat{\mathcal{V}}$. We then introduce in Section 5 a weak version \mathcal{V}_0 of the second order stochastic target problem, the corresponding relaxations $\bar{\mathcal{V}}_0$ and $\hat{\mathcal{V}}_0$, and the corresponding dual problem v_0 . The main result is that $\mathcal{V}_0 = \bar{\mathcal{V}}_0 = \hat{\mathcal{V}}_0 = v_0$, and under a convenient continuity condition all these value functions agree with the dual value function v . Finally sections 6 and 7 report some examples and some technical proofs.

NOTATIONS In the paper, we consider multiple dimensional processes. Given $d, k \in \mathbb{N}$, the set of d by k matrices is denoted $\mathbb{M}(d, k)$. The subset of $\mathbb{M}(d, d)$ of all symmetric matrices is denoted by \mathbb{S}_d , and the corresponding positive orthant by \mathbb{S}_d^+ . Given two matrices A and B in $\mathbb{M}(d, k)$, we denote by $A : B := \text{Tr}[A^T B]$. When $k = 1$, this notation reduces to the usual Euclidean scalar product, and we simply write $AB := A : B$.

Given a filtered probability space $(\Omega, \mathcal{G}, \mathbb{P}, \mathbb{G} = \{\mathcal{G}_t, 0 \leq t \leq 1\})$, we denote by $\mathcal{H}^0(\mathbb{P}, \mathbb{G})$ the space of all \mathbb{G} -adapted and $\mathcal{B}([0, 1]) \times \mathcal{G}$ -progressively measurable processes; $\mathcal{D}^0(\mathbb{P}, \mathbb{G})$ the subspace of $\mathcal{H}^0(\mathbb{P}, \mathbb{G})$ whose elements have càdlàg paths, \mathbb{P} -a.s.; $\mathcal{I}^0(\mathbb{P}, \mathbb{G})$ the subspace of $\mathcal{D}^0(\mathbb{P}, \mathbb{G})$ whose elements K have nondecreasing paths with $K_0 = 0$, \mathbb{P} -a.s.; and $\mathcal{S}^0(\mathbb{P}, \mathbb{G})$ the subspace of $\mathcal{D}^0(\mathbb{P}, \mathbb{G})$ whose elements have continuous paths, \mathbb{P} -a.s. We consider $Z^1, Z^2 \in \mathcal{H}^0(\mathbb{P}, \mathbb{G})$ as the same elements of $\mathcal{H}^0(\mathbb{P}, \mathbb{G})$ if they are \mathbb{P} -modifications, that is, $Z^1 = Z^2, dt \times d\mathbb{P}$ -a.s. For $p > 0$, let

$$\begin{aligned} \mathcal{H}^p(\mathbb{P}, \mathbb{G}) &:= \left\{ Z \in \mathcal{H}^0(\mathbb{P}, \mathbb{G}) : \mathbb{E}^{\mathbb{P}} \left[\left(\int_0^1 |Z_t|^2 dt \right)^{p/2} \right] < \infty \right\}, \\ \mathcal{D}^p(\mathbb{P}, \mathbb{G}) &:= \left\{ Y \in \mathcal{D}^0(\mathbb{P}, \mathbb{G}) : \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq 1} |Y_t|^p \right] < \infty \right\}, \end{aligned} \tag{1.8}$$

and $\mathcal{I}^p(\mathbb{P}, \mathbb{G}) := \mathcal{D}^p(\mathbb{P}, \mathbb{G}) \cap \mathcal{I}^0(\mathbb{P}, \mathbb{G})$, $\mathcal{S}^p(\mathbb{P}, \mathbb{G}) := \mathcal{D}^p(\mathbb{P}, \mathbb{G}) \cap \mathcal{S}^0(\mathbb{P}, \mathbb{G})$. The case $p = 2$ will be particularly interesting for us. Moreover, let

$$\mathcal{SM}^2(\mathbb{P}, \mathbb{G}) := \left\{ Z \in \mathcal{D}^2(\mathbb{P}, \mathbb{G}) : Z \text{ is a } (\mathbb{P}, \mathbb{G})\text{-semi-martingale} \right\}. \tag{1.9}$$

2 BSDEs under singular probability measures

Let $\Omega := C([0, 1])^d$ be the canonical space, B the canonical process, \mathbb{P}_0 the Wiener measure, and $\mathbb{F} := \{\mathcal{F}_t\}_{0 \leq t \leq 1}$ the filtration generated by B . For notational convenience, we denote $\mathcal{F}_t := \mathcal{F}_0$ for $t < 0$ and $\mathcal{F}_t := \mathcal{F}_1$ for $t > 1$. Since B is continuous, we have $B_t \in \mathcal{F}_{t-}$

and thus $\mathcal{F}_t = \mathcal{F}_{t-}$. However, $\mathcal{F}_{t+} \neq \mathcal{F}_t$. We denote $\mathbb{F}^+ := \{\mathcal{F}_{t+}\}_{0 \leq t \leq 1}$. Then \mathbb{F}^+ is right continuous.

Let \mathcal{M} denote the set of all probability measures \mathbb{P} on Ω such that

$$\mathbb{P}(B_0 = 0) = 1 \quad \text{and} \quad B \text{ is a } (\mathbb{P}, \mathbb{F}) \text{ - square integrable martingale.} \quad (2.1)$$

In particular, the Wiener measure $\mathbb{P}_0 \in \mathcal{M}$. Since B is continuous, it is obvious that B is also a $(\mathbb{P}, \mathbb{F}^+)$ -martingale for any $\mathbb{P} \in \mathcal{M}$.

We denote by $\overline{\mathcal{A}}$ the space of all \mathbb{F} -adapted processes a with values in $\mathbb{S}_d^+(\mathbb{R})$ satisfying

$$\underline{a}I_d \leq a_t(\omega) \leq \overline{a}I_d, \quad dt \times d\mathbb{P}_0 \text{ - a.s.} \quad \text{for some constants } \overline{a} \geq \underline{a} > 0 \quad (2.2)$$

where I_d is the d -dimensional identity matrix. Note that the bounds \overline{a} and \underline{a} may depend on $a \in \overline{\mathcal{A}}$. For any $a \in \overline{\mathcal{A}}$, there exists a unique symmetric matrix-valued process $a^{1/2}$, such that $a^{1/2}a^{1/2} = a$. Let

$$X_t^a := \int_0^t a_s^{1/2} dB_s, \quad 0 \leq t \leq 1, \quad \mathbb{P}_0 \text{ - a.s. and denote } \mathbb{P}^a := \mathbb{P}_0 \circ (X^a)^{-1}, \quad (2.3)$$

the measure induced by the process X^a under \mathbb{P}_0 . Notice that $\mathbb{P}^a \in \mathcal{M}$, and the measures \mathbb{P}^a and $\mathbb{P}^{a'}$ are mutually singular for different a and a' in $\overline{\mathcal{A}}$.

By a slight abuse of terminology, we define the following notion introduced by Denis and Martini [7].

Definition 2.1 *We say a property holds quasi-surely, abbreviated as q.s., if it holds \mathbb{P}^a -a.s. for all $a \in \overline{\mathcal{A}}$.*

For any σ -algebra $\mathcal{G} \subset \mathcal{F}_1$ and any $\mathbb{P} \in \mathcal{M}$, denote

$$\mathcal{N}^{\mathbb{P}} := \left\{ E \subset \Omega : \exists \tilde{E} \in \mathcal{F}_1 \text{ s.t. } E \subset \tilde{E} \text{ and } \mathbb{P}[\tilde{E}] = 0 \right\}, \quad \mathcal{G}^{\mathbb{P}} := \mathcal{G} \vee \mathcal{N}^{\mathbb{P}},$$

and $\mathbb{F}^{\mathbb{P}} := \{(\mathcal{F}_{t+})^{\mathbb{P}}\}_{0 \leq t \leq 1}$. The filtration $\mathbb{F}^{\mathbb{P}}$ is the so-called augmented filtration and satisfies the usual hypotheses.

Remark 2.2 For $a \in \overline{\mathcal{A}}$, the \mathbb{P}_0 -augmented filtrations $(\mathbb{F}^{X^a})^{\mathbb{P}_0}$ and $\mathbb{F}^{\mathbb{P}_0}$ are equal, and therefore

$$B_t = \beta_t(X^a), \quad dt \times d\mathbb{P}_0 \text{ - a.s. for some measurable map } \beta.$$

To see this, notice that X^a is $\mathbb{F}^{\mathbb{P}_0}$ -adapted and therefore $(\mathbb{F}^{X^a})^{\mathbb{P}_0} \subset \mathbb{F}^{\mathbb{P}_0}$. Moreover, since $\langle X^a \rangle_t = \int_0^t a_s ds$ \mathbb{P}_0 -a.s. we see that a is $(\mathbb{F}^{X^a})^{\mathbb{P}_0}$ -adapted. Then, it follows from the fact that a takes positive semidefinite values that $B_t = \int_0^t a_s^{-1/2} dX_s^a$ is $(\mathbb{F}^{X^a})^{\mathbb{P}_0}$ -measurable, thus proving that $\mathbb{F}^{\mathbb{P}_0} \subset (\mathbb{F}^{X^a})^{\mathbb{P}_0}$. \square

Now for any $a \in \overline{\mathcal{A}}$, we provide a construction of a Brownian motion under \mathbb{P}^a which generates the same \mathbb{P}^a -augmented filtration as $\mathbb{F}^{\mathbb{P}^a}$. Since $a \in \mathbb{F}$, we can write $a_t = \mathbf{a}_t(B)$, $0 \leq t \leq 1$, for some measurable map \mathbf{a} . We claim that the stochastic differential equation

$$W_t^a := \int_0^t [\mathbf{a}_s(W^a)]^{-1/2} dB_s, \quad 0 \leq t \leq 1, \quad \mathbb{P}^a - \text{a.s.} \quad (2.4)$$

has a strong solution given by $W_t^a := \beta_t(B)$, where β is given in Remark 2.2. Indeed, the \mathbb{P}^a -distribution of B is the same as the \mathbb{P}_0 -distribution of X^a and, by the definition of X^a , $B_t = \int_0^t [\mathbf{a}_s(B)]^{-1/2} dX_s^a, \mathbb{P}_0$ -a.s. Moreover,

$$d\langle B \rangle_t = \tilde{a}_t dt, \quad \mathbb{P}^a - \text{a.s.} \quad \text{where} \quad \tilde{a}_t := \mathbf{a}_t(W^a), \quad (2.5)$$

and

$$W^a \text{ is a } \mathbb{P}^a - \text{Brownian motion and } (\mathbb{F}^{W^a})^{\mathbb{P}^a} = \mathbb{F}^{\mathbb{P}^a}. \quad (2.6)$$

2.1 Existence of \mathbb{F}^+ -adapted version

In the literature of SDE and BSDE theories, it is usually assumed that the filtered probability space satisfies the *usual hypotheses*. When singular measures are considered simultaneously, we do not have a good filtration satisfying the usual hypotheses under all the measures. Instead, we will use the filtration \mathbb{F}^+ , which is right continuous but is not augmented by the null sets under each \mathbb{P}^a . As we see next, all the properties we need still hold under this filtration. The following remark collects some standard results, see e.g. [2].

Remark 2.3 (i) An \mathcal{F}_1 -measurable r.v. X is $\mathcal{G}^{\mathbb{P}}$ -measurable if and only if there exists a \mathcal{G} -measurable random variable \tilde{X} such that $\{X \neq \tilde{X}\} \in \mathcal{N}^{\mathbb{P}}$. In particular, $X = \tilde{X}, \mathbb{P}$ -a.s.
(ii) A $(\mathbb{P}, \mathbb{F}^+)$ -martingale is also a $(\mathbb{P}, \mathbb{F}^{\mathbb{P}})$ -martingale. \square

The following lemma is important for us and its proof is postponed to the Appendix.

Lemma 2.4 *Let $\mathbb{P} \in \mathcal{M}$ be arbitrary.*

- (i) *For any $Y \in \mathcal{H}^0(\mathbb{P}, \mathbb{F}^{\mathbb{P}})$, there exists unique $\tilde{Y} \in \mathcal{H}^0(\mathbb{P}, \mathbb{F}^+)$ such that \tilde{Y} and Y are \mathbb{P} -modifications. In addition, if $Y \in \mathcal{D}^0(\mathbb{P}, \mathbb{F}^{\mathbb{P}})$ (resp. $\mathcal{I}^0(\mathbb{P}, \mathbb{F}^{\mathbb{P}}), \mathcal{S}^0(\mathbb{P}, \mathbb{F}^{\mathbb{P}})$), then $\tilde{Y} \in \mathcal{D}^0(\mathbb{P}, \mathbb{F}^+)$ (resp. $\mathcal{I}^0(\mathbb{P}, \mathbb{F}^+), \mathcal{S}^0(\mathbb{P}, \mathbb{F}^+)$) and they are \mathbb{P} -indistinguishable.*
- (ii) *For any $\mathbb{F}^{\mathbb{P}}$ -stopping time τ , there exists an \mathbb{F}^+ -stopping time $\tilde{\tau}$ such that $\mathbb{P}(\tilde{\tau} \neq \tau) = 0$.*

2.2 Forward and backward SDEs under $(\mathbb{P}^a, \mathbb{F}^+)$

Denote $\mathbb{F}^a := \mathbb{F}^{\mathbb{P}^a}$, the augmented filtration of the canonical process under \mathbb{P}^a , and $\mathbb{E}^a := \mathbb{E}^{\mathbb{P}^a}$ the corresponding expectation operator. We denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel σ -field of \mathbb{R}^d , and \mathcal{P} the σ -field of progressively measurable sets of $\Omega \times [0, 1]$ with respect to \mathbb{F}^+ .

Remark 2.5 It is well known that \mathbb{F}^a is continuous, and thus $\mathbb{F}^+ \subset \mathbb{F}^a$. Therefore, for every \mathcal{F}_{0+} -measurable random variable X and $a \in \bar{\mathcal{A}}$, we have $X = \mathbb{E}^a[X]$, \mathbb{P}^a -a.s. This is the so-called zero-one Blumenthal law under \mathbb{P}^a . Notice however that an \mathcal{F}_{0+} -measurable random variable X is constant \mathbb{P}^a -a.s., but the constant depends on a . \square

For any $Z \in \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^a)$, let $Y_t := \int_0^t Z_s dB_s$ be the stochastic integral defined in the standard way. Then since $Y \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^a)$, we deduce from Lemma 2.4 that there exists a unique $\tilde{Y} \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+)$ which is \mathbb{P}^a -indistinguishable from Y . Therefore, we can always assume a stochastic integral is \mathbb{F}^+ -adapted, as we do in the sequel of this paper. In particular, for any $Z \in \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+) \subset \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^a)$, $Y_t := \int_0^t Z_s dB_s$ is well defined and we always assume $Y \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+)$.

We now state that the martingale representation theorem holds true under \mathbb{F}^+ .

Lemma 2.6 *Let $a \in \bar{\mathcal{A}}$ and $\xi \in \mathbb{L}^2(\mathbb{P}^a, \mathcal{F}_1)$ be given. Then, there exists a unique process $Z^a \in \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+)$ such that*

$$\xi = \mathbb{E}^a[\xi] + \int_0^1 Z_t^a dB_t, \quad \mathbb{P}^a - a.s.$$

Proof. For $a \in \bar{\mathcal{A}}$, let W^a and \tilde{a} be defined as in (2.4) and (2.5), and recall from (2.6) that W^a is a \mathbb{P}^a -Brownian motion and $(\mathbb{F}^{W^a})^{\mathbb{P}^a} = \mathbb{F}^a$. Then $\xi \in (\mathcal{F}_1^{W^a})^{\mathbb{P}^a}$ and it follows from the standard martingale representation result that there exists a unique process $\tilde{Z}^a \in \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^a)$ such that $\xi = \mathbb{E}^a[\xi] + \int_0^1 \tilde{Z}_t^a dW_t^a$, \mathbb{P}^a -a.s. By (2.4), we have $\xi = \mathbb{E}^a[\xi] + \int_0^1 \tilde{Z}_t^a \tilde{a}_t^{-1/2} dB_t$, \mathbb{P}^a -a.s. By using Lemma 2.4, we then define Z^a as the unique \mathbb{F}^+ -modification of the process $\tilde{Z}^a \tilde{a}^{-1/2}$. The required integrability of Z^a follows from that of \tilde{Z}^a and the uniform ellipticity condition (2.2) on a . \square

Next, consider the following forward and backward SDEs under \mathbb{P}^a , for $a \in \bar{\mathcal{A}}$,

$$X_t^a = x + \int_0^t b_s(X_s^a) ds + \int_0^t \sigma_s(X_s^a) dB_s; \quad (2.7)$$

$$Y_t^a = \xi + \int_t^T f_s(Y_s^a, Z_s^a) ds - \int_t^T Z_s^a dB_s, \quad 0 \leq t \leq 1, \quad \mathbb{P}^a - a.s. \quad (2.8)$$

where $b : \Omega \times [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sigma : \Omega \times [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{S}_d$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable, and $f : \Omega \times [0, 1] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable.

Recall the bounds $\bar{a} \geq \underline{a} > 0$ defined in (2.2).

Lemma 2.7 (i) *Assume b, σ are uniformly Lipschitz continuous in x and $b(0), \sigma(0) \in \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+)$. Then the SDE (2.7) has a unique solution $X^a \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+)$ satisfying*

$$\mathbb{E}^a \left[\sup_{0 \leq t \leq 1} |X_t^a|^2 \right] \leq C(\bar{a}) \mathbb{E}^a \left[|x|^2 + \int_0^1 (|b_t(0)|^2 + |\sigma_t(0)|^2) dt \right], \quad (2.9)$$

where $C(\bar{a}) \geq 0$ depends only on \bar{a} , the Lipschitz constant of b, σ and the dimension d .

(ii) Assume f is uniformly Lipschitz continuous in (y, z) , $f(0, 0) \in \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+)$. Then for any $\xi \in \mathbb{L}^2(\mathbb{P}^a, \mathcal{F}_1)$, the BSDE (2.8) has a unique solution $(Y^a, Z^a) \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+) \times \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+)$ satisfying

$$\mathbb{E}^a \left[\sup_{0 \leq t \leq 1} |Y_t^a|^2 + \int_0^1 |Z_t^a|^2 dt \right] \leq C(\underline{a}) \mathbb{E}^a \left[|\xi|^2 + \int_0^1 |f_t(0, 0)|^2 dt \right]. \quad (2.10)$$

where $C(\underline{a}) \geq 0$ depends only on \underline{a} , the Lipschitz constant of f and the dimension d .

(iii) Let $(\tilde{Y}^a, \tilde{Z}^a)$ be the unique solution to BSDE (2.8) with coefficients $(\tilde{f}, \tilde{\xi})$ satisfying the above conditions. Then, with $\delta Y^a := Y^a - \tilde{Y}^a$ and $\delta Z^a, \delta \xi$ and δf defined similarly:

$$\mathbb{E}^a \left[\sup_{0 \leq t \leq 1} |\delta Y_t^a|^2 + \int_0^1 |\delta Z_t^a|^2 dt \right] \leq C(\underline{a}) \mathbb{E}^a \left[|\delta \xi|^2 + \int_0^1 |\delta f_t(Y_t^a, Z_t^a)|^2 dt \right]. \quad (2.11)$$

(iv) Assume further that $\tilde{f}(y, z) \leq f(y, z), dt \times \mathbb{P}^a$ -a.s. for any (y, z) , and $\tilde{\xi} \leq \xi, \mathbb{P}^a$ -a.s. Then $\tilde{Y}^a \leq Y^a$ for all t, \mathbb{P}^a -a.s.

Proof. For $a \in \bar{\mathcal{A}}$, define the \mathbb{P}^a -Brownian motion W^a as in (2.4) and \tilde{a} as in (2.5), and recall that $(\mathbb{F}^{W^a})^{\mathbb{P}^a} = \mathbb{F}^a$ by (2.6).

