SIAM J. CONTROL OPTIM.
Vol. 41, No. 4, pp. 1073–1088

CONTINUOUS-TIME DYNKIN GAMES WITH MIXED STRATEGIES*

NIZAR TOUZI† AND NICOLAS VIEILLE‡

Abstract. Let (X, Y, Z) be a triple of payoff processes defining a Dynkin game

$$\tilde{R}(\sigma, \tau) = E\left[X_\sigma \mathbf{1}_{\{\tau > \sigma\}} + Y_\tau \mathbf{1}_{\{\tau < \sigma\}} + Z_\tau \mathbf{1}_{\{\tau = \sigma\}}\right],$$

where σ and τ are stopping times valued in $[0, T]$. In the case $Z = Y$, it is well known that the condition $X \leq Y$ is needed in order to establish the existence of value for the game, i.e., $\inf_\tau \sup_\sigma \tilde{R}(\sigma, \tau) = \sup_\sigma \inf_\tau \tilde{R}(\sigma, \tau)$.

In order to remove the condition $X \leq Y$, we introduce an extension of the Dynkin game by allowing for an extended set of strategies, namely, the set of mixed strategies. The main result of the paper is that the extended Dynkin game has a value when $Z \leq Y$, and the processes X and Y are restricted to be semimartingales continuous at the terminal time T.

Key words. optimal stopping, Dynkin games, stochastic analysis, minimax theorem

AMS subject classifications. Primary, 60G40, 90D15; Secondary, 60H30, 46N10

PII. S0363012900369812

1. Introduction.

Dynkin games have been introduced by Dynkin (1967) as a generalization of optimal stopping problems. Since then, many authors contributed to solve the problem both in discrete and continuous-time models; see, e.g., Dynkin and Yushkevich (1968), Bensoussan and Friedman (1974), Neveu (1975), Bismut (1977), Stettner (1982), Alario, Lepeltier, and Marchal (1982), Morimoto (1984), Lepeltier and Maingueneau (1984), Cvitanić and Karatzas (1996), and Karatzas and Wang (2001), among others.

The setting of the problem is very simple. There are two players, labeled Player 1 and Player 2, who observe two payoff processes X and Y defined on a probability space $(\Omega, \mathcal{F}, P)$. Player 1 (resp., 2) chooses a stopping time σ (resp., τ) as control for this optimal stopping problem. At (stopping) time $\sigma \wedge \tau$ the game is over, and Player 2 pays the amount $X_\sigma \mathbf{1}_{\{\tau > \sigma\}} + Y_\tau \mathbf{1}_{\{\tau < \sigma\}} + Z_\tau \mathbf{1}_{\{\tau = \sigma\}}$ to Player 1. Therefore the objective of Player 1 is to maximize this payment, while Player 2 wishes to minimize it. It is then natural to introduce the lower and upper values of the game:

$$\sup_\sigma \inf_\tau E\left[X_\sigma \mathbf{1}_{\{\tau > \sigma\}} + Y_\tau \mathbf{1}_{\{\tau < \sigma\}} + Z_\tau \mathbf{1}_{\{\tau = \sigma\}}\right],$$

$$\inf_\tau \sup_\sigma E\left[X_\sigma \mathbf{1}_{\{\tau > \sigma\}} + Y_\tau \mathbf{1}_{\{\tau < \sigma\}} + Z_\tau \mathbf{1}_{\{\tau = \sigma\}}\right].$$

If the above values are equal, then the game is said to have a value. In the previously cited literature, it is proved that the game has a value essentially under the conditions $X_. \leq Y_. = Z_.$, P-a.s. A precise discussion of this is given in section 2.

The purpose of this paper is to remove the condition $X_. \leq Y_. = Z_.$, P-a.s. by suitably convexifying the set of strategies of the players. This is achieved by introducing

*Received by the editors March 22, 2000; accepted for publication (in revised form) January 23, 2002; published electronically October 8, 2002.
http://www.siam.org/journals/sicon/41-4/36981.html

†Centre de Recherche, en Economie et Statistique, 15 Bd. Gabriel Péri, 92245 Malakoff, France (touzi@ensae.fr).