(i) Let $\tilde{\sigma}_t(x) := \sigma_t(x) \tilde{a}_t^{-1/2}$. Then $\tilde{\sigma}$ is Lipschitz continuous with respect to x with Lipschitz constant $L\sqrt{\tilde{a}}$, where L is the Lipschitz constant of σ with respect to x . By classical results, there is a unique solution $X^a \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^a)$ of the SDE

$$X_t^a = x + \int_0^t b_s(X_s^a) ds + \int_0^t \tilde{\sigma}_s(X_s^a) dW_s^a$$

satisfying the required estimate (2.9). By Lemma 2.4, this induces a unique solution $X^a \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+)$.

(ii) Let $\tilde{f}_t(y, z) := f_t(y, z \tilde{a}_t^{-1/2})$. Then $\tilde{f}_t(0, 0) = f_t(0, 0)$ and \tilde{f} is Lipschitz continuous in (y, z) with Lipschitz constant $\underline{a}^{-1/2}L$. By classical results the BSDE

$$Y_t^a = \xi + \int_t^T \tilde{f}_s(Y_s^a, \tilde{Z}_s^a) ds - \int_0^1 \tilde{Z}_t^a dW_t^a, \quad \mathbb{P}^a - \text{a.s.},$$

has a unique solution $(Y^a, \tilde{Z}^a) \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^a) \times \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^a)$ satisfying the required estimate (2.10). Set $Z_t^a := \tilde{Z}_t^a \tilde{a}_t^{-1/2}$. Then $|Z_t^a| \leq \underline{a}^{-1/2} |\tilde{Z}_t^a|$, and thus $Z^a \in \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^a)$. By Lemma 2.4, this induces a unique solution $(Y^a, \tilde{Z}^a) \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+) \times \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+)$.

(iii) If $(\tilde{Y}^a, \tilde{Z}^a)$ satisfies (2.8), then

$$\delta Y_t^a = \delta \xi + \int_t^1 (\delta f_s(Y_s^a, Z_s^a) + \alpha_s \delta Y_s^a + \beta_s \delta Z_s^a) ds - \int_t^1 \delta Z_s^a dB_s, \quad (2.12)$$

where $|\alpha|, |\beta| \leq L$. Denote $\bar{f}_t(y, z) := \alpha_t y + \beta_t z + \delta f_t(Y_t^a, Z_t^a)$. Then, $(\bar{f}, \delta \xi)$ satisfies the conditions of the lemma and $(\delta Y, \delta Z)$ is the unique solution of the BSDE (2.8) with coefficients $(\bar{f}, \delta \xi)$. Then (2.11) follows from (2.10).

(iv) is a direct consequence of (2.12). \square

2.3 Nonlinear Doob-Meyer decomposition

In preparation for the proof of Theorem 4.4, we adapt the concept of f -supermartingale from [15] and [4]. By \mathcal{T}_1 we denote the set of all \mathbb{F}^+ -stopping times with values in $[0, 1]$. For $\tau \in \mathcal{T}_1$, we denote by \mathcal{T}_τ the subset of stopping times with values in $[0, \tau]$.

Throughout this subsection, we fix $a \in \overline{\mathcal{A}}^F$ and a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{S}_d)$ -measurable map $f : \Omega \times [0, 1] \times \mathbb{R}^d \times \mathbb{S}_d \rightarrow \mathbb{R}$ uniformly Lipschitz-continuous in (y, z) with $f(0, 0) \in \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+)$. Then, the BSDE:

$$dy_t = f_t(y_t, z_t)dt - z_t dB_t, \quad \mathbb{P}^a - \text{a.s.} \quad (2.13)$$

has a unique solution $(y, z) \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+) \times \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+)$ for every final condition in $\mathbb{L}^2(\mathbb{P}^a, \mathcal{F}_1)$.

Definition 2.8 *Let $\theta \in \mathcal{T}_1$ and $\mathcal{Y} \in \mathcal{H}^0(\mathbb{P}^a, \mathbb{F}^+)$. For every $\tau \in \mathcal{T}$, let (Y^τ, Z^τ) be the unique solution of the BSDE (2.13) with terminal condition $Y_\tau^\tau = \mathcal{Y}_\tau$.*

(i) \mathcal{Y} is a weak f -supermartingale under \mathbb{P}^a if $\mathbb{E}^a\{|\mathcal{Y}_t|^2\} < \infty$ for every $t \in [0, 1]$ and

$$Y_{t_1}^{t_2} \leq \mathcal{Y}_{t_1} \quad \mathbb{P}^a - \text{a.s. for every } 0 \leq t_1 \leq t_2 \leq 1.$$

(ii) \mathcal{Y} is a strong f -supermartingale under \mathbb{P}^a up to θ if $\mathbb{E}^a\{\sup_{0 \leq t \leq \theta} |\mathcal{Y}_t|^2\} < \infty$ and

$$Y_{\tau_1}^{\tau_2} \leq \mathcal{Y}_{\tau_1} \quad \mathbb{P}^a - \text{a.s. for every } \tau_1, \tau_2 \in \mathcal{T}_\theta \text{ with } \tau_1 \leq \tau_2.$$

(iii) \mathcal{Y} is a local f -supermartingale under \mathbb{P}^a up to θ if there is a sequence $\mathcal{T}_\theta \ni \theta_n \uparrow \theta$, \mathbb{P}^a -a.s. such that, for every n , \mathcal{Y} is a strong f -supermartingale under \mathbb{P}^a up to θ_n .

We note that here we require all the processes and the stopping times to be \mathbb{F}^+ -adapted while in [15] they are all $\mathbb{F}^{\mathbb{P}^a}$ -adapted. However, thanks to Lemma 2.4, this makes no real difference (see the proof of Theorem 2.10).

Lemma 2.9 *Let \mathcal{Y} be a weak f -supermartingale under \mathbb{P}^a . Then*

$$\mathcal{Y}_t^+ := \lim_{r \in \mathbb{Q} \cap (t, 1], r \downarrow t} \mathcal{Y}_r \text{ exists } \mathbb{P}^a - \text{a.s.} \quad (2.14)$$

Moreover, if we assume further that $\mathbb{E}^a\{\sup_{r \in \mathbb{Q} \cap [0, 1]} |\mathcal{Y}_r|^2\} < \infty$, then \mathcal{Y}^+ is a càdlàg strong f -supermartingale under \mathbb{P}^a .

Proof. (i) With the notations of Definition 2.8, set

$$\begin{aligned} \tilde{\mathcal{Y}}_\tau &:= \mathcal{Y}_\tau - Y_\tau^1, \quad \tau \in \mathcal{T}_1, \quad \tilde{Y}_s^\tau = Y_s^\tau - Y_s^1, \quad 0 \leq s \leq \tau, \\ \text{and } \tilde{f}(t, \omega, y, z) &:= f(t, \omega, y + Y_t^1(\omega), z + Z_t^1(\omega)) - f(t, \omega, Y_t^1(\omega), Z_t^1(\omega)). \end{aligned}$$

Then $\tilde{\mathcal{Y}}_\tau \geq 0$ for all $\tau \in \mathcal{T}_1$, $\tilde{Y}_{\tau_1}^{\tau_2} \leq \tilde{\mathcal{Y}}_{\tau_1}$ for any $\tau_1 \leq \tau_2 \in \mathcal{T}_1$, \mathbb{P}^a -a.s. and \tilde{f} satisfies the conditions of Lemma 2.7. One can easily check that \tilde{Y}^τ is the unique solution to BSDE

(2.13) with generator \tilde{f} and terminal condition $\tilde{Y}_\tau^\tau = \tilde{\mathcal{Y}}_\tau$. Now by a line by line analogy of [4], we know $\lim_{n \rightarrow \infty} \tilde{\mathcal{Y}}_{\tau_n \wedge \theta}$ exists \mathbb{P}^a -a.s. Then (2.14) follows from the fact that Y^1 is continuous \mathbb{P}^a -a.s.

(ii) Assume further that $\mathbb{E}^a\{\sup_{r \in \mathbb{Q} \cap [0,1]} |\mathcal{Y}_r|^2\} < \infty$. For any $0 \leq t_1 \leq t_2 \leq 1$ and $r_2 \in \mathbb{Q} \cap (t_2, 1], r_1 \in \mathbb{Q} \cap (t_1, r_2]$, we have $\mathcal{Y}_{r_1} \geq Y_{r_1}^{r_2}$. By first sending $r_1 \downarrow t_1$ and then $r_2 \downarrow t_2$ we know \mathcal{Y}^+ is a weak f -supermartingale under \mathbb{P}^a . The rest of the claim follows [4] Theorem 7. \square

The following decomposition from [15] is crucial for our duality result.

Theorem 2.10 *Let $\mathcal{Y} \in \mathbb{F}^+$ be a strong f -supermartingale under \mathbb{P}^a . Then there exist unique processes $Z \in \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+)$ and $K \in \mathcal{I}^2(\mathbb{P}^a, \mathbb{F}^+)$ such that*

$$\mathcal{Y}_t = \mathcal{Y}_0 - \int_0^t f_s(\mathcal{Y}_s, Z_s) ds + \int_0^t Z_s dB_s - K_t, \quad 0 \leq t \leq 1, \quad \mathbb{P}^a - a.s. \quad (2.15)$$

We shall also make use of

Lemma 2.11 *Let $\theta \in \mathcal{T}_1$ and $\{\mathcal{Y}_\tau\}_{\tau \in \mathcal{T}_\theta}$ a family of \mathcal{F}_τ^+ -measurable r.v. such that $\mathcal{Y}_{\tau_1} = \mathcal{Y}_{\tau_2}$ on $\{\tau_1 = \tau_2\}$, \mathbb{P}^a -a.s. for every $\tau_1, \tau_2 \in \mathcal{T}_\theta$, and $\mathbb{E}^a[\text{ess sup}_{\tau \in \mathcal{T}_\theta} |\mathcal{Y}_\tau|^2] < \infty$. For every $\tau \in \mathcal{T}_\theta$, denote by (Y^τ, Z^τ) the unique solution of (2.13) with terminal condition $Y_\tau^\tau = \mathcal{Y}_\tau$.*

Assume that $Y_{\tau_1}^{\tau_2} \leq \mathcal{Y}_{\tau_1}$ for every $\tau_1, \tau_2 \in \mathcal{T}_\theta$ with $\tau_1 \leq \tau_2$. Then, for every $\tau \in \mathcal{T}_\theta$ and every sequence $(\tau_n)_n \subset \mathcal{T}_1$ with values in $(\tau, 1] \cap \mathbb{Q}$ and converging to τ \mathbb{P}^a -a.s.

$$\mathcal{Y}_\tau^+ := \lim_{n \rightarrow \infty} \mathcal{Y}_{\tau_n \wedge \theta} \text{ exists } \mathbb{P}^a - a.s. \quad (2.16)$$

and the limit is independent of the choice of the sequence $(\tau_n)_n$. Moreover, the family $\{\mathcal{Y}_\tau^+, \tau \in \mathcal{T}_\theta\}$ can be aggregated into the càdlàg process $\{\mathcal{Y}_t^+, t \leq \theta\}$ which is a strong f -supermartingale under \mathbb{P}^a up to θ .

Proof. (i) We first assume $\theta = 1, \mathbb{P}^a$ -a.s. By the same arguments as in Lemma (2.9) (i), we can assume without loss of generality that $\mathcal{Y} \geq 0$. Then follow the arguments in [4], together with the standard arguments in [12] Appendix D, we can prove all the claims.

(ii) In the general case, define the process $\tilde{\mathcal{Y}}$ on the interval $[\theta, 1]$ by solving the following ODE $\tilde{\mathcal{Y}}_t = \mathcal{Y}_\theta + \int_\theta^t \tilde{f}_\theta(\tilde{\mathcal{Y}}_s, 0) ds$, $\theta < t \leq 1$, and extend the family of random variables $\mathcal{Y}_\tau := \mathcal{Y}_\tau \mathbf{1}_{\{\tau \leq \theta\}} + \tilde{\mathcal{Y}}_\tau \mathbf{1}_{\{\tau > \theta\}}$ for every stopping time $\theta \in \mathcal{T}_1$. Then $\{\mathcal{Y}_\tau\}_{0 \leq \tau \leq 1}$ satisfies the conditions in Step (i), and the required results follows. \square

Combining Theorem 2.10 and Part (ii) of the proof of Lemma 2.11 above, we have

Theorem 2.12 *Let $\mathcal{Y} \in \mathbb{F}^+$ be a strong f -supermartingale under \mathbb{P}^a up to θ . Then there exist unique (up to time θ) processes $Z \in \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+)$ and $K \in \mathcal{I}^2(\mathbb{P}^a, \mathbb{F}^+)$ such that*

$$\mathcal{Y}_t = \mathcal{Y}_0 - \int_0^t f_s(\mathcal{Y}_s, Z_s) ds + \int_0^t Z_s dB_s - K_t, \quad 0 \leq t \leq \theta, \quad \mathbb{P}^a - a.s. \quad (2.17)$$

2.4 Patching processes quasi-surely

One important issue in this paper is to patch processes defined \mathbb{P}^a -a.s. to a universal process. The following direct consequence of Karandikar [9] is very useful for us.

Lemma 2.13 *Let X, M be two \mathbb{F}^+ -adapted càdlàg processes q.s. with M a \mathbb{P}^a -semimartingale for every $a \in \bar{\mathcal{A}}$. Then there exists a càdlàg process N such that $N_t = \int_0^t X_s - dM_s$, \mathbb{P}^a -a.s. for every $a \in \bar{\mathcal{A}}$.*

Corollary 2.14 *Assume M is \mathbb{F}^+ -adapted and càdlàg q.s. and is \mathbb{P}^a -semimartingale for every $a \in \bar{\mathcal{A}}$. Then there exists a càdlàg process X such that $X_t = \langle M, B \rangle_t$, \mathbb{P}^a -a.s. for every $a \in \bar{\mathcal{A}}$. In particular, $\langle B \rangle$ can be defined q.s. and there exists a process \hat{a} such that*

$$d\langle B \rangle_t = \hat{a}_t dt = \tilde{a}_t dt, \quad \mathbb{P}^a - a.s. \text{ for every } a \in \bar{\mathcal{A}}. \quad (2.18)$$

Proof. Define $X_t := M_t B_t - \int_0^t M_s dB_s - \int_0^t B_s dM_s$. One can check straightforwardly that X satisfies all the requirements. The existence of \hat{a} is obvious. \square

3 Second Order Target Problem and Relaxations

3.1 Definitions

Let $H_t(\omega, y, z, \gamma) : \Omega \times [0, 1] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d(\mathbb{R}) \rightarrow \mathbb{R} \cup \{\infty\}$ be a given $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{S}^d)$ -measurable map. We denote by $\text{dom}(H_t)$ the domain of H_t as a function of γ , and we assume throughout that

Assumption 3.1 *$\text{dom}(H_t)$ is independent of (y, z) and contains 0. Moreover, the process $H^0 := H(0, 0, 0)$ satisfies $\int_0^1 |H_t^0| dt < \infty$, q.s., H is Lipschitz in (y, z) uniformly in $\omega \in \Omega$, $t \in [0, 1]$, $\gamma \in \text{dom}(H_t)$, continuous in the γ -component, and $|H(0, 0, \gamma)| \leq C (|H^0| + |\gamma|)$ q.s. for some constant $C > 0$.*

We define the corresponding conjugate with respect to γ by:

$$F_t(\omega, y, z, a) := \sup_{\gamma \in \mathbb{S}^d} \left\{ \frac{1}{2} a : \gamma - H_t(\omega, y, z, \gamma) \right\}, \quad (3.1)$$

which takes values in $\mathbb{R} \cup \{\infty\}$, and we define

$$\hat{F}_t(y, z) := F_t(y, z, \hat{a}_t), \quad \text{and} \quad \hat{F}_t^0 := \hat{F}_t(0, 0). \quad (3.2)$$

Since $H(y, z, 0) \in \mathbb{R}$, we have

$$F_t(y, z, a) \geq H_t(y, z, 0) \quad \text{for all} \quad (y, z, a) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d(\mathbb{R}). \quad (3.3)$$

Since H is uniformly Lipschitz in (y, z) by Assumption 3.1, we have:

$$F_t(y, z, a) \quad \text{is uniformly Lipschitz-continuous in} \quad (y, z) \quad \text{for} \quad t \in [0, \hat{\tau}], \quad \text{q.s.} \quad (3.4)$$

In particular $\text{dom}(F_t)$ does not depend on (y, z) :

$$\text{dom}(F_t) = \{a : F_t(0, 0, a) < \infty\}.$$

We now introduce an increasing sequence of \mathbb{F}^+ -stopping times

$$\hat{\tau}_n := 1 \wedge \inf \left\{ t \geq 0 : \int_0^t \hat{F}_s^0 ds \geq n \right\}, \quad n \geq 1; \quad \text{and} \quad \hat{\tau} := \lim_{n \rightarrow \infty} \hat{\tau}_n. \quad (3.5)$$

Notice that

$$\int_0^1 \hat{F}_s^0 ds < \infty \quad \text{on} \quad \bigcup_{n \geq 1} \{\hat{\tau}_n = 1\} \quad \text{and} \quad \int_0^1 \hat{F}_s^0 ds = \infty \quad \text{on} \quad \bigcap_{n \geq 1} \{\hat{\tau}_n < 1\}. \quad (3.6)$$

Our next condition on the nonlinearity H is:

Assumption 3.2 *For every $a \in \bar{\mathcal{A}}$ and n , we have $\mathbb{E}^a \left[\int_0^{\hat{\tau}_n} (|\hat{F}_s^0|^2 + |H_s^0|^2) ds \right] < \infty$.*

We finally define the spaces which will be needed for the formulation of the second order target problems in the subsequent subsections: $\hat{\mathbb{L}}^2 := \bigcap_{a \in \bar{\mathcal{A}}} \mathbb{L}^2(\mathbb{P}^a, \mathcal{F}_1)$,

$$\hat{\mathcal{H}}^2 := \bigcap_{a \in \bar{\mathcal{A}}} \mathcal{H}_{loc}^2(\mathbb{P}^a, \mathbb{F}^+) := \bigcap_{a \in \bar{\mathcal{A}}} \bigcap_{n \geq 1} \left\{ (X_t)_{t \in [0, 1]} : X \mathbf{1}_{[0, \hat{\tau}_n]} \in \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+) \right\},$$

and the corresponding subsets of càdlàg, continuous processes, nondecreasing processes, and semimartingales: $X \in \hat{\mathcal{D}}^2 := \bigcap_{a \in \bar{\mathcal{A}}} \mathcal{D}_{loc}^2(\mathbb{P}^a, \mathbb{F}^+)$ (resp. $\hat{\mathcal{S}}^2 := \bigcap_{a \in \bar{\mathcal{A}}} \mathcal{S}_{loc}^2(\mathbb{P}^a, \mathbb{F}^+)$), $\hat{\mathcal{I}}^2 := \bigcap_{a \in \bar{\mathcal{A}}} \mathcal{I}_{loc}^2(\mathbb{P}^a, \mathbb{F}^+)$, $\widehat{\mathcal{SM}}^2 := \bigcap_{a \in \bar{\mathcal{A}}} \mathcal{SM}_{loc}^2(\mathbb{P}^a, \mathbb{F}^+)$) iff for every $n \geq 1$ and $a \in \bar{\mathcal{A}}$, $X_{\cdot \wedge \hat{\tau}_n} \in \mathcal{D}^2(\mathbb{P}^a, \mathbb{F}^+)$ (resp. $\mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+)$, $\mathcal{I}^2(\mathbb{P}^a, \mathbb{F}^+)$, $\mathcal{SM}^2(\mathbb{P}^a, \mathbb{F}^+)$).

3.2 Second order target problems

For $Z \in \widehat{\mathcal{SM}}^2$, denote by Γ the density of the quadratic covariation between Z and B :

$$d\langle Z, B \rangle_t = \Gamma_t d\langle B \rangle_t = \Gamma_t : \hat{a}_t dt, \quad q.s., \quad (3.7)$$

where the existence of Γ is due to Corollary 2.14. Let

$$\hat{\mathcal{H}}_H^2 := \left\{ X \in \hat{\mathcal{H}}^2 : X \in \text{dom}(H) dt - q.s. \right\} \text{ and } \widehat{\mathcal{SM}}_H^2 := \left\{ Z \in \widehat{\mathcal{SM}}^2 : \Gamma \in \hat{\mathcal{H}}_H^2 \right\}.$$

For $y \in \mathbb{R}$ and $Z \in \widehat{\mathcal{SM}}_H^2$, let $Y^{y,Z} \in \mathcal{S}^2$ denote the controlled process defined by the following ODE (with random coefficients):

$$Y_t = y - \int_0^t H_s(Y_s, Z_s, \Gamma_s) ds + \int_0^t Z_s \circ dB_s, \quad t < \hat{\tau}, \quad q.s. \quad (3.8)$$

$$Y_{\hat{\tau}} := \lim_{n \rightarrow \infty} Y_{\hat{\tau}_n} \text{ on } \bigcup_{n \geq 1} \{\hat{\tau}_n = 1\}, \quad Y_t := \infty \text{ for } t \in [\hat{\tau}, 1] \text{ on } \bigcap_{n \geq 1} \{\hat{\tau}_n < 1\}. \quad (3.9)$$

where \circ denotes the Stratonovich stochastic integral:

$$Z_s \circ dB_s = Z_s dB_s + \frac{1}{2} \Gamma_s : d\langle B \rangle_s, \quad q.s.$$

Here we used the fact that $\int_0^t Z_s dB_s$ has a universal version, thanks to Lemma 2.13. To see that (3.8) has a unique solution Y , for each $n \geq 1$ and $a \in \bar{\mathcal{A}}$, by definitions of Z and Γ , we know $\mathbb{E}^a[\int_0^{\hat{\tau}_n} \Gamma_s \Gamma'_s : \hat{a}_s ds] < \infty$. Since a has a lower bound, we get $\mathbb{E}^a[\int_0^{\hat{\tau}_n} |\Gamma_s \Gamma'_s| ds] < \infty$. Then by Assumption 3.1, it follows that $\mathbb{E}^a\{\int_0^{\hat{\tau}_n} |H_t(0, Z_s, \Gamma_s)|^2 ds\} < \infty$. This implies that $\int_0^{\hat{\tau}_n} |H_t(0, Z_s, \Gamma_s)|^2 ds < \infty$, q.s. and therefore, (3.8) has a universal solution Y in $[0, \hat{\tau}]$. Moreover, by Lemma 2.7, $Y_{\cdot \wedge \hat{\tau}_n} \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+)$ for all $n \geq 1$ and $a \in \bar{\mathcal{A}}$. That is, $Y \in \hat{\mathcal{S}}^2$. \square

Let ξ be a given \mathcal{F}_1 -measurable random variable. Following Soner and Touzi [19] we introduce the second order stochastic target problem:

$$\mathcal{V}(\xi) := \inf \left\{ y : \exists Z \in \widehat{\mathcal{SM}}_H^2 \text{ such that } Y_1^{y,Z} \geq \xi, \quad q.s. \right\}. \quad (3.10)$$

3.3 Relaxations and dual formulation

We first rewrite (3.8) in the Itô stochastic integral form:

$$Y_t = y + \int_0^t \left(\frac{1}{2} \Gamma_s : \hat{a}_s - H_s(Y_s, Z_s, \Gamma_s) \right) ds + \int_0^t Z_s dB_s, \quad 0 \leq t < \hat{\tau}, \quad (3.11)$$

where we notice that by (2.2), $\Gamma : \hat{a} \in \mathcal{H}_{loc}^2(\mathbb{P}^a, \mathbb{F}^+)$ for every $a \in \bar{\mathcal{A}}$. We next relax the target problem (3.10) by removing the constraint that Z is a semimartingale. We note that in this case in general $\int_0^t Z_s dB_s$ does not have a universal version, and thus we can only

study (3.11) under each fixed $a \in \bar{\mathcal{A}}$. To be precise, for any $y \in \mathbb{R}$, $\bar{Z} \in \hat{\mathcal{H}}^2$, $\bar{G} \in \hat{\mathcal{H}}_H^2$, and $a \in \bar{\mathcal{A}}^F$, let $\bar{Y} := \bar{Y}^{a,y,\bar{Z},\bar{G}} \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+)$ denote the unique solution of:

$$\bar{Y}_t = y + \int_0^t \left(\frac{1}{2} \bar{G}_s : \hat{a}_s - H_s(\bar{Y}_s, \bar{Z}_s, \bar{G}_s) \right) ds + \int_0^t \bar{Z}_s dB_s, \quad t < \hat{\tau}, \quad \mathbb{P}^a - \text{a.s.} \quad (3.12)$$

$$\bar{Y}_{\hat{\tau}} := \lim_{n \rightarrow \infty} \bar{Y}_{\hat{\tau}_n} \text{ on } \bigcup_{n \geq 1} \{\hat{\tau}_n = 1\}, \quad \bar{Y}_t := \infty \text{ for } t \in [\hat{\tau}, 1] \text{ on } \bigcap_{n \geq 1} \{\hat{\tau}_n < 1\}, \quad (3.13)$$

Our relaxed target problem is

$$\bar{\mathcal{V}}(\xi) := \inf \left\{ y : \exists (\bar{Z}, \bar{G}) \in \hat{\mathcal{H}}^2 \times \hat{\mathcal{H}}_H^2 \text{ s.t. } \bar{Y}_1^{a,y,\bar{Z},\bar{G}} \geq \xi, \quad \mathbb{P}^a - \text{a.s. for all } a \in \bar{\mathcal{A}} \right\}. \quad (3.14)$$

The main duality result of this paper relies on the following further relaxation of the above target problems. For $y \in \mathbb{R}$, $\hat{Z} \in \hat{\mathcal{H}}^2$, and $a \in \bar{\mathcal{A}}$, let $\hat{Y} := \hat{Y}^{a,y,\hat{Z}} \in \mathcal{H}_{loc}^2(\mathbb{P}^a, \mathbb{F}^+)$ be the unique solution of:

$$\hat{Y}_t = y + \int_0^t \hat{F}_s(\hat{Y}_s, \hat{Z}_s) ds + \int_0^t \hat{Z}_s dB_s, \quad t < \hat{\tau}, \quad \mathbb{P}^a - \text{a.s.} \quad (3.15)$$

$$\hat{Y}_{\hat{\tau}} := \lim_{n \rightarrow \infty} \hat{Y}_{\hat{\tau}_n} \text{ on } \bigcup_{n \geq 1} \{\hat{\tau}_n = 1\}, \quad \hat{Y}_t := \infty \text{ for } t \in [\hat{\tau}, 1] \text{ on } \bigcap_{n \geq 1} \{\hat{\tau}_n < 1\}. \quad (3.16)$$

To see that (3.15) has a unique solution \hat{Y} on $[0, \hat{\tau}_n]$ for each n , recall from Assumption 3.2 that $\hat{F}^0 \in \mathcal{H}_{loc}^2(\mathbb{P}^a, \mathbb{F}^+)$, so that existence and uniqueness in $\mathcal{S}_{loc}^2(\mathbb{P}^a, \mathbb{F}^+)$ follow from Lemma 2.7 and (3.4). Our further relaxed second order target problem does not involve the processes Γ and \bar{G} , and is defined by:

$$\hat{\mathcal{V}}(\xi) := \inf \left\{ y : \exists \hat{Z} \in \hat{\mathcal{H}}^2 \text{ s.t. } \hat{Y}_1^{a,y,\hat{Z}} \geq \xi, \quad \mathbb{P}^a - \text{a.s. for all } a \in \bar{\mathcal{A}} \right\}. \quad (3.17)$$

Finally, denote

$$\bar{\mathcal{A}}^F := \{a \in \bar{\mathcal{A}} : a_t \in \text{dom}(F_t), 0 \leq t \leq 1\}.$$

For any $\xi \in \hat{\mathbb{L}}^2$ and $a \in \bar{\mathcal{A}}^F$, by Assumption 3.2, (3.4), and Lemma 2.7, there is a unique solution $(Y^a, Z^a) \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+) \times \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+)$ to the backward SDE

$$Y_t^a = \xi - \int_t^1 \hat{F}_s(Y_s^a, Z_s^a) ds - \int_t^1 Z_s^a dB_s, \quad \mathbb{P}^a - \text{a.s.} \quad (3.18)$$

Similar to Remark 2.5, we know Y_0^a is a constant. Our dual formulation is:

$$v(\xi) := \sup_{a \in \bar{\mathcal{A}}^F} Y_0^a. \quad (3.19)$$

3.4 Some preliminary results

Note that

$$\mathcal{V}(\xi) \geq \bar{\mathcal{V}}(\xi) \geq \hat{\mathcal{V}}(\xi) \geq v(\xi). \quad (3.20)$$

The first inequality holds true by definition of \mathcal{V} and $\bar{\mathcal{V}}$. To see that the second inequality holds true, let $y \in \mathbb{R}$, $\bar{Z} \in \hat{\mathcal{H}}^2$, $\bar{G} \in \hat{\mathcal{H}}_H^2$ be such that $\bar{Y}_1^{a,y,\bar{Z},\bar{G}} \geq \xi$, \mathbb{P}^a -a.s. for every $a \in \bar{\mathcal{A}}$. Notice, by the definition of the conjugate function F , that

$$\frac{1}{2}\hat{a}_s : \bar{G}_s - H_s(y, \bar{Z}_s, \bar{G}_s) \leq \hat{F}_s(y, \bar{Z}_s) \quad \text{for all } y \in \mathbb{R}.$$

By the comparison theorem for ODEs, we get $\bar{Y}_1^{a,y,\bar{Z},\bar{G}} \leq \hat{Y}_1^{a,y,\bar{Z}}$, \mathbb{P}^a -a.s. Thus $\hat{Y}_1^{a,y,\bar{Z}} \geq \xi$, \mathbb{P}^a -a.s. and therefore $y \geq \hat{\mathcal{V}}(\xi)$. Finally, for $a \in \bar{\mathcal{A}}^F$, we have $\mathbb{P}^a(\cup_{n \geq 1} \{\hat{\tau}_n = 1\}) = 1$. Then the third inequality can be proved similarly, by using the comparison theorem for BSDEs.

In the Markov case, we have

Theorem 3.3 *Let Assumptions 3.1 and 3.2 hold true. Assume further that (1.3) holds with $X = B$, and the PDE (1.4) has a solution $u \in C^{1,2}$ with $\{Du(t, B_t), t \in [0, 1]\} \in \widehat{\mathcal{SM}}^2$. Then $\mathcal{V}(\xi) = \bar{\mathcal{V}}(\xi) = \hat{\mathcal{V}}(\xi) = v(\xi) = u(0, 0)$, and existence holds for all problems $\mathcal{V}(\xi)$, $\bar{\mathcal{V}}(\xi)$ and $\hat{\mathcal{V}}(\xi)$. Moreover, an optimal a^* (if exists) for the problem $v(\xi)$ satisfies:*

$$\frac{1}{2}\tilde{a}_t^* : D^2u(t, B_t) - h_t(B_t, u(t, B_t), Du(t, B_t), D^2u(t, B_t)) = F_t(B_t, u(t, B_t), Du(t, B_t), \tilde{a}_t^*).$$

Proof. By standard arguments we know $v(\xi) = u(0, 0)$. On the other hand, $Y_t := u(t, B_t)$, $Z_t := Du(t, B_t)$, $\Gamma_t := D^2u(t, B_t)$ hedges ξ . This implies that $V(\xi) \leq u(0, 0)$. Then by (3.20) we know $\mathcal{V}(\xi) = \bar{\mathcal{V}}(\xi) = \hat{\mathcal{V}}(\xi) = v(\xi) = u(0, 0)$. The existence of the optimal arguments are immediate now. \square

In the non Markov case, we shall prove in the next section that $\hat{\mathcal{V}}(\xi) = v(\xi)$ and that the optimal \hat{Z} for the $\hat{\mathcal{V}}(\xi)$ problem exists. However, we are not able to prove $\mathcal{V}(\xi) = \bar{\mathcal{V}}(\xi)$ in general. In order to obtain a result of this type, we shall introduce a slight modification of these problems by restricting a to smaller sets, see Section 5 below.

We conclude this section by observing that the target problems can be restricted to only those $a \in \bar{\mathcal{A}}^F$. To do that, we shall strengthen Assumption 3.2 slightly.

Assumption 3.4 *For every $a \in \bar{\mathcal{A}}$, we have $\mathbb{E}^a \left[\int_0^{\hat{\tau}} (|\hat{F}_s^0|^2 + |H_s^0|^2) ds \right] < \infty$.*

Define $\mathbb{L}_F^2 := \bigcap_{a \in \bar{\mathcal{A}}^F} \mathbb{L}^2(\mathbb{P}^a, \mathcal{F}_1)$, $\mathcal{H}_F^2 := \bigcap_{a \in \bar{\mathcal{A}}^F} \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+)$, and \mathcal{D}_F^2 , \mathcal{S}_F^2 , \mathcal{I}_F^2 , \mathcal{SM}_F^2 corresponding to càdlàg, continuous, nondecreasing processes, and semimartingales, respectively, in an obvious way. Moreover, we let

$$\mathcal{H}_{F,H}^2 := \bigcap_{a \in \bar{\mathcal{A}}^F} \{X \in \mathcal{H}^2 : X \in \text{dom}(H) \text{ dt} \times d\mathbb{P}^a - \text{a.s.}\}, \quad \mathcal{SM}_{F,H}^2 := \{Z \in \mathcal{SM}_F^2 : \Gamma \in \mathcal{H}_H^2\}.$$

Theorem 3.5 *Let Assumptions 3.1 and 3.4 hold true. Then*

$$\begin{aligned}\mathcal{V}(\xi) &= \inf \left\{ y : \exists Z \in \mathcal{SM}_{F,H}^2 \text{ s.t. } Y_1^{a,y,Z} \geq \xi, \mathbb{P}^a - \text{a.s. for all } a \in \bar{\mathcal{A}}^F \right\} \\ \bar{\mathcal{V}}(\xi) &= \inf \left\{ y : \exists (\bar{Z}, \bar{G}) \in \mathcal{H}_F^2 \times \mathcal{H}_{F,H}^2 \text{ s.t. } \bar{Y}_1^{a,y,\bar{Z},\bar{G}} \geq \xi, \mathbb{P}^a - \text{a.s. for all } a \in \bar{\mathcal{A}}^F \right\}, \\ \hat{\mathcal{V}}(\xi) &= \inf \left\{ y : \exists \hat{Z} \in \mathcal{H}_F^2 \text{ s.t. } \hat{Y}_1^{a,y,\hat{Z}} \geq \xi, \mathbb{P}^a - \text{a.s. for all } a \in \bar{\mathcal{A}}^F \right\}.\end{aligned}$$

Proof. We prove the theorem only for the $\mathcal{V}(\xi)$ problem. The other two problems can be proved similarly. Let $\mathcal{V}_F(\xi)$ denote the right side of the first equality in Theorem 3.5. Note that $\mathbb{P}^a[\cup_{n \geq 1} \{\hat{\tau}_n = 1\}] = 1$ for every $a \in \bar{\mathcal{A}}^F$, then $\mathcal{V}_F(\xi) \leq \mathcal{V}(\xi)$.

To prove $\mathcal{V}(\xi) \leq \mathcal{V}_F(\xi)$, we assume $\mathcal{V}_F(\xi) < \infty$, without loss of generality. For any $\varepsilon > 0$, by definition of $\mathcal{V}_F(\xi)$ there exist $y < \mathcal{V}_F(\xi) + \varepsilon$ and $Z \in \mathcal{SM}_{F,H}^2$ such that $Y_1^{a,y,Z} \geq \xi$, \mathbb{P}^a -a.s. for all $a \in \bar{\mathcal{A}}^F$. Define $Z^F := Z \mathbf{1}_{[0,\hat{\tau})} \in \mathcal{SM}_H^2$, and fix $a^0 \in \bar{\mathcal{A}}^F$. Now for any $a \in \bar{\mathcal{A}}$, define $a^F := a \mathbf{1}_{[0,\hat{\tau})} + a^0 \mathbf{1}_{[\hat{\tau},1]} \in \bar{\mathcal{A}}^F$. On $\cup_{n \geq 1} \{\hat{\tau}_n = 1\}$, we have $Z_t^F = Z_t$ and $a_t^F = a_t$ for all $0 \leq t \leq 1$. This implies that $Y_1^{a,y,Z^F} = Y_1^{a^F,y,Z} \geq \xi$ on $\cup_{n \geq 1} \{\hat{\tau}_n = 1\}$, \mathbb{P}^a -a.s. Since by definition $Y_1^{a,y,Z^F} = \infty$ on $\cap_{n \geq 1} \{\hat{\tau}_n < 1\}$, we get $Y_1^{a,y,Z^F} \geq \xi$, \mathbb{P}^a -a.s. That is, $\mathcal{V}(\xi) \leq y < \mathcal{V}_F(\xi) + \varepsilon$. Since ε is arbitrary, we prove the result. \square

We shall provide a counterexample in Section 7 which shows that, in general,

$$\mathcal{V}(\xi) \neq \inf \left\{ y : \exists Z \in \mathcal{SM}_H^2 \text{ s.t. } Y_1^{a,y,Z} \geq \xi, \mathbb{P}^a - \text{a.s. for all } a \in \bar{\mathcal{A}}^F \right\}. \quad (3.21)$$

4 Dual formulation of the relaxed target problem

4.1 Conditional expectation

We first introduce some notations. For $0 \leq t_1 \leq t_2 \leq 1$, denote $\Omega_{t_1,t_2} := \{\omega \in C([t_1, t_2]) : \omega(t_1) = 0\}$, and for $\omega \in \Omega_{t_1,t_2}$, $|\omega|_{t_1,t_2} := \sup_{t_1 \leq t \leq t_2} |\omega(t)|$. We simply write $|\omega| := |\omega|_{0,1}$.

For $0 \leq t_1 \leq t_2 \leq t_3 \leq 1$ and $\omega_1 \in \Omega_{t_1,t_2}, \omega_2 \in \Omega_{t_2,t_3}$, we define $(\omega_1, \omega_2) := \omega$ by $\omega(t) := \omega_1(t)$ for $t \in [t_1, t_2]$ and $\omega(t) = \omega_1(t_2) + \omega_2(t)$ for $t \in [t_2, t_3]$. On the other hand, for $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 \leq 1$ and $\omega \in \Omega_{t_1,t_4}$, $\omega^{t_2,t_3} \in \Omega_{t_2,t_3}$ is defined by $\omega^{t_2,t_3}(t) := \omega(t) - \omega(t_2)$, $t \in [t_2, t_3]$.

For $0 \leq t_1 \leq t_2 \leq 1$, B^{t_1,t_2} is the coordinate process in Ω_{t_1,t_2} . That is, $B^{t_1,t_2}(\omega) := \omega$ for all $\omega \in \Omega_{t_1,t_2}$. Let \mathbb{F}^{t_1,t_2} denote the filtration generated by B^{t_1,t_2} and $\mathbb{F}^{t_1,t_2,+}$ its right limit. Let $\mathbb{P}_0^{t_1,t_2}$ denote the Wiener measure on Ω_{t_1,t_2} , and $\bar{\mathcal{A}}^{t_1,t_2}$ the space of all \mathbb{F}^{t_1,t_2} -adapted processes $a : [t_1, t_2] \times \Omega_{t_1,t_2} \rightarrow \mathbb{S}_d^+(\mathbb{R})$ satisfying (2.2) $dt \times d\mathbb{P}_0^{t_1,t_2}$ -a.s. For $a \in \bar{\mathcal{A}}^{t_1,t_2}$, define \tilde{a} , as in (2.5), and \mathbb{P}^a on $\mathcal{F}_2^{t_1,t_2}$ in an obvious way. For any $a \in \bar{\mathcal{A}}^{0,t_2}$, since a is adapted, we have $a_{t_1}(\omega_1) = a_{t_1}(\omega_2)$ for any $\omega_1, \omega_2 \in \Omega_{0,t_2}$ such that $\omega_1^{0,t_1} = \omega_2^{0,t_1}$. Thus we can define $a^{0,t_1} \in \bar{\mathcal{A}}^{0,t_1}$ by $a^{0,t_1}(\omega_1) := a(\omega_1, \omega_2)$ for any $\omega_1 \in \Omega_{0,t_1}, \omega_2 \in \Omega_{t_1,t_2}$.