‡HEC, 78351 Jouy en Josas, France, and Laboratoire d'Econométrie de l'Ecole Polytechnique, Paris, France (vieille@poly.polytechnique.fr).

the notion of mixed strategies, standard in (discrete-time) game theory literature. Loosely speaking, instead of choosing a stopping time, we shall allow both players to choose a distribution on the set of stopping times. Namely, at each time, both players fix a probability of stopping and decide whether or not to stop according to this probability.

This leads us to define mixed strategies as nondecreasing right-continuous processes with zero initial data and final data less than 1. In section 7 of this paper, we provide two justifications of this definition. The first is obtained by enlarging the probability space in order to allow for an independent randomizing device for each player. The second justification consists of defining the notion of randomized stopping time by means of functional analysis arguments, as in Bismut (1979).

Section 3 reports the precise definition of the extended Dynkin game and the main result of the paper: the extended Dynkin game has a value, provided the payoff processes X and Y are semimartingales continuous at the terminal time T, and $Z_. \leq Y_.$, P-a.s. For ease of presentation, we split the proof as follows. Section 4 provides the main steps of the proof, which basically relies on the two following technical results. In the first one, reported in section 5, we prove that the players' strategy sets can be reduced without affecting the lower and the upper values of the game. The second one states that the game with restricted strategies has a value. The proof of the last claim, reported in section 6, is obtained by an application of Sion's min-max theorem.

Before concluding this introduction, let us set up some notation which will be extensively used in the paper.

Given a right-continuous process with left limits S, we denote $S_{t-} := \lim_{s\uparrow t} S_s$. The jumps of S are denoted by $\Delta S_t := S_t - S_{t-}$. We shall denote by ΔS the process of jumps of S, and by S_- the process of left limits of S.

We shall denote by λ the Lebesgue measure on $[0, T]$, and by E_λ the associated expectation operator. For a nondecreasing process A, we denote by m_A the positive finite measure induced by A. If S is a semimartingale, then it admits a decomposition $S = M + A$, where A is a finite variation process and M is a martingale. We shall denote by m_M the measure induced by the (nondecreasing) predictable quadratic variation process $\langle M, M\rangle$ of M, i.e., $m_M(B) = E_\lambda\left[\mathbf{1}_B \langle M\rangle_\infty\right]$. We abuse the latter notation by saying that some property holds m_S-a.s. whenever it holds both m_A-a.s. and m_M-a.s.

2. Dynkin game with pure strategies. In this section, we recall the classical formulation of a Dynkin game, as suggested by Dynkin and Yushkevich (1968), Neveu (1975), and Bismut (1977).

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and let $T > 0$ be a fixed terminal time. Let $X = \{X_t,\, 0 \leq t \leq T\}$, $Y = \{Y_t,\, 0 \leq t \leq T\}$, and $Z = \{Z_t,\, 0 \leq t \leq T\}$ be real-valued càdlàg processes, satisfying the integrability condition

$$E\left[\sup_t |X_t| + \sup_t |Y_t| + \sup_t |Z_t|\right] < +\infty. \tag{2.1}$$

We denote by $\mathbb{F} = \{\mathcal{F}_t,\, 0 \leq t \leq T\}$ the P-augmentation of the filtration generated by (X, Y, Z), and by $\mathcal{T}$ the set of all stopping times for $\mathbb{F}$.

The structure of a Dynkin game is the following. Two players observe the triple of stochastic processes (X, Y, Z). Player 1 chooses a stopping time $\sigma \in \mathcal{T}$, and Player 2 chooses a stopping time $\tau \in \mathcal{T}$. Player 2 pays Player 1 the amount

$$X_\sigma \mathbf{1}_{\{\tau>\sigma\}} + Y_\tau \mathbf{1}_{\{\tau<\sigma\}} + Z_\tau \mathbf{1}_{\{\tau=\sigma\}}.$$

The payoff of the game is then defined by the expected value of the above payoff:

$$\tilde{R}(\sigma,\tau) := E\left[X_\sigma \mathbf{1}_{\{\tau>\sigma\}} + Y_\tau \mathbf{1}_{\{\tau<\sigma\}} + Z_\tau \mathbf{1}_{\{\tau=\sigma\}}\right].$$

Player 1 wishes to maximize $\tilde{R}(\sigma,\tau)$, while Player 2 wishes to minimize it. It is then natural to define the lower and upper values of the game:

$$\underline{V} \;:=\; \sup_\sigma \inf_\tau \tilde{R}(\sigma,\tau) \qquad \text{and} \qquad \overline{V} \;:=\; \inf_\tau \sup_\sigma \tilde{R}(\sigma,\tau),$$

which satisfy $\underline{V} \le \overline{V}$. If it happens that

$$\underline{V} = \overline{V},$$

then the above Dynkin game is said to have a value.