Moreover, for any $\omega_0 \in \Omega$, $B^{t_1, t_2, \omega_0} \in \Omega_{0, t_2}$ is defined by $B^{t_1, t_2, \omega_0}(\omega) := (\omega_0^{0, t_1}, \omega)$ for all $\omega \in \Omega_{t_1, t_2}$. For $a \in \overline{\mathcal{A}}^{0, t_2}$, define $a^{t_1, \omega_0} \in \overline{\mathcal{A}}^{t_1, t_2}$ by $a^{t_1, \omega_0}(\omega)_t := a(\omega_0^{0, t_1}, \omega)_t$, for all $\omega \in \Omega_{t_1, t_2}$ and $t \in [t_1, t_2]$.

The following lemma is obvious.

Lemma 4.1 *Assume $0 \leq t_1 < t_2 \leq 1$, $\varphi : \Omega_{0, t_2} \rightarrow \mathbb{R}$ is $\mathcal{F}_{t_2}^{0, t_2}$ -measurable and bounded. Then for any $a \in \overline{\mathcal{A}}^{0, t_2}$, we have*

$$\mathbb{E}^{a^{t_1, \omega_0}}[\varphi(B^{t_1, t_2, \omega_0})] = \mathbb{E}^a[\varphi(B^{0, t_2}) | \mathcal{F}_{t_1}^{0, t_2}](\omega_1), \quad \mathbb{P}^{a^{0, t_1}} - a.s. \omega_1 \in \Omega_{0, t_1}.$$

4.2 The duality result

We first construct a universal process $\{V_t, t \in [0, 1]\}$. Denote $\xi = g(B.)$ and $F_s(y, z, a) := \phi(s, B., y, z, a)$ for some deterministic functions g and ϕ .

Assumption 4.2 (i) *For any $0 \leq t_0 \leq 1$, $\omega_0 \in \Omega$, and $a \in \overline{\mathcal{A}}^{t_0, 1, F}$,*

$$\xi_{t_0}^2(\omega_0) := \mathbb{E}^a \left\{ |g(B^{t_0, 1, \omega_0})|^2 + \int_{t_0}^1 |\phi(s, B^{t_0, 1, \omega_0}, 0, 0, \hat{a}_s)|^2 ds \right\} < \infty. \quad (4.1)$$

(ii) *For any $a \in \overline{\mathcal{A}}^F$, $\mathbb{E}^a \{\sup_{0 \leq t \leq 1} \xi_t^2\} < \infty$.*

(iii) *For any (t, y, z) and $a \in \text{dom}(F_t)$: $|\phi(t, y, z, a)| \leq C [|\phi(t, 0, 0, a)| + |y| + |\sqrt{a}z|]$.*

(iv) *$\text{dom}(F)$ is independent of ω , and there exist a constant C and a modulus of continuity function ρ such that for all $a \in \text{dom}(F_t)$:*

$$|g(\omega_1) - g(\omega_2)| + |\phi(t, \omega_1, y_1, z_1, a) - \phi(t, \omega_2, y_2, z_2, a)| \leq C \left[\rho(|\omega_1 - \omega_2|_\infty) + |y_1 - y_2| + |\sqrt{a}(z_1 - z_2)| \right],$$

For any $0 \leq t_0 \leq 1$, $\omega_0 \in \Omega$, and $a \in \overline{\mathcal{A}}^{t_0, 1, F}$, let $(Y_t^{a, t_0, \omega_0}, Z_t^{a, t_0, \omega_0})_{t_0 \leq t \leq 1}$ denote the unique solution of the following BSDE under \mathbb{P}^a :

$$Y_t^{a, t_0, \omega_0} = g(B^{t_0, 1, \omega_0}) - \int_t^1 \phi(s, B^{t_0, 1, \omega_0}, Y_s^{a, t_0, \omega_0}, Z_s^{a, t_0, \omega_0}, \hat{a}_s) ds - \int_t^1 Z_s^{a, t_0, \omega_0} dB_s^{t_0, 1}.$$

By Remark 2.5, $Y_{t_0}^{a, t_0, \omega_0}$ is a constant. Define

$$V_t(\omega) := \sup_{a \in \overline{\mathcal{A}}^{t, 1, F}} Y_t^{a, t, \omega}, \quad \text{for all } 0 \leq t \leq 1, \omega \in \Omega. \quad (4.2)$$

We note that ξ_t^2 and V_t are defined for all $\omega \in \Omega$. Our main result of this section involves the following additional

Assumption 4.3 *For any $\varepsilon > 0$, there is an ε -maximizer γ^ε of (3.1) in the domain, i.e.*

$$\frac{1}{2}a : \gamma_t^\varepsilon(y, z, a) - H_t(y, z, \gamma_t^\varepsilon(y, z, a)) \geq F_t(y, z, a) - \varepsilon \quad \text{for } a \in \text{dom}(F_t).$$

such that γ^ε is progressively measurable over all variables, \mathbb{F}^+ -adapted for fixed (y, z, a) . Moreover, for every $\delta > 0$, there is a constant $C_{\varepsilon, \delta}$ such that:

$$|\gamma_t^\varepsilon(y, z, a)| \leq C_{\varepsilon, \delta} (1 + |y| + |z|) \quad \text{for every } a \geq \delta I_d.$$

Theorem 4.4 (i) Under Assumptions 3.1, 3.4, and 4.2, we have $\hat{V}(\xi) = v(\xi)$, and existence holds for the problem $\hat{V}(\xi)$.

(ii) Under the additional Assumption 4.3, we have $\bar{V}(\xi) = \hat{V}(\xi)$ for all $\xi \in \hat{\mathbb{L}}^2$. Moreover, if Assumption 4.3 holds with $\varepsilon = 0$, then existence holds for the problem $\bar{V}(\xi)$.

In this section Assumptions 3.1, 3.4, and 4.2 are always in force.

Lemma 4.5 For all (t, ω_1, ω_2) , we have $|V_t(\omega_1)|^2 \leq C\xi_t^2(\omega_1)$, and $|V_t(\omega_1) - V_t(\omega_2)| \leq C\rho(|\omega_1 - \omega_2|_{0,t})$. Consequently, V_t is \mathcal{F}_t -measurable for every $t \in [0, 1]$.

Proof. (i) For any $0 \leq t_0 \leq 1$, $\omega_0 \in \Omega_{0,t_0}$, and $a \in \bar{\mathcal{A}}^{t_0,1}$,

$$Y_t^{a,t_0,\omega_0} = g(B^{t_0,1,\omega_0}) - \int_t^1 \left[\phi(s, B^{t_0,1,\omega_0}, 0, 0, \tilde{a}_s) + \alpha_s Y_s^{a,t_0,\omega_0} + \beta_s \sqrt{\tilde{a}_s} Z_s^{a,t_0,\omega_0} \right] ds - \int_t^1 Z_s^{a,t_0,\omega_0} dB_s^{t_0,1},$$

where α, β are bounded, thanks to Assumption (4.2) (ii). Now by standard arguments, it follows from (2.5) that $|Y_{t_0}^{a,t_0,\omega_0}|^2 \leq C\xi_{t_0}^2(\omega_0)$. Since a is arbitrary, we know V is bounded.

(ii) For any $0 \leq t_0 \leq 1$, $\omega_1, \omega_2 \in \Omega_{0,t_0}$, and any $a \in \bar{\mathcal{A}}^{t_0,1}$, denote

$$\begin{aligned} \Delta Y_t &:= Y_t^{a,t_0,\omega_1} - Y_t^{a,t_0,\omega_2}, \quad \Delta Z_t := Z_t^{a,t_0,\omega_1} - Z_t^{a,t_0,\omega_2}, \quad \Delta g := g(B^{t_0,1,\omega_1}) - g(B^{t_0,1,\omega_2}), \\ \Delta F_t &:= \phi(t, B^{t_0,1,\omega_1}, Y_t^{a,t_0,\omega_1}, Z_t^{a,t_0,\omega_1}, \tilde{a}_t) - \phi(t, B^{t_0,1,\omega_2}, Y_t^{a,t_0,\omega_1}, Z_t^{a,t_0,\omega_1}, \tilde{a}_t). \end{aligned}$$

Then $|\Delta g| + |\Delta F_t| \leq C\rho(\|\omega_1 - \omega_2\|_{0,t_0})$ and

$$\Delta Y_t = \Delta g - \int_t^1 [\Delta F_s + \alpha_s \Delta Y_s + \sqrt{\tilde{a}_s} \beta_s \Delta Z_s] ds - \int_t^1 \Delta Z_s dB_s^{t_0,1}, \quad t_0 \leq t \leq 1, \mathbb{P}^a - \text{a.s.}$$

where α, β are bounded, thanks to Assumption (4.2) (iii). By standard arguments, it follows from (2.5) that $|\Delta Y_{t_0}| \leq C\rho(\|\omega_1 - \omega_2\|_{0,t_0})$. Since a is arbitrary, we prove the lemma. \square

Lemma 4.6 For any $t_0, a^0 \in \bar{\mathcal{A}}^{0,t_0,F}$ and any $\varepsilon > 0$, there exists $a^\varepsilon \in \bar{\mathcal{A}}^F = \bar{\mathcal{A}}^{0,1,F}$ such that $a^{\varepsilon,0,t_0} = a^0$ and

$$V_{t_0}(\omega_0) \leq Y_{t_0}^{a^\varepsilon, t_0, \omega_0, t_0, \omega_0} + \varepsilon, \quad \text{for all } \omega_0 \in \Omega_{0,t_0}.$$

Proof. Let $\delta > 0$ be a constant which will be specified later. Since Ω_{0,t_0} is separable, there exists a partition $\Omega_{0,t_0}^i \in \mathcal{F}_{t_0}^{0,t_0}, i = 1, 2, \dots$ such that $|\omega_1 - \omega_2|_{0,t_0} \leq \delta$ for any i

and any $\omega_1, \omega_2 \in \Omega_{0,t_0}^i$. For each i , fix an $\hat{\omega}_i \in \Omega_{0,t_0}^i$, and let $a^{\varepsilon,i} \in \bar{\mathcal{A}}^{t_0,1,F}$ be an $\varepsilon/2$ -optimizer of $\hat{Y}_{t_0}(\hat{\omega}_i)$, i.e. $V_{t_0}(\hat{\omega}_i) \leq Y_{t_0}^{a^{\varepsilon,i}, t_0, \hat{\omega}_i} + \frac{\varepsilon}{2}$. Now define $a^\varepsilon \in \bar{\mathcal{A}}^F$ as follows. For $\omega_1 \in \Omega_{0,t_0}, \omega_2 \in \Omega_{t_0,1}$,

$$a_t^\varepsilon(\omega_1, \omega_2) := \begin{cases} a_t^0(\omega_1), & t \in [0, t_0]; \\ \sum_{i=1}^\infty a_t^{\varepsilon,i}(\omega_2) 1_{\Omega_{0,t_0}^i}(\omega_1), & t \in [t_0, 1]. \end{cases}$$

Then for all $\omega_1 \in \Omega_{0,t_0}^i$, we have $a^{\varepsilon,t_0,\omega_1} = a^{\varepsilon,i}$. By Lemma 4.5, we get

$$\begin{aligned} & V_{t_0}(\omega_1) - V_{t_0}^{a^{\varepsilon,t_0,\omega_0}, t_0, \omega_0} \\ &= V_{t_0}(\omega_1) - V_{t_0}(\hat{\omega}_i) + V_{t_0}(\hat{\omega}_i) - Y_{t_0}^{a^{\varepsilon,i}, t_0, \hat{\omega}_i} + Y_{t_0}^{a^{\varepsilon,i}, t_0, \hat{\omega}_i} - V_{t_0}^{a^{\varepsilon,t_0,\omega_0}, t_0, \omega_0} \leq 2C\rho(\delta) + \frac{\varepsilon}{2}. \end{aligned}$$

Now by choosing δ small enough we prove the result. \square

The following dynamic programming principle is important.

Proposition 4.7 *For any $0 \leq t_1 < t_2 \leq 1$, we have*

$$V_{t_1}(\omega_1) = \sup_{a \in \bar{\mathcal{A}}^{t_1,t_2,F}} \hat{V}_{t_1}^{a,t_1,t_2,\omega_1}, \quad \text{for all } \omega_1 \in \Omega_{0,t_1},$$

where, for $t_1 \leq t \leq t_2$, $(\hat{V}^{a,t_1,t_2,\omega_1}, \hat{Z}^{a,t_1,t_2,\omega_1})$ is the solution of the BSDE under \mathbb{P}^a ,

$$\hat{V}_t^{a,t_1,t_2,\omega_1} = V_{t_2}(B^{t_1,t_2,\omega_1}) - \int_t^{t_2} \phi(s, B^{t_1,t_2,\omega_1}, \hat{V}_s^{a,t_1,t_2,\omega_1}, \hat{Z}_s^{a,t_1,t_2,\omega_1}, \tilde{a}_s) ds - \int_t^{t_2} \hat{Z}_s^{a,t_1,t_2,\omega_1} dB_s^{t_1,t_2}.$$

Proof. Fix $\omega_1 \in \Omega_{0,t_1}$. On one hand, for any $a \in \bar{\mathcal{A}}^{t_1,1,F}$ and $t_1 \leq t \leq t_2$, we have

$$Y_t^{a,t_1,\omega_1} = Y_{t_2}^{a,t_1,\omega_1} - \int_t^{t_2} F_s(B^{t_1,t_2,\omega_1}, Y_s^{a,t_1,t_2,\omega_1}, Z_s^{a,t_1,t_2,\omega_1}, \tilde{a}_s) ds - \int_t^{t_2} Z_s^{a,t_1,t_2,\omega_1} dB_s^{t_1,t_2}.$$

Note that $Y_{t_2}^{a,t_1,\omega_1} \leq V_{t_2}(B^{t_1,t_2,\omega_1})$. By the comparison theorem for BSDEs we get $Y_t^{a,t_1,\omega_1} \leq \hat{V}_t^{a,t_1,t_2,\omega_1}$. Since a is arbitrary, we obtain that

$$V_{t_1}(\omega_1) \leq \sup_{a \in \bar{\mathcal{A}}^{t_1,t_2,F}} \hat{V}_{t_1}^{a,t_1,t_2,\omega_1}, \quad \text{for all } \omega_1 \in \Omega_{0,t_1}.$$

On the other hand, for any $a \in \bar{\mathcal{A}}^{t_1,t_2,F}$ and any $\varepsilon > 0$, by Lemma 4.6, there exists $a^\varepsilon \in \bar{\mathcal{A}}^{t_1,1,F}$ such that $a^{\varepsilon,t_1,t_2} = a$ and

$$V_{t_2}(\omega_1, \omega_2) \leq Y_{t_2}^{a^\varepsilon,t_2,(\omega_1,\omega_2)} + \varepsilon, \quad \text{for all } \omega_2 \in \Omega_{t_1,t_2}.$$

By standard arguments for BSDEs we have $\hat{V}_{t_1}^{a,t_1,t_2,\omega_1} \leq Y_{t_1}^{a^\varepsilon,t_1,t_2,t_1,\omega_1} + C\varepsilon \leq V_{t_1}(\omega_1) + C\varepsilon$, which implies that

$$\sup_{a \in \bar{\mathcal{A}}^{t_1,t_2,F}} \hat{V}_{t_1}^{a,t_1,t_2,\omega_1} \leq V_{t_1}(\omega_1) + C\varepsilon.$$

Since ε is arbitrary, we prove the result. \square

Proof of Theorem 4.4. (i) Notice that the limit in (2.14) depends on \mathbb{P}^a . We first define a universal version of V_t^+ :

$$V_t^+ := \overline{\lim}_{r \in \mathbb{Q} \cap (t, 1], r \downarrow t} V_r, \quad \text{and denote } N := \{\omega \in \Omega : \hat{Y}^+(\omega) \text{ is not càdlàg}\}.$$

Then by Lemma 2.9, $\mathbb{P}^a(N) = 0$ for every $a \in \overline{\mathcal{A}}^F$. Moreover, following the standard arguments as in [11] and [3], we have $V_t^+ = V_t$, \mathbb{P}^a -a.s. Therefore, by choosing V^+ as its universal modification, we may assume V is a càdlàg strong \hat{F} -supermartingale under \mathbb{P}^a for every $a \in \overline{\mathcal{A}}^F$. Therefore, by Theorem 2.10 there exist unique processes $Z^a \in \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+)$ and $K^a \in \mathcal{I}^2(\mathbb{P}^a, \mathbb{F}^+)$ such that

$$V_t = V_0 - \int_0^t \hat{F}_s(V_s, Z_s^a) ds + \int_0^t Z_s^a dB_s - K_t^a, \quad 0 \leq t \leq 1, \quad \mathbb{P}^a - \text{a.s.}$$

Moreover, by Lemma 2.14, we can define uniquely a universal process Z by $d\langle V, B \rangle_t = Z_t d\langle B \rangle_t$, so that $Z = Z^a$, $dt \times d\mathbb{P}^a$ -a.s. for all $a \in \overline{\mathcal{A}}^F$. Thus we have

$$V_t = V_0 - \int_0^t \hat{F}_s(V_s, Z_s) ds + \int_0^t Z_s dB_s - K_t^a, \quad 0 \leq t \leq 1, \quad \mathbb{P}^a - \text{a.s.} \quad (4.3)$$

Now by Theorem 3.5, we get $\hat{V}(\xi) \leq V_0 = v(\xi)$, and the required result follows from (3.20).

(ii) We already know that $\bar{V}(\xi) \geq \hat{V}(\xi)$ by (3.20). It remains to show that $\bar{V}(\xi) \leq \hat{V}(\xi)$ when $\hat{V}(\xi) < \infty$. Define the process \bar{G}^ε for any $\varepsilon > 0$ by:

$$\bar{G}_t^\varepsilon := \gamma_t^\varepsilon(V_t, Z_t, \hat{a}_t), \quad \text{for } t < \hat{\tau},$$

where γ^ε is introduced in Assumption 4.3. By (2.2), we have

$$|\bar{G}_t^\varepsilon| \leq C_{\varepsilon, a} \left(1 + |\hat{Y}_t| + |\hat{Z}_t|\right) \quad t < \hat{\tau}, \quad dt \times d\mathbb{P}^a - \text{a.s. for every } a \in \overline{\mathcal{A}}.$$

Here $C_{\varepsilon, a}$ may depend on \underline{a} , the lower bound of a . Then $\bar{G}^\varepsilon \in \hat{\mathcal{H}}_H^2$. Let $\bar{Z} := Z$ and $\bar{Y}^a := \bar{Y}^{a, V_0 + \varepsilon, \bar{Z}, \bar{G}^\varepsilon}$ be defined by (3.12). On $\{t < \hat{\tau}\}$, set $\delta Y^a := \bar{Y}^a - V$, then

$$\begin{aligned} \delta Y_t^a &= \varepsilon + \int_0^t \left[\frac{1}{2} \hat{a}_s : \bar{G}_s^\varepsilon - H_s(\bar{Y}_s^a, \bar{Z}_s, \bar{G}_s^\varepsilon) - \hat{F}_s(V_s, Z_s) \right] ds + K_t^a \\ &\geq \varepsilon + \int_0^t \left[H_s(V_s, Z_s, \bar{G}_s^\varepsilon) - H_s(\bar{Y}_s^a, Z_s, \bar{G}_s^\varepsilon) - \varepsilon \right] ds \geq \int_0^t \alpha_s \delta Y_s ds, \quad \mathbb{P}^a - \text{a.s.} \end{aligned}$$

where α is bounded by Assumption 3.1. Thus $\delta Y_1^a \geq 0$ on $\{\hat{\tau} = 1\}$, \mathbb{P}^a -a.s. Since $\bar{Y}_1^a = V_1 = \infty$ on $\{\hat{\tau} < 1\}$, this implies that $\bar{Y}_1^a \geq V_1 = \xi$, \mathbb{P}^a -a.s. and therefore $\bar{V}(\xi) \leq V_0 + \varepsilon = \hat{V}(\xi) + \varepsilon$. Since ε is arbitrary, this proves the required result.

The existence for the problem \bar{V} when Assumption 4.3 holds with $\varepsilon = 0$ is obvious. \square

5 A weaker version of the second order target problem

Our goal for the rest of the paper is to suggest a slight modification of the second order stochastic target problem so that its value is not affected by the relaxations of Section 3.3. The key-tool for this is to define a dominating measure for any countable family of diffusion processes $a \in \overline{\mathcal{A}}$.

5.1 The reference probability measures $\hat{\mathbb{P}}$

Let T_0 be a dense subset of $[0, 1]$ containing the end-points $\{0, 1\}$, and $\mathcal{A}_0 = (a^i)_{i \geq 1}$ a sequence in $\overline{\mathcal{A}}$ satisfying the concatenation property:

$$a^i \mathbf{1}_{[0, t_0)} + a^j \mathbf{1}_{[t_0, 1]} \in \mathcal{A}_0 \quad \text{for every } i, j \geq 1 \text{ and } t_0 \in T_0. \quad (5.1)$$

Remark 5.1 A typical construction of such a family \mathcal{A}_0 is as follows. Let $A = (A_i)_{i \geq 1}$ be an arbitrary sequence in $\overline{\mathcal{A}}$ satisfying (5.2), and let T_0 be as above. Then, there exists a minimal countable family \mathcal{A}_0 which contains A and satisfies the concatenation condition (5.1). This follows immediately by complementing the family A with all possible concatenations. \square

By standard arguments, we may find a sequence of positive scalars ν_i , $i \geq 1$, such that

$$\sum_{i=1}^{\infty} \nu_i = 1 \quad \text{and} \quad \sum_{i=1}^{\infty} \nu_i \mathbb{E}^{\mathbb{P}^0} \left[\int_0^1 \text{Tr}[a_t^i] dt \right] < \infty. \quad (5.2)$$

Let $\mathbb{P}^i := \mathbb{P}^{a^i} \in \mathcal{M}$ and $\mathbb{E}^i := \mathbb{E}^{\mathbb{P}^i}$, so that $\mathbb{E}^i[|B_1|^2] = \mathbb{E}^{\mathbb{P}^0}[\int_0^1 \text{Tr}[a_t^i] dt]$. Define

$$\hat{\mathbb{P}} := \hat{\mathbb{P}}^{\mathcal{A}_0, T_0} := \sum_{i=1}^{\infty} \nu_i \mathbb{P}^i. \quad (5.3)$$

The following lemma, together with (5.2), shows that $\hat{\mathbb{P}} \in \mathcal{M}$.

Lemma 5.2 *Let $\mathbb{P}_i \in \mathcal{M}$, $\alpha_i \in [0, 1]$, $i \geq 1$ with $\sum_{i \geq 1} \alpha_i = 1$ and $\sum_{i \geq 1} \alpha_i \mathbb{E}^{\mathbb{P}_i}[|B_1|^2] < \infty$. Then $\mathbb{P} := \sum_{i=1}^{\infty} \alpha_i \mathbb{P}_i \in \mathcal{M}$. In particular, $\hat{\mathbb{P}}$ defined above is in \mathcal{M} .*

Proof. First, it is obvious that \mathbb{P} is a probability measure and $\mathbb{P}(B_0 = 0) = 1$. Next, $\mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq 1} |B_t|^2 \right] = \sum_{i=1}^{\infty} \alpha_i \mathbb{E}^{\mathbb{P}_i} \left[\sup_{0 \leq t \leq 1} |B_t|^2 \right] \leq C \sum_{i=1}^{\infty} \alpha_i \mathbb{E}^{\mathbb{P}_i}[|B_1|^2] < \infty$. Moreover, for any $s < t$ and any bounded $\xi \in \mathcal{F}_s$ taking values in \mathbb{R}^d , $\mathbb{E}^{\mathbb{P}}[(B_t - B_s) \cdot \xi] = \sum_{i=1}^{\infty} \alpha_i \mathbb{E}^{\mathbb{P}_i}[(B_t - B_s) \cdot \xi] = 0$. So $\mathbb{E}^{\mathbb{P}}[B_t | \mathcal{F}_s] = B_s$ and thus $\mathbb{P} \in \mathcal{M}$. \square

Notice that the probability measure $\hat{\mathbb{P}}$ and all the above construction relies crucially on the choice of the countable families \mathcal{A}_0 and T_0 . Theorem 5.8 below provides conditions under which our main results do not depend on this choice.

5.2 Aggregation of processes under $\hat{\mathbb{P}}$

Let $t_l^n := l 2^{-n}$, and $\Delta_l^n B_t := B_{t \wedge t_l^n} - B_{t \wedge t_{l-1}^n}$, $l, n \geq 0$. In order to better understand the structure of the supports of the singular measures \mathbb{P}^i , $i \geq 1$, we introduce for every $\tau \in \mathcal{T}_1$:

$$\Omega_\tau^i := \left\{ \langle B \rangle_t = \int_0^t a_s^i ds, \quad t \leq \tilde{\tau} \text{ for some } \mathcal{T}_1 \ni \tilde{\tau} > \tau \right\} \quad (5.4)$$

Clearly $\Omega_\tau^i \in \mathcal{F}_\tau^+$ and $\mathbb{P}^i(\Omega_\tau^i) = 1$. Consequently, $\hat{\mathbb{P}}(\cup_{i \geq 1} \Omega_1^i) = 1$. We also introduce:

$$\mathcal{A}_0(i, \tau) := \left\{ a^j \in A : a^j = a^i \text{ on } [0, \tilde{\tau}] dt \times d\mathbb{P}_0 - \text{a.s. for some } \mathcal{T}_1 \ni \tilde{\tau} > \tau \right\}, \quad (5.5)$$

so that $\mathcal{A}_0 = \cup_i \mathcal{A}_0(i, 0)$. The following lemma is obvious, and collects the main properties of the Ω_τ^i 's.

Lemma 5.3 *For any $\tau \in \mathcal{T}_1$ and $i \geq 1$, we have:*

- (i) $\Omega_\tau^i \subset \Omega_\theta^i$, and $\Omega_{\theta+}^i = \Omega_\theta^i$ for every $\theta \in \mathcal{T}_\tau$,
- (ii) If $a^j \in \mathcal{A}_0(i, \tau)$, then $\Omega_\tau^i = \Omega_\tau^j$ and $\mathbb{P}^i = \mathbb{P}^j$ on \mathcal{F}_τ^+ .

Note that $\Omega_{\tau-}^i \neq \Omega_\tau^i$, in general. The next important result shows that one can aggregate processes defined \mathbb{P}^i -a.s., $i \geq 1$, under some consistency condition.

Lemma 5.4 *Let $X^i \in \mathcal{H}^0(\mathbb{P}^i, \mathbb{F}^+)$ for any $i \geq 1$, and assume that for every $\tau \in \mathcal{T}_1$:*

$$X^i = X^j, \quad ds \times d\mathbb{P}^i - \text{a.s. on } [0, \tau] \quad \text{for any } i, j \text{ such that } a^j \in \mathcal{A}_0(i, \tau).$$

Then there is a unique $X \in \mathcal{H}^0(\hat{\mathbb{P}}, \mathbb{F}^+)$ s.t. $X = X^i, dt \times d\mathbb{P}^i$ -a.s. on $[0, 1] \times \Omega$, $i \geq 1$. Moreover, if $X^i \in \mathcal{D}^0(\mathbb{P}^i, \mathbb{F}^+)$ (resp. $\mathcal{I}^0(\mathbb{P}^i, \mathbb{F}^+), \mathcal{S}^0(\mathbb{P}^i, \mathbb{F}^+)$) for any i , then $X \in \mathcal{D}^0(\hat{\mathbb{P}}, \mathbb{F}^+)$ (resp. $\mathcal{I}^0(\hat{\mathbb{P}}, \mathbb{F}^+), \mathcal{S}^0(\hat{\mathbb{P}}, \mathbb{F}^+)$).

Proof. The uniqueness is obvious. We shall only prove the existence.

1. Assume $X^i \in \mathcal{H}^0(\mathbb{P}^i, \mathbb{F}^+)$ for all i . Define

$$X_t := \sum_{i=1}^{\infty} X_t^i \mathbf{1}_{\Omega_i^i \setminus \cup_{j \leq i-1} \Omega_t^j}. \quad (5.6)$$

By Lemma 5.3, $X \in \mathcal{H}^0(\hat{\mathbb{P}}, \mathbb{F}^+)$. Now fix i , and define the stopping times

$$\theta_0 := 0, \quad \theta_j := \inf \left\{ t : \int_0^t a_s^i ds \neq \int_0^t a_s^j ds \right\} \vee \theta_{j-1}, \quad j = 1, \dots, i-1, \quad \theta_i := 1.$$

One can easily check that

$$X^i = X^j \quad \text{on } [\theta_{j-1}, \theta_j]. \quad (5.7)$$

By our assumption, $X^i = X^j$, $dt \times d\mathbb{P}^i$ -a.s. in $[\theta_{j-1}, \theta_j) \times \Omega$. Then $X^i = X$, $dt \times d\mathbb{P}^i$ -a.s. in $[\theta_{j-1}, \theta_j) \times \Omega$ for any j . Thus $X^i = X$, $dt \times d\mathbb{P}^i$ -a.s. in $[0, 1) \times \Omega$.

2. Assume $X^i \in \mathcal{D}^0(\mathbb{P}^i, \mathbb{F}^+)$ (resp. $\mathcal{I}^0(\mathbb{P}^i, \mathbb{F}^+)$, $\mathcal{S}^0(\mathbb{P}^i, \mathbb{F}^+)$) for every $i \geq 1$. By Lemma 5.3, it suffices to show that X is càdlàg (resp. nondecreasing, continuous) in Ω_1^i , \mathbb{P}^i -a.s., for any i . For any $j < i$, since X^i, X^j are \mathbb{P}^i -modifications in $[\theta_{j-1}, \theta_j)$ and both are càdlàg, we know they are \mathbb{P}^i -indistinguishable in $[\theta_{j-1}, \theta_j)$. Then by (5.7) we know X and X^i are \mathbb{P}^i -indistinguishable on $[0, 1]$. Therefore, X is càdlàg (resp. nondecreasing, continuous) \mathbb{P}^i -a.s. \square

5.3 More singular probability measures dominated by $\hat{\mathbb{P}}$

By $\bar{\mathcal{A}}_0$, we denote the set of all processes $a \in \bar{\mathcal{A}}$ such that, for some non-decreasing sequence $(\tau_n)_{n \geq 1} \subset \mathcal{T}_1$ with values in T_0 , such that $\inf\{n : \tau_n = 1\} < \infty$, \mathbb{P}_0 -a.s.

$$\int_{\tau_n}^{\cdot} a_s ds = \int_{\tau_n}^{\cdot} a_s^i ds \quad \text{on } [\tau_n, \tau_{n+1}] \text{ for some } i \geq 1, \mathbb{P}_0 - \text{a.s.} \quad (5.8)$$

We also introduce the subset

$$\bar{\mathcal{A}}_0(i, \tau) := \left\{ a \in \bar{\mathcal{A}}_0 : a = a^i \text{ on } [0, \tilde{\tau}] \text{ for some } \mathcal{T} \ni \tilde{\tau} > \tau \text{ } dt \times d\mathbb{P}_0 - \text{a.s.} \right\}. \quad (5.9)$$

Clearly, $\bar{\mathcal{A}}_0 = \cup_i \bar{\mathcal{A}}_0(i, 0)$. It is obvious that $\bar{\mathcal{A}}_0(i, \tau)$ is decreasing in τ , and

$$\mathbb{P}^a = \mathbb{P}^i \quad \text{on } \mathcal{F}_\tau^+ \text{ whenever } a \in \bar{\mathcal{A}}_0(i, \tau).$$

By definition, $\hat{\mathbb{P}}$ is a dominating measure for all \mathbb{P}^i , $i \geq 1$. We first show that this property extends to all of $\bar{\mathcal{A}}_0$.

Proposition 5.5 *For any $a \in \bar{\mathcal{A}}_0$, \mathbb{P}^a is absolutely continuous with respect to $\hat{\mathbb{P}}$.*

Proof. Fix $a \in \bar{\mathcal{A}}_0$, define the disjoint subsets of Ω :

$$\mathcal{O}(i) := \{a = a^i \text{ on } [0, 1]\}, \quad \text{for } i \geq 1. \quad (5.10)$$

Notice that $\Omega = \cup_{i \geq 1} \mathcal{O}(i)$, as a consequence of the crucial concatenation condition (5.1). We also observe that $X^a = X^{a^i}$, \mathbb{P}_0 -a.s. on $\mathcal{O}(i)$, and therefore \mathbb{P}^a and \mathbb{P}^i coincide on $\mathcal{O}(i)$. To see this, let ξ be an arbitrary bounded \mathcal{F}_1 -measurable random variable, and write $\xi \mathbf{1}_{\mathcal{O}(i)} = \mathbb{E}^{\mathbb{P}_0}[\xi \mathbf{1}_{\mathcal{O}(i)}] + \int_0^1 \eta_s dB_s$ for some process $\eta \in \mathbb{H}^2(\mathbb{P}_0, \mathbb{F})$. Then $\mathbb{E}^{\mathbb{P}_0}[(X^a - X^{a^i}) \xi \mathbf{1}_{\mathcal{O}(i)}] = 0$, and $(X^a - X^{a^i}) \mathbf{1}_{\mathcal{O}(i)} = 0$ \mathbb{P}_0 -a.s. by the arbitrariness of ξ .

We now compute for any $E \in \mathcal{F}_1^+$ that $\mathbb{P}^a[E] = \sum_{i \geq 1} \mathbb{P}^a[E \cap \mathcal{O}(i)] = \sum_{i \geq 1} \mathbb{P}^i[E \cap \mathcal{O}(i)]$. Since $\mathbb{P}^i \ll \hat{\mathbb{P}}$ for each $i \geq 1$, this provides the required result. \square

5.4 The second order target problem under $\hat{\mathbb{P}}$

Define

$$\hat{\mathbb{L}}_0^2 := \bigcap_{i \geq 1} \mathbb{L}^2(\mathbb{P}^i, \mathcal{F}_1), \quad \hat{\mathcal{H}}_0^2 := \bigcap_{n \geq 1} \left\{ (X_t)_{t \in [0,1]} : X \mathbf{1}_{[0, \hat{\tau}_n]} \in \bigcap_{i \geq 1} \mathcal{H}^2(\mathbb{P}^i, \mathbb{F}^+) \right\},$$

and the corresponding subsets of càdlàg, continuous processes, and semimartingales: $X \in \hat{\mathcal{D}}_0^2$ (resp. $\hat{\mathcal{S}}_0^2$, $\hat{\mathcal{I}}_0^2$, $\widehat{\mathcal{SM}}_0^2$) iff for every $i, n \geq 1$, $X_{\cdot \wedge \hat{\tau}_n} \in \mathcal{D}^2(\mathbb{P}^i, \mathbb{F}^+)$ (resp. $\mathcal{S}^2(\mathbb{P}^i, \mathbb{F}^+)$, $\mathcal{I}^2(\mathbb{P}^i, \mathbb{F}^+)$, $\widehat{\mathcal{SM}}^2(\mathbb{P}^i, \mathbb{F}^+)$). Moreover, let

$$\hat{\mathcal{H}}_{0_H}^2 := \left\{ X \in \hat{\mathcal{H}}^2 : X \in \text{dom}(H) \, dt \times d\hat{\mathbb{P}} - \text{a.s.} \right\} \text{ and } \widehat{\mathcal{SM}}_{0_H}^2 := \left\{ Z \in \widehat{\mathcal{SM}}_0^2 : \Gamma \in \hat{\mathcal{H}}_H^2 \right\}.$$

For $y \in \mathbb{R}$ and $Z \in \widehat{\mathcal{SM}}_{0_H}^2$, let $Y^{y,Z} \in \hat{\mathcal{S}}_0^2$ denote the controlled process defined by the following ODE (with random coefficients):

$$Y_t = y - \int_0^t H_s(Y_s, Z_s, \Gamma_s) ds + \int_0^t Z_s \circ dB_s, \quad t < \hat{\tau}, \quad \hat{\mathbb{P}} - \text{a.s.} \quad (5.1)$$

$$Y_{\hat{\tau}} := \lim_{n \rightarrow \infty} Y_{\hat{\tau}_n} \text{ on } \bigcup_{n \geq 1} \{\hat{\tau}_n = 1\}, \quad Y_t := \infty \text{ for } t \in [\hat{\tau}, 1] \text{ on } \bigcap_{n \geq 1} \{\hat{\tau}_n < 1\}. \quad (5.2)$$

The existence and uniqueness of solution to (5.1) is obvious by applying Lemma 5.4. Now for any $\xi \in \mathcal{F}_1$, our second order target problem under $\hat{\mathbb{P}}$ is:

$$\mathcal{V}_0(\xi) := \inf \left\{ y : Y_1^{y,Z} \geq \xi, \quad \hat{\mathbb{P}} - \text{a.s. for some } Z \in \widehat{\mathcal{SM}}_{0_H}^2 \right\}. \quad (5.3)$$

Since $\hat{\mathbb{P}}$ depends on the particular choice of \mathcal{A}_0 and T_0 , this definition also depends on this choice. As a by-product of the duality result in next subsection, we shall provide sufficient conditions under which \mathcal{V}_0 does not depend on the choice of these dense subsets.

As in Section 3, we relax the problem. For any $y \in \mathbb{R}$, $\bar{Z} \in \hat{\mathcal{H}}_0^2$ and $\bar{G} \in \hat{\mathcal{H}}_{0_H}^2$, as a simple application of Lemma 5.4, the processes $\{\bar{Y}^{a^i, y, \bar{Z}, \bar{G}}\}_{i \geq 1}$ solving (3.12) can be patched together into one process $\bar{Y} := \bar{Y}^{y, \bar{Z}, \bar{G}} \in \hat{\mathcal{S}}_0^2$, which is the unique solution of:

$$\bar{Y}_t = y + \int_0^t \left(\frac{1}{2} \bar{G}_s \cdot \hat{a}_s - H_s(\bar{Y}_s, \bar{Z}_s, \bar{G}_s) \right) ds + \int_0^t \bar{Z}_s dB_s, \quad t < \hat{\tau}, \quad \hat{\mathbb{P}} - \text{a.s.} \quad (5.4)$$

$$\bar{Y}_{\hat{\tau}} := \lim_{n \rightarrow \infty} \bar{Y}_{\hat{\tau}_n} \text{ on } \bigcup_{n \geq 1} \{\hat{\tau}_n = 1\}, \quad \bar{Y}_t := \infty \text{ for } t \in [\hat{\tau}, 1] \text{ on } \bigcap_{n \geq 1} \{\hat{\tau}_n < 1\}, \quad (5.5)$$

Our relaxed target problem is

$$\bar{\mathcal{V}}_0(\xi) := \inf \left\{ y : \bar{Y}_1^{y, \bar{Z}, \bar{G}} \geq \xi, \quad \hat{\mathbb{P}} - \text{a.s. for some } (\bar{Z}, \bar{G}) \in \hat{\mathcal{H}}_0^2 \times \hat{\mathcal{H}}_{0_H}^2 \right\}. \quad (5.6)$$

Similarly, for $y \in \mathbb{R}$, $\hat{Z} \in \hat{\mathcal{H}}^2$, let $\hat{Y} := \hat{Y}^{y, \hat{Z}}$ be the solution of:

$$\hat{Y}_t = y + \int_0^t \hat{F}_s(\hat{Y}_s, \hat{Z}_s) ds + \int_0^t \hat{Z}_s dB_s, \quad t < \hat{\tau}, \quad \hat{\mathbb{P}} - \text{a.s.} \quad (5.7)$$

$$\hat{Y}_{\hat{\tau}} := \lim_{n \rightarrow \infty} \hat{Y}_{\hat{\tau}_n} \text{ on } \bigcup_{n \geq 1} \{\hat{\tau}_n = 1\}, \quad \hat{Y}_t := \infty \text{ for } t \in [\hat{\tau}, 1] \text{ on } \bigcap_{n \geq 1} \{\hat{\tau}_n < 1\}. \quad (5.8)$$

Our further relaxed second order target problem is defined by:

$$\hat{\mathcal{V}}_0(\xi) := \inf \left\{ y : \hat{Y}_1^{y, \hat{Z}} \geq \xi, \hat{\mathbb{P}} - \text{a.s. for some } \hat{Z} \in \hat{\mathcal{H}}_0^2 \right\}. \quad (5.9)$$

Finally, for any $i \geq 1$ and $\tau \in \mathcal{T}_1$, we define:

$$\bar{\mathcal{A}}_0^F := \bar{\mathcal{A}}^F \cap \bar{\mathcal{A}}_0, \quad \bar{\mathcal{A}}_0^F(i, \tau) := \{a \in \bar{\mathcal{A}}_0^F : a = a^i \text{ on } [0, \tilde{\tau}] \text{ for some } \mathcal{T}_1 \ni \tilde{\tau} > \tau\}.$$

Our dual formulation is, recalling (3.18):

$$v_0(\xi) := \sup_{a \in \bar{\mathcal{A}}_0^F} Y_0^a. \quad (5.10)$$

Similar to (3.20), it is obvious that

$$V_0(\xi) \geq \bar{V}_0(\xi) \geq \hat{V}_0(\xi) \geq v_0(\xi). \quad (5.11)$$

5.5 The main results

We shall adopt the following analogue of Assumption 4.2 (i)-(iii) in the present context.