There is extensive literature providing sufficient conditions for the existence of the value for the continuous-time Dynkin game in the case $Z = Y$. Bismut (1977) proved existence of the value under the condition

$$X_. \le Y_. \;=\; Z_., \qquad P\text{-a.s.} \tag{2.2}$$

as well as some regularity conditions and Mokobodski's hypothesis (namely, that there exist positive bounded supermartingales Z and Z' satisfying $X \le Z - Z' \le Y$). The regularity assumption was weakened by Alario, Lepeltier, and Marchal (1982), and then Lepeltier and Maingueneau (1984) established the existence of the value without Mokobodski's hypothesis, assuming only $X_. \le Y_. = Z_.$.

We also mention the paper by Cvitanić and Karatzas (1996), which derives the latter result in the context of a Brownian filtration by means of doubly reflected backward stochastic differential equations.

3. Dynkin game with mixed strategies. The chief goal of this paper is to remove condition (2.2) by "convexifying" the set of stopping times. A precise discussion of the problem of extending the set of strategies is provided in section 7. In this section, we give only the main intuition in order to obtain an extended version of the Dynkin game, and we state the main result of the paper.

The main idea is to identify stopping times with $\{0,1\}$-valued, nondecreasing processes. Then convexifying the set of these processes leads naturally to considering the set $\mathcal{V}^+$ of all adapted, nondecreasing, right-continuous processes A with $A_{0-} = 0$ and $A_T \le 1$.

More precisely, let $\mathcal{V}_{0,1}$ be the subset of $\{0,1\}$-valued processes of $\mathcal{V}^+$. For every stopping time τ, define the process F^τ by

$$F_t^\tau := \mathbf{1}_{\{\tau\le t\}}, \qquad 0 \le t \le T.$$

It is clear that $F^\tau \in \mathcal{V}_{0,1}$. Conversely, given $F \in \mathcal{V}_{0,1}$, let

$$\tau_F := \inf\{t \in [0,T] :\; F_t > 0\}$$

with the usual convention $\inf \emptyset = +\infty$. From the right-continuity of F, it is clear that τ_F is a stopping time for $\mathbb{F}$. This provides an identification of $\mathcal{V}_{0,1}$ and $\mathcal{T}$. Clearly, the payoff function $\tilde{R}$ can be written in terms of $F, G \in \mathcal{V}_{0,1}$ as

$$R(F,G) \;:=\; \tilde{R}(\tau_F,\tau_G) = E\left[\int_0^T X(1-G)dF + \int_0^T Y(1-F)dG + \sum_{[0,T]} Z\Delta F \Delta G\right].$$

Observe that the right-hand side expression is well defined for $F, G \in \mathcal{V}^+$. Our interest is in the extended Dynkin game, in which players choose elements of $\mathcal{V}^+$, and the payoff is given by R. A rigorous justification of the set $\mathcal{V}^+$ as being the set of mixed strategies is reported in section 7, as well as the extension of the payoff function $\tilde{R}$ to $\mathcal{V}^+$.

The following is the main result of the paper.

THEOREM 3.1. *Let (X, Y, Z) be a triple of payoff processes satisfying* (2.1)*. Suppose that X and Y are semimartingales with trajectories continuous at time T, P-a.s. Assume further that $Z \leq Y$. Then*

$$\sup_{F \in \mathcal{V}^+} \inf_{G \in \mathcal{V}^+} R(F, G) = \inf_{G \in \mathcal{V}^+} \sup_{F \in \mathcal{V}^+} R(F, G),$$

i.e., the extended Dynkin game has a value.

This theorem states that the Dynkin game has a value when the set of strategies $\mathcal{V}_{0,1}$ is convexified in the natural way. The only conditions required for this result are $Z \leq Y$, and X and Y are semimartingales continuous at the terminal time T. The reason for the restriction to semimartingales is explained in Remark 5.1.