Assumption 5.6 *The nonlinearity F satisfies:*

$$\begin{aligned} |F_t(0, z, a)| &\leq C (1 + |F(0, 0, a)| + |a|^{1/2}|z|), \quad dt \times d\mathbb{P}^a - \text{a.s. for } z \in \mathbb{R}^d, a \in \bar{\mathcal{A}}^F \\ \text{and } \mathbb{E}^i \left\{ \text{ess sup}_{\tau \in \mathcal{T}_{\tau_n}}^{\mathbb{P}^i} \text{ess sup}_{a \in \bar{\mathcal{A}}^F(i, \tau)}^{\mathbb{P}^i} \mathbb{E}_\tau^a [|\xi|^2 + \int_\tau^1 |\hat{F}_s^0|^2 ds] \right\} &< \infty \text{ for each } i \geq 1, n \geq 1. \end{aligned}$$

Theorem 5.7 *Let Assumption 3.1 hold true and consider an arbitrary $\xi \in \hat{\mathbb{L}}_0^2$.*

- (i) *If Assumption 3.2 holds, then $\mathcal{V}_0(\xi) = \bar{\mathcal{V}}_0(\xi)$.*
- (ii) *Assume further that Assumption 4.3 holds, then $\bar{\mathcal{V}}_0(\xi) = \hat{\mathcal{V}}_0(\xi)$.*
- (iii) *If Assumption 5.6 also holds, then $\hat{\mathcal{V}}_0(\xi) = v_0(\xi)$.*

We shall prove Theorem 5.7 in the next two subsections. Moreover, we have

Theorem 5.8 *In addition to the conditions of Theorem 5.7, assume*

- (i) $\xi = g(B.)$, $F_s(y, z, a) = \phi(s, B., y, z, a)$ whenever $a \in \text{dom}(F_t)$ for some uniformly continuous maps $g : C([0, 1])^d \rightarrow \mathbb{R}$, and $\phi_t(\cdot, y, z, \cdot) : C([0, 1])^d \times \mathbb{S}_+^d \rightarrow \mathbb{R}$.
- (ii) *For any $a \in \bar{\mathcal{A}}^F$ and any $\varepsilon > 0$, there exists $a^\varepsilon \in \bar{\mathcal{A}}_0^F$ such that $\mathbb{E}^{\mathbb{P}^0} \{ \int_0^1 |a_t^\varepsilon - a_t|^2 dt \} \leq \varepsilon$.*

Then $v_0(\xi) = v(\xi)$, so that $\mathcal{V}_0(\xi)$, $\bar{\mathcal{V}}_0(\xi)$ and $\hat{\mathcal{V}}_0(\xi)$ are independent from the choice of the sets \mathcal{A}_0 and T_0 .

Proof. For any $a \in \bar{\mathcal{A}}^F$, we define the processes X^a, W^a by (2.3), (2.4), and recall from (2.6) that W^a is a \mathbb{P}^a -Brownian motion with $(\mathbb{F}^{W^a})^{\mathbb{P}^a} = \mathbb{F}^a$. By the definition of \tilde{a} in (2.5), recall that (Y^a, Z^a) is defined by the BSDE:

$$Y_t^a = g(B_\cdot) + \int_t^1 \phi(s, B_\cdot, Y_s^a, Z_s^a, \tilde{a}_s) ds - \int_t^1 Z_s^a \tilde{a}_s^{1/2} dW_s^a, \quad 0 \leq t \leq 1, \quad \mathbb{P}^a - \text{a.s.}$$

Then $Y_0^a = \tilde{Y}_0^a$, where $(\tilde{Y}^a, \tilde{Z}^a)$ is the solution to the following BSDE under \mathbb{P}_0 :

$$\tilde{Y}_t^a = g(X_\cdot^a) + \int_t^1 \phi(s, X_\cdot^a, \tilde{Y}_s^a, \tilde{Z}_s^a, a_s) ds - \int_t^1 \tilde{Z}_s^a a_s^{1/2} dB_s, \quad 0 \leq t \leq 1, \quad \mathbb{P}_0 - \text{a.s.},$$

where we used the fact that $a_t = \mathbf{a}_t(B_\cdot)$ by the definition of the map \mathbf{a} . By Proposition 2.1 from El Karoui, Peng and Quenez [10], we deduce that for $a, b \in \bar{\mathcal{A}}^F$:

$$|Y_0^a - Y_0^b|^2 \leq \mathbb{E} \left[\left| g(X_\cdot^a) - g(X_\cdot^b) \right|^2 + \int_0^1 \left| \phi(t, X_\cdot^a, \tilde{Y}_t^b, \tilde{Z}_t^b, a_t) - \phi(t, X_\cdot^b, \tilde{Y}_t^b, \tilde{Z}_t^b, b_t) \right|^2 dt \right].$$

From the uniform continuity condition on g and ϕ , this shows that the map $a \mapsto Y_0^a$ is continuous from $\bar{\mathcal{A}}^F \subset \mathcal{H}^2(\mathbb{P}_0, \mathbb{F})$ to \mathbb{R} . This, together with condition (iii), proves that the maximization problem v_0 can be extended to the closure of the set $\bar{\mathcal{A}}^F$. \square

Remark 5.9 As a by-product of the previous proof, we have $Y_0^a = \tilde{Y}_0^a$ for each $a \in \bar{\mathcal{A}}^F$, where \tilde{Y}^a is the solution to a BSDE under \mathbb{P}_0 . In particular,

$$v_0(\xi) = \sup_{a \in \bar{\mathcal{A}}_0^F} \tilde{Y}_0^a \quad \text{and} \quad v(\xi) := \sup_{a \in \bar{\mathcal{A}}^F} \tilde{Y}_0^a.$$

5.6 The original problem and its relaxations are equivalent

The proof of Theorem 5.7 requires the following extension of Bank and Baum [1] to the nonlinear case, whose proof is postponed to the Appendix.

Lemma 5.10 *Let θ_n be an increasing sequence of \mathbb{F}^+ -stopping times taking values in $[0, 1]$, $\theta := \lim_{n \rightarrow \infty} \theta_n$, and $h_t(\omega, y, z) : [0, 1] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ be uniformly Lipschitz-continuous in y , uniformly continuous in z , and $h(0, 0) \mathbf{1}_{[0, \tau_n]} \in \cap_{i \geq 1} \mathcal{H}^2(\mathbb{P}^i, \mathbb{F}^+)$ for each $n \geq 1$. For a process Z satisfying $Z \mathbf{1}_{[0, \theta_n]} \in \cap_{i \geq 1} \mathcal{H}^2(\mathbb{P}^i, \mathbb{F}^+)$ for each $n \geq 1$, define a process \mathcal{Y}^Z satisfying $\mathcal{Y}_{\cdot \wedge \theta_n}^Z \in \cap_{i \geq 1} \mathcal{S}^2(\mathbb{P}^i, \mathbb{F}^+)$ by*

$$\mathcal{Y}_t^Z = y + \int_0^t h_s(\mathcal{Y}_s^Z, Z_s) ds + \int_0^t Z_s dB_s, \quad t < \tau, \quad \hat{\mathbb{P}} - \text{a.s.}$$

Then for any $\varepsilon > 0$, there is Z^ε such that $Z^\varepsilon \mathbf{1}_{[0, \theta_n]} \in \cap_{i \geq 1} \mathcal{S}^2(\mathbb{P}^i, \mathbb{F}^+)$ with finite variation for each $n \geq 1$, and

$$\sup_{0 \leq t < \theta} |\mathcal{Y}_t^{Z^\varepsilon} - \mathcal{Y}_t^Z| \leq \varepsilon, \quad \hat{\mathbb{P}} - \text{a.s.}$$

Proof of Theorem 5.7 (i) In view of (5.11), we only need to show that $\mathcal{V}_0(\xi) \leq \bar{\mathcal{V}}_0(\xi)$ when $\bar{\mathcal{V}}_0(\xi) < \infty$. For any $\varepsilon > 0$, there exist $y < \bar{\mathcal{V}}_0(\xi) + \varepsilon$ and $\bar{Z} \in \hat{\mathcal{H}}_0^2, \bar{G} \in \hat{\mathcal{H}}_{0_H}^2$ such that the corresponding $\bar{Y}_1 := \bar{Y}_1^{y, \bar{Z}, \bar{G}} \geq \xi$, $\hat{\mathbb{P}}$ -a.s. Denote for $t < \hat{\tau}$:

$$Z_t^0 := \int_0^t \bar{G}_s dB_s, \quad \mathcal{Z}_t := \bar{Z}_t - Z_t^0, \quad Y_t^0 := \int_0^t Z_s^0 dB_s, \quad \mathcal{Y}_t := \bar{Y}_t - Y_t^0,$$

and

$$h_s(\omega, y, z) := \frac{1}{2} \hat{a}_s(\omega) : \bar{G}_s(\omega) - H_s(\omega, y + Y_s^0(\omega), z + Z_s^0(\omega), \bar{G}_s(\omega)).$$

Then, with the notation of the previous lemma, $\mathcal{Y} = \mathcal{Y}^{\mathcal{Z}}$, and one easily checks that $h, \hat{\tau}_n$ and $\hat{\tau}$ satisfy the conditions of Lemma 5.10. Then, there is $\mathcal{Z}^\varepsilon \in \hat{\mathcal{S}}_0^2$ with finite variation over each $[0, \hat{\tau}_n]$ so that:

$$\sup_{0 \leq t < \hat{\tau}} |\mathcal{Y}_t^{\mathcal{Z}^\varepsilon} - \mathcal{Y}_t| \leq \varepsilon, \quad \hat{\mathbb{P}} - \text{a.s.}$$

Set $Z_t^\varepsilon := \mathcal{Z}_t^\varepsilon + Z_t^0$, $Y_t^\varepsilon := \mathcal{Y}_t^{\mathcal{Z}^\varepsilon} + Y_t^0$ for $t < \hat{\tau}$, and observe that $Z^\varepsilon \in \widehat{\mathcal{SM}}_0^2$ and $d\langle Z^\varepsilon, B \rangle_t = d\langle Z^0, B \rangle_t = \bar{G}_t dt$. Then with the notation of (5.1),

$$\mathcal{Y}_t^{\mathcal{Z}^\varepsilon} + Y_t^0 = Y^{y, Z^\varepsilon} \quad \text{and} \quad Y^{y, Z^\varepsilon} - \bar{Y} = \mathcal{Y}^{\mathcal{Z}^\varepsilon} - \mathcal{Y}.$$

Y^ε satisfies (5.4) for given $(y, Z^\varepsilon, \bar{\Gamma})$, and

$$\sup_{0 \leq t < \hat{\tau}} |Y_t^\varepsilon - \bar{Y}_t| \leq \varepsilon, \quad \hat{\mathbb{P}} - \text{a.s.}$$

Now let L denote the uniform Lipschitz constant of H with respect to y , set $y^\varepsilon := y + e^L \varepsilon$. On $\cap_{n \geq 1} \{\hat{\tau}_n < 1\}$, we have $Y_1^{y^\varepsilon, Z^\varepsilon} = \infty \geq \xi$ $\hat{\mathbb{P}}$ -a.s. To prove that this inequality holds on $\cup_{n \geq 1} \{\hat{\tau}_n = 1\}$, we directly calculate that

$$Y_t^{y^\varepsilon, Z^\varepsilon} - Y_t^{y, Z^\varepsilon} = e^L \varepsilon + \int_0^t \alpha_s (Y_s^{y^\varepsilon, Z^\varepsilon} - Y_s^{y, Z^\varepsilon}) ds,$$

where $|\alpha_s| \leq L$. This leads to $Y_1^{y^\varepsilon, Z^\varepsilon} - Y_1^{y, Z^\varepsilon} = e^L \varepsilon e^{\int_0^1 \alpha_t dt} \geq \varepsilon$, so that:

$$Y_1^{y^\varepsilon, Z^\varepsilon} \geq Y_1^{y, Z^\varepsilon} + \varepsilon \geq \bar{Y}_1 \geq \xi \quad \text{on} \quad \cup_{n \geq 1} \{\hat{\tau}_n = 1\}, \quad \hat{\mathbb{P}} - \text{a.s.}$$

Therefore $\mathcal{V}_0(\xi) \leq y + e^L \varepsilon \leq \bar{\mathcal{V}}_0(\xi) + (1 + e^L) \varepsilon \rightarrow \bar{\mathcal{V}}_0(\xi)$ by sending ε to zero.

(ii) is proved exactly as in the proof of Part (ii) of Theorem 4.4, and (iii) is postponed to next subsection. \square

5.7 The duality result

This subsection is dedicated to Theorem 5.7 (iii). Let $\hat{\mathcal{T}} := \cup_{n \geq 1} \mathcal{T}_{\hat{\tau}_n}$. As a main ingredient for our duality result, we introduce for every $i \geq 1$ the family of random variables:

$$V_\tau^i := \text{ess sup}_{a \in \bar{\mathcal{A}}_0^F(i, \tau)} \mathbb{P}^i Y_\tau^a, \quad \tau \in \hat{\mathcal{T}}, \quad (5.12)$$

where we always take \mathcal{F}_τ^+ -measurable version. Clearly, $V_{\tau_1}^i = V_{\tau_2}^i$ on $\{\tau_1 = \tau_2\}$ \mathbb{P}^i -a.s. for every $\tau_1, \tau_2 \in \hat{\mathcal{T}}$, and $v_0(\xi) = \sup_{a^i \in \bar{\mathcal{A}}_0^F} V_0^i$.

Lemma 5.11 *Under Assumptions 3.1, 4.3, and 5.6, there is a constant $C > 0$ such that:*

$$|V_\tau^i|^2 \leq C \operatorname{ess\,sup}_{a \in \bar{\mathcal{A}}_0^F(i, \tau)} \mathbb{P}^i \mathbb{E}_\tau^{\mathbb{P}^a} \left[\xi^2 + \int_\tau^1 \left(1 + |\hat{F}_s^0|^2 \right) ds \right], \quad \mathbb{P}^i - \text{a.s. for all } \tau \in \hat{\mathcal{T}}.$$

Proof. Let $a \in \bar{\mathcal{A}}_0^F$ be fixed. Since F is Lipschitz-continuous in y , we may define the bounded process α by $\hat{F}_t(Y_t^a, Z_t^a) = \hat{F}_t(0, Z_t^a) + \alpha_t Y_t^a$. Let $\beta > 0$ be a constant to be fixed later, and set $\Lambda_t := \exp \int_0^t (\beta^2 - 2\alpha_s) ds$, $\phi_t := \beta Y_t^a - \beta^{-1} \hat{F}_t(0, Z_t^a)$. It follows from Itô's formula that

$$\begin{aligned} \Lambda_\tau |Y_\tau^a|^2 &= \mathbb{E}_\tau^{\mathbb{P}^a} \left[\Lambda_1 \xi^2 + \int_\tau^1 \Lambda_s \left(\beta^{-2} |\hat{F}_s(0, Z_s^a)|^2 - |\hat{a}_s^{1/2} Z_s^a|^2 - \phi_t^2 \right) ds \right] \\ &\leq \mathbb{E}_\tau^{\mathbb{P}^a} \left[\Lambda_1 \xi^2 + \int_\tau^1 \Lambda_s \left(\beta^{-2} |\hat{F}_s(0, Z_s^a)|^2 - |\hat{a}_s^{1/2} Z_s^a|^2 \right) ds \right] \\ &\leq \mathbb{E}_\tau^{\mathbb{P}^a} \left[\Lambda_1 \xi^2 + \int_\tau^1 \Lambda_s \left(C\beta^{-2} (1 + |\hat{F}_s^0|^2 + |a_s^{1/2} Z_s^a|^2) - |\hat{a}_s^{1/2} Z_s^a|^2 \right) ds \right], \end{aligned}$$

where we used the growth condition on F contained in Assumption 5.6. Set $\beta := \sqrt{C}$. Since Λ is bounded both from above and from below away from 0, this implies that:

$$|Y_\tau^a|^2 \leq C \mathbb{E}_\tau^{\mathbb{P}^a} \left[\xi^2 + \int_\tau^1 \left(1 + |\hat{F}_s^0|^2 \right) ds \right], \quad \mathbb{P}^a - \text{a.s.}$$

and the required estimate follows from (5.12). \square

We next prove that the family $\{V_\tau^i, \tau \in \hat{\mathcal{T}}, i \geq 1\}$ satisfies a partial dynamic programming principle. For every $\tau \in \hat{\mathcal{T}}$, denote by $(\hat{V}^{i, \tau}, \hat{Z}^{i, \tau})$ the solution of the BSDE (3.18) with $a = a^i$ and terminal condition $\hat{V}_\tau^{i, \tau} = V_\tau^i$, i.e.

$$\hat{V}_t^{i, \tau} = V_\tau^i - \int_t^\tau \hat{F}_s(\hat{V}_s^{i, \tau}, \hat{Z}_s^{i, \tau}) ds - \int_t^\tau \hat{Z}_s^{i, \tau} dB_s, \quad t \leq \tau, \quad \mathbb{P}^i - \text{a.s.} \quad (5.13)$$

Lemma 5.12 *Let Assumptions 3.1, 4.3, and 5.6 hold true. Then, for every $i \geq 1$:*

$$\hat{V}_{\tau_1}^{i, \tau_2} \leq V_{\tau_1}^i \quad \mathbb{P}^i - \text{a.s. for every } \tau_1, \tau_2 \in \hat{\mathcal{T}} \text{ satisfying } \tau_1 \leq \tau_2, \quad (5.14)$$

Moreover, the family $\{V_\tau^i\}_{\tau \in \hat{\mathcal{T}}}$ can be aggregated into a càdlàg process which is a local \hat{F} -supermartingale under \mathbb{P}^i up to $\hat{\tau}$.

Proof. (i) By the definition of the essential supremum, see e.g. Neveu [13], there exists a sequence $(b^j)_{j \geq 1} \subset \bar{\mathcal{A}}^F(i, \tau_2)$ such that $V_{\tau_2}^i = \operatorname{ess\,sup}_j^{\mathbb{P}^i} Y_{\tau_2}^{b^j}$. For $\varepsilon > 0$, denote

$$A_j := \{\hat{n}(i) = j\} \quad \text{where} \quad \hat{n}(i) := \inf \left\{ k : \hat{Y}_{\tau_2}^i < Y_{\tau_2}^{b^k} + \varepsilon \right\}.$$

Then $(A_j)_{j \geq 1}$ is a partition of Ω and we define:

$$a_s^\varepsilon := a_s^i \text{ for } s \leq \tau_2, \quad \text{and} \quad a_s^\varepsilon := \sum_{j \geq 1} b_s^j \mathbf{1}_{A_j} \text{ for } s > \tau_2.$$

By the definition of $\bar{\mathcal{A}}^F(i, \tau_2)$ together with the fact that T_0 is dense in $[0, 1]$, we may find a stopping time $\tilde{\tau}_2 > \tau_2$, with values in T_0 , such that $a^\varepsilon = a^i$ on $[0, \tilde{\tau}_2]$. Then, it follows from the concatenation property (5.1) of \mathcal{A}_0 that $a^\varepsilon \in \bar{\mathcal{A}}^F(i, \tau_2)$.

By the definition of (A_j) , we have $V_{\tau_2}^i \leq Y_{\tau_2}^{a^\varepsilon} + \varepsilon$, \mathbb{P}^i -a.s. Note that

$$Y_t^{a^\varepsilon} = Y_{\tau_2}^{a^\varepsilon} - \int_t^{\tau_2} \hat{F}_s(Y_s^{a^\varepsilon}, Z_s^{a^\varepsilon}) ds - \int_t^{\tau_2} Z_r^{a^\varepsilon} dB_r, \quad \tau_1 \leq t \leq \tau_2, \mathbb{P}^i - \text{a.s.}$$

i.e. $(Y^{a^\varepsilon}, Z^{a^\varepsilon})$ solves the same BSDE as $(\hat{V}^{\tau_2, i}, \hat{Z}^{\tau_2, i})$ but with terminal data $Y_{\tau_2}^{a^\varepsilon}$ instead of $V_{\tau_2}^i$. By Lemma 2.7 (iii) and (iv), we have $0 \leq \hat{V}_{\tau_1}^{\tau_2, i} - Y_{\tau_1}^{a^\varepsilon} \leq C_i \varepsilon$, where C_i depends on the lower bound of a^i . Since $a^\varepsilon \in \bar{\mathcal{A}}^F(i, \tau_2) \subset \bar{\mathcal{A}}^F(i, \tau_1)$, we have $V_{\tau_1}^i \geq Y_{\tau_1}^{a^\varepsilon} \geq \hat{V}_{\tau_1}^{\tau_2, i} - C_i \varepsilon$, \mathbb{P}^i -a.s. for any $\varepsilon > 0$, and the proof of (5.14) is complete.

(ii) First, by Assumption 5.6 and Lemma 5.11 we have

$$\mathbb{E}^i \left[\text{ess sup}_{\tau \in \mathcal{T}_{\hat{\tau}_n}}^{\mathbb{P}^i} |V_\tau^i|^2 \right] < \infty \quad \text{for every } n \geq 1.$$

Fix n . By Lemma 2.11 and part (i) of this proof, the limit $V_t^{i, n} := \lim_{\mathbb{Q} \ni r \downarrow t} V_{r \wedge \hat{\tau}_n}^i$ exists, \mathbb{P}^i -a.s. Following the standard arguments as in [11] and [3], we claim that $V_\tau^{i, n} = V_\tau^i$, \mathbb{P}^i -a.s. for every $\tau \in \mathcal{T}_{\hat{\tau}_n}$. Indeed, let τ^m be a sequence of stopping times taking values in $\mathbb{Q} \cap (\tau, 1]$ such that $\tau^m \downarrow \tau$, \mathbb{P}^i -a.s. By (i), we have $\hat{V}_{\tau^m \wedge \hat{\tau}_n}^{i, \tau^m} \leq V_{\tau^m}^i$. Send $m \rightarrow \infty$, by the stability of BSDEs we obtain $V_{\tau^m \wedge \hat{\tau}_n}^{i, n} \leq V_{\tau^m}^i$, \mathbb{P}^i -a.s. On the other hand, for each $a \in \bar{\mathcal{A}}_0^F(i, \tau)$, there exists $\tilde{\tau} > \tau$ such that $a \in \bar{\mathcal{A}}_0^F(i, \tilde{\tau} \wedge \tau_n)$. Let $\tilde{\tau}^m$ be a sequence of stopping times taking values in \mathbb{Q} such that $\tau < \tilde{\tau}^m \leq \tilde{\tau}$ and $\tilde{\tau}^m \downarrow \tau$. Then $a \in \bar{\mathcal{A}}_0^F(i, \tilde{\tau}^m \wedge \tau_n)$ for each m , and thus $Y_{\tilde{\tau}^m \wedge \tau_n}^a \leq V_{\tilde{\tau}^m \wedge \tau_n}^i$. By the comparison theorem of BSDEs, we get $Y_\tau^a \leq \hat{V}_\tau^{i, \tilde{\tau}^m \wedge \tau_n}$. Send $m \rightarrow \infty$ and by the stability of BSDEs we obtain $Y_\tau^a \leq V_\tau^{i, n}$. Since a is arbitrary, we prove $V_\tau^i \leq V_\tau^{i, n}$ and thus, $V_\tau^{i, n} = V_\tau^i$, \mathbb{P}^i -a.s. Then $V^{i, n}$ inherits from V^i the F -supermartingale under \mathbb{P}^i up to $\hat{\tau}_n$.

Finally, it is obvious that $V_t^{i, n} = V_t^{i, n+1}$, \mathbb{P}^i -a.s. on $\{t \leq \hat{\tau}_n\}$. Then we can aggregate them into one càdlàg process in $[0, \hat{\tau})$ which is a local F -supermartingale under \mathbb{P}^i up to $\hat{\tau}$. \square

Proposition 5.13 *Let Assumptions 3.1, 4.3, and 5.6 hold true. Then, there exists an aggregating process $\{V_t, t \in [0, 1]\}$ in $\hat{\mathcal{D}}_0^2$ such that for every $\tau \in \hat{\mathcal{T}}$,*

$$V_{\tau(\omega)}(\omega) = V_\tau^i(\omega) \quad \text{for } \mathbb{P}^i - \text{a.e. } \omega \in \Omega. \quad (5.15)$$

Moreover, there exist $\hat{Z} \in \hat{\mathcal{H}}_0^2$ and $\hat{K} \in \hat{\mathcal{I}}_0^2$, such that the process V has the decomposition

$$V_t = V_0 + \int_0^t \hat{F}_s(V_s, \hat{Z}_s) ds + \int_0^t \hat{Z}_s dB_s - \hat{K}_t, \quad 0 \leq t < \hat{\tau}, \quad \hat{\mathbb{P}} - a.s. \quad (5.16)$$

Proof. For each $i \geq 1$ and $n \geq 1$, applying Lemma 2.11 and Theorem 2.12, we may assume V^i is càdlàg in $[0, \hat{\tau}_n]$ and there exist unique $\hat{Z}^{i,n} \mathbf{1}_{[0, \hat{\tau}_n]} \in \mathcal{H}^2(\mathbb{P}^i, \mathbb{F}^+)$, $\hat{K}_{\cdot \wedge \hat{\tau}_n}^{i,n} \in \mathcal{I}^2(\mathbb{P}^i, \mathbb{F}^+)$ such that

$$V_t^i = V_0^i + \int_0^t \hat{F}_s(V_s^i, \hat{Z}_s^{i,n}) ds + \int_0^t \hat{Z}_s^{i,n} dB_s - \hat{K}_s^{i,n}, \quad 0 \leq t \leq \hat{\tau}_n, \quad \mathbb{P}^i - a.s.$$

By uniqueness, $\hat{Z}^{i,n} = \hat{Z}^{i,n+1}$ and $\hat{K}^{i,n} = \hat{K}^{i,n+1}$, $dt \times d\mathbb{P}^i$ -a.s. for $0 \leq t \leq \tau_n$. Denote

$$\hat{Z}^i := \sum_{n=1}^{\infty} \hat{Z}^{i,n} \mathbf{1}_{[\hat{\tau}_{n-1}, \hat{\tau}_n)} \quad \text{and} \quad \hat{K}^i := \sum_{n=1}^{\infty} \hat{K}^{i,n} \mathbf{1}_{[\hat{\tau}_{n-1}, \hat{\tau}_n)}.$$

Then

$$V_t^i = V_0^i + \int_0^t \hat{F}_s(V_s^i, \hat{Z}_s^i) ds + \int_0^t \hat{Z}_s^i dB_s - \hat{K}_s^i, \quad 0 \leq t < \hat{\tau}, \quad \mathbb{P}^i - a.s.$$

Moreover, by Lemma 5.11 and Assumption 5.6 again, we know the càdlàg process V^i satisfies

$$\mathbb{E}^i \left\{ \sup_{0 \leq t \leq \hat{\tau}_n} |V_t^i|^2 \right\} < \infty \quad \text{and therefore} \quad \mathbb{E}^i \left\{ \int_0^{\hat{\tau}_n} |\hat{Z}_t^i|^2 + |\hat{K}_{\hat{\tau}}^i|^2 \right\} < \infty, \quad (5.17)$$

for every $i, n \geq 1$. Finally, by the construction of V^i , it is obvious that $V^i = V^j$, \mathbb{P}^i -a.s. on $[0, \tau \wedge \hat{\tau})$ whenever $a^j \in \mathcal{A}_0^F(i, \tau \wedge \hat{\tau})$. Since both are càdlàg, V^i and V^j are \mathbb{P}^i -indistinguishable in $[0, t] \cap [0, \hat{\tau})$, for each i . By the uniqueness in Theorem 2.12, we know \hat{Z}^i and \hat{Z}^j are \mathbb{P}^i -modifications and \hat{K}^i and \hat{K}^j are \mathbb{P}^i -indistinguishable in $[0, \tau \wedge \hat{\tau})$. Then, by Lemma 5.4, there exist patching processes $V \in \hat{\mathcal{D}}_0^2$, $\hat{Z} \in \hat{\mathcal{H}}_0^2$ and $\hat{K} \in \hat{\mathcal{I}}_0^2$ such that the required decomposition (5.16) holds true. \square

Proof of Theorem 5.7 (iii) In view of (5.11), we only need to prove $\hat{V}_0(\xi) \leq v_0(\xi)$. Recall the process V defined in Proposition 5.13 above. Note that $v_0(\xi) \geq V_0^i$ for each i , then $v_0(\xi) \geq V_0$, $\hat{\mathbb{P}}$ -a.s. By comparison, it follows from (5.16) that $V_1^{v_0(\xi), \hat{Z}} \geq V_1 = \xi$, $\hat{\mathbb{P}}$ -a.s. on $\cup_{n \geq 1} \{\hat{\tau}_n = 1\}$, which completes the proof by the definition of the problem $\hat{V}_0(\xi)$. \square

6 Examples and discussion

Example 1 *Linear generator.* Assume that H is linear in γ :

$$H_t(y, z, \gamma) = f_t(y, z) + \frac{1}{2} \sigma_t \sigma_t^T : \gamma,$$

where $f_t(y, z)$ and σ_t are so that our assumptions on the nonlinearity H and its conjugate F hold true. Notice that the domain of F is reduced to a one-point set:

$$F_t(y, z, a) = f_t(y, z) \mathbf{1}_{\{a = \sigma_t \sigma_t^T\}} + \infty \mathbf{1}_{\{a \neq \sigma_t \sigma_t^T\}}.$$

If $\sigma \sigma^T \notin \bar{\mathcal{A}}_0$, all $a \in \bar{\mathcal{A}}_0$ are outside the domain of F , and $\hat{\tau} = 0$ $\hat{\mathbb{P}}$ -a.s. so that the value function of the second order target problems are $\mathcal{V}_0(\xi) = \bar{\mathcal{V}}_0(\xi) = \hat{\mathcal{V}}_0(\xi) = -\infty$. Notice that we also have $v_0(\xi) = -\infty$ so that the duality result of Theorem 5.7 holds true.

On the other hand, if $\sigma \sigma^T \in \bar{\mathcal{A}}_0$, then the present formulation of the second order target problem is clearly equivalent to the classical formulation under the reference measure $\mathbb{P}^{\sigma \sigma^T}$ which ignores any uncertainty on the diffusion coefficient. \square

Example 2 *Uncertain volatility models.* Set $H_t(y, z, \gamma) := H(\gamma) := \frac{1}{2}[\bar{\sigma}^2 \gamma^+ - \underline{\sigma}^2 \gamma^-]$, where $\bar{\sigma} > \underline{\sigma} \geq 0$. This is the context studied by Denis and Martini [7]. By straightforward calculation, we find $\text{dom}(F_t) = [\underline{\sigma}^2, \bar{\sigma}^2]$, and for any $a \in [\underline{\sigma}^2, \bar{\sigma}^2]$, $F(a) = 0$. It is easily seen that all our assumptions are satisfied. Moreover, under the additional density condition (ii) of Theorem 5.8, we have $\hat{\mathcal{V}}(\xi) = \mathbb{E}^G(\xi)$ for ξ satisfying certain technical conditions, where \mathbb{E}^G is the G -expectation defined in Peng [16]. \square

Example 3 *Hedging under gamma constraints.* Let $\underline{\Gamma}, \bar{\Gamma} \geq 0$ be two given constants. The problem of superhedging under Gamma constraint as introduced in [17], [5], and [18], corresponds to the specification $H_s(y, z, \gamma) = H(\gamma) = \frac{1}{2} \sigma^2 \gamma$ for $\gamma \in [-\underline{\Gamma}, \bar{\Gamma}]$, and $+\infty$ otherwise. By straightforward calculation, we see that $F(a) = \frac{1}{2} (\bar{\Gamma}(a - \sigma^2)^+ + \underline{\Gamma}(a - \sigma^2)^-)$. If both bounds are finite, the domain of the dual function F is the non-negative real line. The dual formulation of this paper coincides with that of [18]. \square

Our formulation of the second order stochastic target problem uses crucially the stopping time $\hat{\tau}$ introduced in (3.5). Similarly, the dual problem is formulated by restricting the dual variables to those which lie in the domain of F on the whole time interval $[0, 1]$. Our final example shows that such restrictions are necessary for the duality result of Theorem 5.7 (iii). This also gives an example for (3.21).

Example 4 *On the importance of $\hat{\tau}$.* Consider the nonlinearity $H_t(y, z, \gamma) := H(\gamma) := \frac{1}{2}[4\gamma^+ - 1\gamma^-]$ of Example 2 above. We have $\text{dom}(F_t) = [1, 4]$ for any (t, ω) , and for any $a \in [1, 4]$, $F(a) = 0$. Let $T_0 := [0, 1] \cap \mathbb{Q}$, and \mathcal{A}_0 the collection of all piecewise constant deterministic functions with time steps in T_0 and values in \mathbb{Q}_+ . Then $\bar{\mathcal{A}}_0^F := \{a \in \bar{\mathcal{A}}_0 : 1 \leq a_t \leq 4, 0 \leq t \leq 1\}$. Let

$$\xi := \mathbf{1}_{\Omega_0^c} \quad \text{where} \quad \Omega_0 := \bigcap_{r \in \mathbb{Q} \cap [0, 1]} \left\{ \overline{\lim}_{\varepsilon \rightarrow 0} \frac{B_{r+\varepsilon} - B_r}{\sqrt{2\varepsilon \ln \ln(1/\varepsilon)}} \in [1, 2] \right\}.$$

By definition, we clearly have for $a \in \overline{\mathcal{A}}_0^F$ that $\xi = 0$, \mathbb{P}^a -a.s. and $Y_0^a = 0$. But, for $a \notin \overline{\mathcal{A}}_0^F$, we have $\mathbb{P}^a[\xi = 1] > 0$ and $Y_0^a = -\infty$. Then, if the dual variables are not restricted to lie in the domain of F , the dual value function is:

$$\sup_{a \in \overline{\mathcal{A}}_0} Y_0^a = 0.$$

Interestingly, we shall now prove that:

$$\overline{\overline{V}}_0(\xi) := \inf \left\{ y : \bar{Y}_T^{y, \bar{Z}, \bar{G}} \geq \xi, \hat{\mathbb{P}} - \text{a.s. for some } Z, G \in \cap_{i \geq 1} \mathbb{H}^2(\mathbb{P}^i, \mathbb{F}^+) \right\} > 0,$$

where the process $\bar{Y}^{y, Z, G}$ is defined in (3.12) for $t \in [0, 1]$, i.e. replacing $\hat{\tau}$ by 1. To see this, we assume to the contrary that $\overline{\overline{V}}_0(\xi) \leq 0$. Then, by using initial value $1/2 > 0$, there exist $Z, G \in \cap_{i \geq 1} \mathbb{H}^2(\mathbb{P}^i, \mathbb{F}^+)$ such that

$$\bar{Y}_1 := \frac{1}{2} + \int_0^1 \left(\frac{1}{2} G_s \hat{a}_s - H(G_s) \right) ds + \int_0^1 Z_s dB_s \geq \xi, \hat{\mathbb{P}} - \text{a.s.}$$

Let $a_n(t) := 9 \mathbf{1}_{[0, \frac{1}{n})}(t) + 4 \mathbf{1}_{[\frac{1}{n}, 1)}(t)$, for $n \geq 1$. Then, \mathbb{P}^{a_n} -a.s., we have $\xi = 1$ and thus

$$\frac{1}{2} - \int_0^1 Z_s dB_s \leq \int_0^1 \left(\frac{1}{2} G_s a_n(s) - H(G_s) \right) ds \leq \int_0^{1/n} \left(\frac{1}{2} 9 G_s - \frac{1}{2} [4G_s^+ - G_s^-] \right) ds \leq 4 \int_0^{1/n} |G_s| ds.$$

Taking expectation under \mathbb{P}^{a_n} , this provides $\mathbb{E}^{\mathbb{P}^{a_n}} \left\{ \int_0^{1/n} |G_s| ds \right\} \geq 1/8$ for all $n \geq 1$. Since $\mathbb{P}^{a_n} = \mathbb{P}^{a_1}$ on $\mathcal{F}_{1/n}$, this implies that $\mathbb{E}^{\mathbb{P}^{a_1}} \left[\int_0^{1/n} |G_s| ds \right] \geq 1/8$, for all $n \geq 1$, contradicting the condition that $G \in \mathbb{H}^2(\mathbb{P}^{a_1}, \mathbb{F}^+)$. \square

7 Appendix

7.1 Proof of Lemma 2.4

Proof. (i) Assume $Y \in \mathcal{D}^0(\mathbb{P}, \mathbb{F}^{\mathbb{P}})$. The uniqueness is obvious. To prove the existence, assume let $A_0 := \{\omega : Y(\omega) \text{ is not càdlàg}\}$. For each $r \in \mathbb{Q} \cap (0, 1]$, by Remark 2.3, there exists $\tilde{Y}_r \in \mathcal{F}_r^+$ such that $A_r := \{\tilde{Y}_r \neq Y_r\} \in \mathcal{N}^{\mathbb{P}}$. Let $A := A_0 \cup (\cup_r A_r)$. Then $\mathbb{P}(A) = 0$. Define

$$Y_t^n := \tilde{Y}_{k+\frac{1}{n}}, \quad t \in \left(\frac{k}{n}, \frac{k+1}{n} \right]; \quad \tilde{Y} := \left(\overline{\lim}_{n \rightarrow \infty} Y^n \right) \mathbf{1}_{\{\overline{\lim}_{n \rightarrow \infty} Y^n \in \mathbb{R}\}}. \quad \text{for } n \geq 1.$$

Then for any $t \in \left(\frac{k}{n}, \frac{k+1}{n} \right]$, $Y_t^n \in \mathcal{F}_{\frac{k+1}{n}}^+$ and $Y^n|_{[0, t]} \in \mathcal{B}([0, t]) \times \mathcal{F}_{\frac{k+1}{n}}^+$. Since \mathbb{F}^+ is right continuous, we get $\tilde{Y}_t \in \mathcal{F}_t^+$ and $\tilde{Y}|_{[0, t]} \in \mathcal{B}([0, t]) \times \mathcal{F}_t^+$. That is, $\tilde{Y} \in \mathbb{F}^+$. Moreover, for any $\omega \notin A$ and any t , for any n , assume $t \in \left(\frac{k}{n}, \frac{k+1}{n} \right]$, we get

$$\lim_{n \rightarrow \infty} Y_t^n(\omega) = \lim_{n \rightarrow \infty} \tilde{Y}_{\frac{k+1}{n}}(\omega) = \lim_{n \rightarrow \infty} Y_{\frac{k+1}{n}}(\omega) = Y_t(\omega).$$

So $\{\omega : \exists t \text{ s.t. } \tilde{Y}_t(\omega) \neq Y_t(\omega)\} \subset A$. Then, \tilde{Y} is \mathbb{P} -indistinguishable from Y and thus \tilde{Y} also has càdlàg paths, \mathbb{P} -a.s.