An alternative way of convexifying the set $\mathcal{T}$ of stopping times is to allow the players to choose a randomized stopping time, i.e., a probability distribution over stopping times. This corresponds to the concept of mixed strategy in game theory. Although in some respect more natural, this approach is more technically demanding, as it requires an abstract construction by means of functional analysis tools (see section 7 and Bismut (1979)).

The connection between the two approaches is that any process in $\mathcal{V}^+$ can intuitively be viewed as the random distribution function of a randomized stopping time. Another interpretation is that each player chooses randomly, at each time t, whether to stop or not. This corresponds to the concept of behavioral strategy in game theory.

There is entensive literature in game theory, starting with Kuhn (1953), on the equivalence between mixed strategies and behavioral strategies. In discrete time, both notions are equivalent under fairly general assumptions (see Mertens, Sorin, and Zamir (1994)).

A by-product of section 7 is that, in the context of the simple game studied in this paper, behavioral strategies and mixed strategies are equivalent.

4. Proof of the main result. We prove the result by applying the following well-known min-max theorem.

THEOREM 4.1 (see Sion (1958)). *Let S and T be convex subsets of topological vector spaces, one of which is compact, and let $g: S \times T \longrightarrow \mathbb{R}$. Assume that for every real c, the sets $\{t : g(s_0, t) \leq c\}$ and $\{s : g(s, t_0) \geq c\}$ are closed and convex for every $(s_0, t_0) \in S \times T$. Then*

$$\sup_{s \in S} \inf_{t \in T} g(s, t) = \inf_{t \in T} \sup_{s \in S} g(s, t).$$

If S (resp., T) is compact, then sup *(resp.,* inf*) may be replaced by* max *(resp.,* min*), i.e., the corresponding player has an optimal strategy.*

The main difficulty in the proof of Theorem 3.1 is that the above min-max theorem does not apply directly to the set of strategies $\mathcal{V}^+$ (see the proofs of Lemmas 6.3 and 6.4). We therefore start by reducing the set of strategies to some subsets of $\mathcal{V}^+$ for which the min-max theorem applies.

We first restrict the strategies of the first player. Define

$$\mathcal{V}_1 := \left\{ F \in \mathcal{V}^+ : \ F \text{ is continuous, } P\text{-a.s.} \right\}.$$

As for the second player, we introduce the subset of strategies:

$$\mathcal{V}_2 := \left\{ G \in \mathcal{V}^+ : \ G_T = 1 \text{ on } \{Y_T < 0 < X_T\}, \text{ and } Y_T \Delta G_T \leq 0 \right\}.$$

We shall prove that the restriction of the strategies of Player 2 from $\mathcal{V}^+$ to $\mathcal{V}_2$ does not change the value of the game. The following is an intuitive justification of this claim. On the event set $\{Y_T < 0 < X_T\}$, it follows from the continuity of the payoff processes X and Y at T that it is optimal for Player 2 to stop the game before time T; recall that $Z_. \leq Y_.$, implying that the situation is even better for Player 2 if Player 1 stops at the same time. On the other hand, on the event set $\{Y_T > 0\}$, Player 2 can obtain the same value of the game by smoothing his strategy at time T, again taking advantage of the continuity at time T of the process Y.

Also, given that the strategies of Player 2 are restricted to $\mathcal{V}_2$, we shall prove that the restriction of the strategies of Player 1 to $\mathcal{V}_1$ does not change the value of the game; i.e., Player 1 can achieve the same value by means of continuous strategies.

For ease of presentation, the proof of the following two propositions will be reported in section 5.

PROPOSITION 4.1. *Let* (X, Y, Z) *be a triple of payoff processes satisfying* (2.1). *Then*

$$\sup_{F \in \mathcal{V}_1} \inf_{G \in \mathcal{V}_2} R(F, G) = \sup_{F \in \mathcal{V}_1} \inf_{G \in \mathcal{V}^+} R(F, G).$$

PROPOSITION 4.2. *Under the assumptions of Theorem* 3.1, *we have*

$$\inf_{G \in \mathcal{V}_2} \sup_{F \in \mathcal{V}_1} R(F, G) = \inf_{G \in \mathcal{V}_2} \sup_{F \in \mathcal{V}^+} R(F, G).$$

We then apply the min-max theorem to the strategy sets $S = \mathcal{V}_1$ and $T = \mathcal{V}_2$.