The case that $Y \in \mathcal{S}^0(\mathbb{P}, \mathbb{F}^{\mathbb{P}})$ can be proved similarly.

(ii) The uniqueness is obvious. To prove the existence, we first assume Z is bounded. Let $Y_t := \int_0^t Z_s ds$. Then $Y \in \mathcal{S}^0(\mathbb{P}, \mathbb{F}^{\mathbb{P}})$. By (i), there exists $\tilde{Y} \in \mathcal{S}^0(\mathbb{P}, \mathbb{F}^+)$ \mathbb{P} -indistinguishable from Y . Let $A_0 := \{\exists t \text{ s.t. } \tilde{Y}_t \neq Y_t\}$, then $\mathbb{P}(A_0) = 0$ and $\tilde{Y}(\omega)$ is continuous for each $\omega \notin A_0$. Define:

$$Z_t^n := n[\tilde{Y}_t - \tilde{Y}_{t-\frac{1}{n}}]; \quad \tilde{Z} := (\overline{\lim}_{n \rightarrow \infty} Z^n) \mathbf{1}_{\{\overline{\lim}_{n \rightarrow \infty} Z^n \in \mathbb{R}\}} \quad \text{for } n \geq 1.$$

As in (i), we see $\tilde{Z} \in \mathbb{F}^+$. Moreover, for each $\omega \notin A_0$, $Z_t^n(\omega) = n \int_{t-\frac{1}{n}}^t Z_s(\omega) ds$. Then $\tilde{Z}(\omega) = Z(\omega)$, dt -a.s. Therefore, $\tilde{Z} = Z$, $dt \times d\mathbb{P}$ -a.s.

Finally, for general Z , for any m , let $Z_t^m := (-m) \vee (Z \wedge m)$. By the above arguments, Z^m has a modification $\tilde{Z}^m \in \mathbb{F}^+$. Then obviously $\tilde{Z} := (\overline{\lim}_{m \rightarrow \infty} \tilde{Z}^m) \mathbf{1}_{\{\overline{\lim}_{m \rightarrow \infty} \tilde{Z}^m \in \mathbb{R}\}}$ satisfies all the requirements.

(iii) For any n , define

$$A_i^n := \left\{ \frac{i-1}{n} \leq \tau < \frac{i}{n} \right\}, \quad \tau^n := \sum_{i=1}^n \frac{i}{n} \mathbf{1}_{A_i^n}.$$

Note that $A_i^n \in \mathcal{F}_{\frac{i}{n}+}^{\mathbb{P}}$. By Remark 2.3 (i), there exists $\tilde{A}_i^n \in \mathcal{F}_{\frac{i}{n}+}^{\mathbb{P}}$ such that $\mathbf{1}_{A_i^n} = \mathbf{1}_{\tilde{A}_i^n}$, \mathbb{P} -a.s. We note that \tilde{A}_i^n are not a partition of Ω anymore. Define $\bar{A}_1^n := \tilde{A}_1^n$, and for $i = 2, \dots, n-1$, $\bar{A}_i^n := \tilde{A}_i^n \setminus \cup_{1 \leq j < i} \tilde{A}_j^n$, and finally $\bar{A}_n^n := \Omega \setminus \cup_{1 \leq j < n} \tilde{A}_j^n$. Then $\bar{A}_1^n, \dots, \bar{A}_n^n$ form a partition of Ω . Define $\tilde{\tau}^n := \sum_{i=1}^n \frac{i}{n} \mathbf{1}_{\bar{A}_i^n}$. One can easily check that $\tilde{\tau}^n$ is an \mathbb{F}^+ -stopping time and $\tilde{\tau}^n = \tau^n$, \mathbb{P} -a.s. Now set $\tilde{\tau} := \inf_n \tilde{\tau}^n$. We see that $\tilde{\tau}$ is an \mathbb{F}^+ -stopping time and $\tilde{\tau} = \tau$, \mathbb{P} -a.s. \square

7.2 Proof of Lemma 5.10

Lemma 7.1 *The result of Lemma 5.10 holds when $\theta_n = 1$, $\hat{\mathbb{P}}$ -a.s., for some n .*

Proof. Recall Lemma 2.7, and define $\tilde{\mathbb{P}} := \sum_{i=1}^{\infty} \alpha_i \mathbb{P}^i$, where $\alpha_1 := 1 - \sum_{i=2}^{\infty} \alpha_i$, and

$$\alpha_i^{-1} := 2^i (1 + C(\underline{a}^i)) \left[1 + \mathbb{E}^i \left\{ \sup_{0 \leq t \leq 1} |\mathcal{Y}_t|^2 + \int_0^1 [|Z_t|^2 + |h_t(0,0)|^2] dt \right\} \right], \quad \text{for } i \geq 2. \quad (7.1)$$

Then $\tilde{\mathbb{P}} \in \mathcal{M}$, $\mathbb{P}^i \leq \alpha_i^{-1} \tilde{\mathbb{P}}$, $\tilde{\mathbb{P}}$ is equivalent to $\hat{\mathbb{P}}$, and $\mathcal{Y} \in \mathcal{S}^2(\tilde{\mathbb{P}}, \mathbb{F}^+)$, $Z \in \mathcal{H}^2(\tilde{\mathbb{P}}, \mathbb{F}^+)$. Obviously it suffices to find $Z^\varepsilon \in \mathcal{H}^2(\tilde{\mathbb{P}}, \mathbb{F}^+)$ such that $\sup_{0 \leq t \leq 1} |\mathcal{Y}_t^\varepsilon - \mathcal{Y}_t| \leq \varepsilon$, $\tilde{\mathbb{P}}$ -a.s.

As in Bank-Baum, we first claim that, for any (\mathbb{F}^+) -stopping time τ and any $\tilde{Y}_\tau, \tilde{Z}_\tau \in \mathcal{F}_\tau$, there exists a process $\{Z_t^{\varepsilon, \tau}, t \in [\tau, 1]\} \in \mathcal{H}^2(\tilde{\mathbb{P}}, \mathbb{F}^+)$ such that $Z_\tau^{\varepsilon, \tau} = \tilde{Z}_\tau$, $Z^{\varepsilon, \tau}$ is absolutely continuous in t with finite variation, and

$$\tilde{\mathbb{P}} \left\{ \sup_{\tau \leq t \leq 1} e^{-L(t-\tau)} |\mathcal{Y}_t^{\varepsilon, \tau} - \mathcal{Y}_t| > \varepsilon + |\tilde{Y}_\tau - \mathcal{Y}_\tau| \right\} \leq \varepsilon \quad (7.2)$$

where L is the uniform Lipschitz constant of h with respect to y , and

$$\mathcal{Y}_t^{\varepsilon, \tau} = \tilde{Y}_\tau + \int_\tau^t h_s(\mathcal{Y}_s^{\varepsilon, \tau}, Z_s^{\varepsilon, \tau}) ds + \int_\tau^t Z_s^{\varepsilon, \tau} dB_s, \quad t \geq \tau, \quad \tilde{\mathbb{P}} - \text{a.s.}$$

1. Assume (7.2) is true. In this step, we construct a sequence of stopping times $(\tau_i)_{i \geq 0}$ which yields the required approximation $(\mathcal{Y}^\varepsilon, Z^\varepsilon)$. We initialize our construction by $\tau_0 := 0$, $\tilde{Y}_0 = \mathcal{Y}_0$ and \tilde{Z}_0 arbitrary. Let $\varepsilon > 0$ be fixed, and set $\varepsilon_n := 2^{-n} e^{-L\varepsilon}$.

Assume τ_i is defined and $(\mathcal{Y}^\varepsilon, Z^\varepsilon)$ have been defined over $[0, \tau_i]$. By (7.2) there exists $\tilde{Z}^{i+1} \in \mathcal{H}^2(\tilde{\mathbb{P}}, \mathbb{F}^+)$ which is absolutely continuous in t and has finite variation such that $\tilde{Z}_{\tau_i} = Z_{\tau_i}^\varepsilon$ and

$$\tilde{\mathbb{P}} \left\{ \sup_{\tau_i \leq t \leq 1} e^{-L(t-\tau_i)} |\tilde{Y}_t^{i+1} - \mathcal{Y}_t| > \varepsilon_{i+1} + |Y_{\tau_i}^\varepsilon - Y_{\tau_i}| \right\} \leq \varepsilon_{i+1},$$

where

$$\tilde{Y}_t^{i+1} = \mathcal{Y}_{\tau_i}^\varepsilon + \int_{\tau_i}^t h_s(\tilde{Y}_s^{i+1}, \tilde{Z}_s^{i+1}) ds + \int_{\tau_i}^t \tilde{Z}_s^{i+1} dB_s, \quad t \geq \tau_i, \quad \tilde{\mathbb{P}} - \text{a.s.}$$

Denote

$$\tau_{i+1} := 1 \wedge \inf \left\{ t \geq \tau_i : e^{-L(t-\tau_i)} |\tilde{Y}_t^{i+1} - \mathcal{Y}_t| = \varepsilon_{i+1} + |Y_{\tau_i}^\varepsilon - \mathcal{Y}_{\tau_i}| \right\},$$

and define

$$\mathcal{Y}_t^\varepsilon := \tilde{Y}_t^{i+1}, \quad Z_t^\varepsilon := \tilde{Z}_t^{i+1}, \quad \forall t \in (\tau_i, \tau_{i+1}].$$

Since $\sum_{i=1}^\infty \tilde{\mathbb{P}}(\tau_i < 1) = \sum_{i=1}^\infty \varepsilon_i < 1$, it follows from the Borel-Cantelli Lemma that $\tilde{\mathbb{P}}(\tau_i < 1, \forall i) = 0$. That is, $(\mathcal{Y}^\varepsilon, Z^\varepsilon)$ is well-defined on $[0, 1]$ and Z^ε is absolutely continuous in t and has finite variation. Moreover, for $t \in [\tau_i, \tau_{i+1}]$,

$$\sup_{\tau_i \leq t \leq \tau_{i+1}} e^{-L(t-\tau_i)} |\mathcal{Y}_t^\varepsilon - \mathcal{Y}_t| \leq \varepsilon_{i+1} + |\mathcal{Y}_{\tau_i}^\varepsilon - \mathcal{Y}_{\tau_i}|.$$

Then

$$\sup_{\tau_i \leq t \leq \tau_{i+1}} e^{-Lt} |\mathcal{Y}_t^\varepsilon - \mathcal{Y}_t| \leq e^{-L\tau_i} \varepsilon_{i+1} + e^{-L\tau_i} |\mathcal{Y}_{\tau_i}^\varepsilon - \mathcal{Y}_{\tau_i}| \leq \varepsilon_{i+1} + e^{-L\tau_i} |\mathcal{Y}_{\tau_i}^\varepsilon - \mathcal{Y}_{\tau_i}|.$$

By induction one can easily see that $\sup_{0 \leq t \leq 1} e^{-Lt} |\mathcal{Y}_t^\varepsilon - \mathcal{Y}_t| \leq \sum_{i=1}^\infty \varepsilon_i = e^{-L}\varepsilon$, and then:

$$\sup_{0 \leq t \leq 1} |\mathcal{Y}_t^\varepsilon - \mathcal{Y}_t| \leq \varepsilon, \quad \tilde{\mathbb{P}} - \text{a.s.}$$

2. We next show that $Z^\varepsilon \in \mathcal{H}^2(\tilde{\mathbb{P}}, \mathbb{F}^+)$. For any i and j , note that

$$\mathcal{Y}_t^\varepsilon = \mathcal{Y}_{\tau_j}^\varepsilon - \int_t^{\tau_j} h_s(\mathcal{Y}_s^\varepsilon, Z_s^\varepsilon, a_s^i) ds - \int_t^{\tau_j} Z_s^\varepsilon dB_s, \quad t \leq \tau_j, \quad \mathbb{P}^i - \text{a.s.}$$

Note that $d\langle B \rangle_t = a_t^i dt$, \mathbb{P}^i -a.s. By standard arguments one can easily see that for all $j \geq 1$:

$$\begin{aligned} \mathbb{E}^i \left\{ \int_0^{\tau_j} |Z_t^\varepsilon|^2 dt \right\} &\leq C(\underline{a}^i) \mathbb{E}^i \left\{ |\mathcal{Y}_{\tau_j}^\varepsilon|^2 + \int_0^{\tau_j} |h_s(0,0)|^2 ds \right\} \\ &\leq C(\underline{a}^i) \mathbb{E}^i \left\{ \sup_{0 \leq t \leq 1} |\mathcal{Y}_t|^2 + \varepsilon + \int_0^1 |h_s(0,0)|^2 ds \right\} \leq \frac{1}{2^i \alpha_i} \end{aligned}$$

by (7.1). Then

$$\mathbb{E}^{\tilde{\mathbb{P}}} \left\{ \int_0^1 |Z_t^\varepsilon|^2 dt \right\} = \sum_{i=1}^{\infty} \alpha_i \mathbb{E}^{\mathbb{P}^i} \left\{ \int_0^1 |Z_t^\varepsilon|^2 dt \right\} < \infty.$$

3. It remains to prove (7.2). For simplicity we assume $\tau = 0$ and $\tilde{Y}_\tau = \tilde{y}$, $\tilde{Z}_\tau = \tilde{z}$. Set $Z_t := \tilde{z}$ for $t < 0$, and define for every $n \geq 1$:

$$Z_t^n := n \int_{t-\frac{1}{n}}^t Z_s ds \quad \text{so that} \quad \lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{\mathbb{P}}} \left\{ \int_0^1 |Z_t^n - Z_t|^2 dt \right\} = 0,$$

Z^n is continuous in t with finite variation, and $Z_0^n = \tilde{z}$. Let Y^n and \tilde{Y} be defined by $Y_0^n = \tilde{Y}_0 = \tilde{y}$ and:

$$dY_t^n = h_t(Y_t^n, Z_t^n) dt + Z_t^n dB_t, \quad d\tilde{Y}_t = h_t(\tilde{Y}_t, Z_t) dt + Z_t dB_t.$$

By stability of SDEs and standard estimates, we get

$$\lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{\mathbb{P}}} \left\{ \sup_{0 \leq t \leq 1} |Y_t^n - \tilde{Y}_t|^2 dt \right\} = 0 \quad \text{and} \quad e^{-Lt} |\tilde{Y}_t - \mathcal{Y}_t| \leq |\tilde{y} - y|.$$

Then, for any $\varepsilon > 0$,

$$\begin{aligned} \tilde{\mathbb{P}} \left[\sup_{0 \leq t \leq 1} e^{-Lt} |Y_t^n - \mathcal{Y}_t| \geq \varepsilon + |\tilde{y} - y| \right] &\leq \tilde{\mathbb{P}} \left[\sup_{0 \leq t \leq 1} e^{-Lt} |Y_t^n - \tilde{Y}_t| \geq \varepsilon \right] \\ &\leq \tilde{\mathbb{P}} \left[\sup_{0 \leq t \leq 1} |Y_t^n - \tilde{Y}_t| \geq \varepsilon \right] \longrightarrow 0, \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which completes the proof of (7.2). \square

Proof of Lemma 5.10. Fix $\varepsilon > 0$. First, notice that $Z \mathbf{1}_{[0, \theta_1]}$ satisfies the conditions in Lemma 7.1 and that $\mathcal{Y}_t^Z = \mathcal{Y}_t^{Z \mathbf{1}_{[0, \theta_1]}}$ for $0 \leq t \leq \theta_1$. Thus there exists $Z^\varepsilon \mathbf{1}_{[0, \theta_1]} \in \cap_{i \geq 1} \mathcal{S}^2(\mathbb{P}^i, \mathbb{F}^+)$ with finite variation such that

$$\sup_{0 \leq t \leq \theta_1} |\mathcal{Y}_t^{Z^\varepsilon} - \mathcal{Y}_t^Z| \leq 2^{-1} e^{-L} \varepsilon \leq 2^{-1} e^{L(\theta_1 - 1)} \varepsilon, \hat{\mathbb{P}} - \text{a.s.},$$

where L is the Lipschitz constant of h with respect to y . Assume we have defined Z^ε on $[0, \theta_n]$ with finite variation such that

$$\sup_{0 \leq t \leq \theta_n} |\mathcal{Y}_t^{Z^\varepsilon} - \mathcal{Y}_t^Z| \leq [1 - 2^{-n}] e^{L(\theta_n - 1)} \varepsilon, \hat{\mathbb{P}} - \text{a.s.}$$

Note that

$$\mathcal{Y}_t^Z = \mathcal{Y}_{\theta_n}^Z + \int_{\theta_n}^t h_s(\mathcal{Y}_s^Z, Z_s) ds + \int_{\theta_n}^t Z_s dB_s, \quad \theta_n \leq t \leq \theta_{n+1}, \quad \hat{\mathbb{P}} - \text{a.s.}$$

Since $Z \mathbf{1}_{[\theta_n, \theta_{n+1}]}$ satisfies the conditions in Lemma 7.1 by restricting to $[\theta_n, 1]$, there exists $Z^\varepsilon \mathbf{1}_{[\theta_n, \theta_{n+1}]} \in \cap_{i \geq 1} \mathcal{S}^2(\mathbb{P}^i, \mathbb{F}^+)$ with finite variation such that

$$\sup_{\theta_n \leq t \leq \theta_{n+1}} |\tilde{\mathcal{Y}}_t^{Z^\varepsilon} - \mathcal{Y}_t^Z| \leq 2^{-(n+1)} e^{L(\theta_{n+1}-1)} \varepsilon, \quad \hat{\mathbb{P}} - \text{a.s.},$$

where

$$\tilde{\mathcal{Y}}_t^Z = \mathcal{Y}_{\theta_n}^Z + \int_{\theta_n}^t h_s(\tilde{\mathcal{Y}}_s^Z, Z_s^\varepsilon) ds + \int_{\theta_n}^t Z_s^\varepsilon dB_s, \quad \theta_n \leq t \leq \theta_{n+1}, \quad \hat{\mathbb{P}} - \text{a.s.}$$

Moreover, since $|\mathcal{Y}_{\theta_n}^{Z^\varepsilon} - \mathcal{Y}_{\theta_n}^Z| \leq [1 - 2^{-n}] e^{L(\theta_n-1)} \varepsilon, \hat{\mathbb{P}} - \text{a.s.}$, by standard arguments we have

$$\sup_{\theta_n \leq t \leq \theta_{n+1}} |\tilde{\mathcal{Y}}_t^{Z^\varepsilon} - \mathcal{Y}_t^{Z^\varepsilon}| \leq e^{L(\theta_{n+1}-\theta_n)} |\mathcal{Y}_{\theta_n}^{Z^\varepsilon} - \mathcal{Y}_{\theta_n}^Z| \leq (1 - 2^{-n}) e^{L(\theta_{n+1}-1)} \varepsilon, \quad \hat{\mathbb{P}} - \text{a.s.}$$

Then $Z^\varepsilon \mathbf{1}_{[0, \theta_{n+1}]}$ has finite variation and

$$\begin{aligned} \sup_{\theta_n \leq t \leq \theta_{n+1}} |\mathcal{Y}_t^{Z^\varepsilon} - \mathcal{Y}_t^Z| &\leq \sup_{\theta_n \leq t \leq \theta_{n+1}} |\tilde{\mathcal{Y}}_t^{Z^\varepsilon} - \mathcal{Y}_t^{Z^\varepsilon}| + \sup_{\theta_n \leq t \leq \theta_{n+1}} |\tilde{\mathcal{Y}}_t^{Z^\varepsilon} - \mathcal{Y}_t^Z| \\ &\leq (1 - 2^{-n}) e^{L(\theta_{n+1}-1)} \varepsilon + 2^{-(n+1)} e^{L(\theta_{n+1}-1)} \varepsilon \\ &= (1 - 2^{-n}) e^{L(\theta_{n+1}-1)} \varepsilon \leq \varepsilon. \end{aligned}$$

Repeating the arguments, and sending $n \rightarrow \infty$, we define Z^ε over $[0, \theta)$ satisfying all the requirements. \square

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