PROPOSITION 4.3. *Let* (X, Y, Z) *be a triple of processes satisfying* (2.1). *Assume further that* X *and* Y *are semimartingales. Then, we have*

$$\sup_{F \in \mathcal{V}_1} \inf_{G \in \mathcal{V}_2} R(F, G) = \inf_{G \in \mathcal{V}_2} \sup_{F \in \mathcal{V}_1} R(F, G).$$

The proof of the last proposition will be carried out in section 6. We now complete the proof of Theorem 3.1. By Proposition 4.2 and the fact that $\mathcal{V}_2 \subset \mathcal{V}^+$, we see that

$$\inf_{G \in \mathcal{V}_2} \sup_{F \in \mathcal{V}_1} R(F, G) = \inf_{G \in \mathcal{V}_2} \sup_{F \in \mathcal{V}^+} R(F, G) \ \geq \ \inf_{G \in \mathcal{V}^+} \sup_{F \in \mathcal{V}^+} R(F, G).$$

Similarly, it follows from Proposition 4.1 and the fact that $\mathcal{V}_1 \subset \mathcal{V}^+$ that

$$\sup_{F \in \mathcal{V}_1} \inf_{G \in \mathcal{V}_2} R(F, G) = \sup_{F \in \mathcal{V}_1} \inf_{G \in \mathcal{V}^+} R(F, G) \ \leq \ \sup_{F \in \mathcal{V}^+} \inf_{G \in \mathcal{V}^+} R(F, G).$$

In view of Proposition 4.3, this provides

$$\inf_{G \in \mathcal{V}^+} \sup_{F \in \mathcal{V}^+} R(F, G) \leq \sup_{F \in \mathcal{V}^+} \inf_{G \in \mathcal{V}^+} R(F, G),$$

which ends the proof, as the reverse inequality is trivial.

5. A priori restrictions on strategies. This section is devoted to the proofs of Propositions 4.1 and 4.2.

5.1. Proof of Proposition 4.1. Let F be a fixed strategy of Player 1 in the set $\mathcal{V}_1$. For each $G \in \mathcal{V}^+$, we define $\overline{G} \in \mathcal{V}_2$ by

$$\begin{aligned} \overline{G}_T = 1 &\quad \text{on the event set } \{X_T > 0 > Y_T\}, \\ \overline{G}_T = G_{T-} &\quad \text{on the event set } \{Y_T > 0\}, \\ \overline{G} = G &\quad \text{otherwise.} \end{aligned}$$

Then it is immediately checked that

$$\begin{aligned} R(F, \overline{G}) - R(F, G) &= E\left[X_T(\Delta G_T - \Delta \overline{G}_T)\Delta F_T\right] \\ &\quad + E\left[Y_T(1 - F_T)(\Delta \overline{G}_T - \Delta G_T)\right] \\ &\quad + E\left[Z_T \Delta F_T(\Delta \overline{G}_T - \Delta G_T)\right] \\ &= E\left[Y_T(1 - F_T)(\Delta \overline{G}_T - \Delta G_T)\right] \end{aligned}$$

since F is continuous. By definition of $\overline{G}$, we have $\Delta\overline{G}_T = 0$ on $\{Y_T > 0\}$ and $\Delta\overline{G}_T \geq \Delta G_T$ on $\{Y_T < 0\}$. It follows that $R(F, \overline{G}) - R(F, G) \leq 0$, and therefore

$$\sup_{F \in \mathcal{V}_1} \inf_{G \in \mathcal{V}_2} R(F, G) \leq \sup_{F \in \mathcal{V}_1} \inf_{G \in \mathcal{V}^+} R(F, G).$$

The required result follows from the fact that $\mathcal{V}_2 \subset \mathcal{V}^+$.

5.2. Proof of Proposition 4.2. We introduce the subset of strategies $\mathcal{W}_1$ defined by

$$\mathcal{W}_1 = \left\{F \in \mathcal{V}^+ : \quad \Delta F_T = 0 \;\text{ on } \{X_T > 0, Y_T \geq 0\}\right\}.$$

In order to prove Proposition 4.2, we first need to prove that the restriction of the strategies of Player 1 from $\mathcal{V}^+$ to $\mathcal{W}_1$ does not change the value of the game. As we shall see in the subsequent proof, this is a consequence of the continuity of the payoff processes X and Y at time T.

LEMMA 5.1. *Let (X, Y, Z) be a triple of processes satisfying* (2.1). *Assume further that X and Y have continuous trajectories at time T. Then, for any $G \in \mathcal{V}_2$ and $F \in \mathcal{V}^+$, there exists a sequence $(F^n)_n$ in $\mathcal{W}_1$ such that*

$$\limsup_{n\to\infty} R(F^n, G) \geq R(F, G).$$

Proof. We organize the proof in four steps.

Step 1. Let $\mathcal{T}_{[t,T]}$ denote the set of $[t, T]$-valued stopping times. We introduce the two Snell envelopes U and V defined by

$$\begin{aligned} U_t &:= \operatorname{ess} \sup_{\zeta \in \mathcal{T}_{[t,T]}} E[X_\zeta | \mathcal{F}_t], \\ V_t &:= \operatorname{ess} \inf_{\zeta \in \mathcal{T}_{[t,T]}} E[Y_\zeta | \mathcal{F}_t]. \end{aligned}$$

In view of our assumptions on X and Y, the processes U and V can be considered in their càdlàg modifications; see, e.g., Appendix D in Karatzas and Shreve (1998). In the rest of this step, we prove that

$$U \text{ and } V \text{ are continuous at } T, \quad P\text{-a.s.}$$

To see this, observe that

$$0 \leq U_t - E\left[X_T|\mathcal{F}_t\right] \leq E\left[\sup_{t\leq s\leq T} X_s - X_T|\mathcal{F}_t\right], \tag{5.1}$$

and, by Theorem VI.6 in Dellacherie and Meyer (1975),

$$E\left[X_T|\mathcal{F}_t\right] \longrightarrow E\left[X_T|\mathcal{F}_{T-}\right] = X_T \quad \text{as } t \nearrow T \tag{5.2}$$

by continuity of X at T. Now, notice that the process $A_t := \sup_{t\leq s\leq T} X_s - X_T$ is decreasing. Then, for fixed $s < T$, we have

$$0 \leq \limsup_{t\nearrow T} E\left[A_t|\mathcal{F}_t\right] \leq E\left[A_s|\mathcal{F}_{T-}\right].$$

By sending s to T, it follows from the dominated convergence theorem that

$$0 \leq \limsup_{t\nearrow T} E\left[A_t|\mathcal{F}_t\right] \leq E\left[A_T|\mathcal{F}_{T-}\right] = 0, \tag{5.3}$$

where we used the continuity of A at T inherited from X. The required continuity result follows from (5.1)–(5.3).

Step 2. For each $\varepsilon > 0$, define

$$\theta^\varepsilon := \inf\{t \geq T - \varepsilon : \ X_t \geq 0,\ U_t - \varepsilon \leq X_t \text{ and } V_t \geq -\varepsilon\} \wedge T.$$

Since X, U, and V are right-continuous, θ^ε is a stopping time. Observe that $\theta^\varepsilon \to T$, P-a.s., as $\varepsilon \to 0$.

Next, for each integer $n \geq 1$, define the sequence of stopping times

$$\theta^{\varepsilon,n} := T \wedge \left(\theta^\varepsilon + \frac{1}{n}\right).$$

We define $(F^{\varepsilon,n}) \in \mathcal{V}^+$ to be a continuous process on $(\theta^\varepsilon, \theta^{\varepsilon,n}]$ such that

$$F^{\varepsilon,n} = F \text{ on } [0,\theta^\varepsilon] \qquad \text{and} \qquad F^{\varepsilon,n} = 1 \text{ on } [\theta^{\varepsilon,n}, T].$$

Since X, Y, U, and V are continuous at T, $F^{\varepsilon,n}$ is a sequence in $\mathcal{W}_1$. We intend to prove that

$$\limsup_{\varepsilon\to 0}\limsup_{n\to\infty} R(F^{\varepsilon,n}, G) \geq R(F, G),$$

which will provide the required result.

First, since $F^{\varepsilon,n}$ is continuous on $(\theta^\varepsilon, T]$ and $F^{\varepsilon,n} = 1$ on $[\theta^{\varepsilon,n}, T]$, we have

$$\begin{aligned} R(F^{\varepsilon,n}, G) &= A + E\left[\xi^\varepsilon \int_{\theta^\varepsilon}^T X(1-G)dF^{\varepsilon,n} + Y(1-F^{\varepsilon,n})dG\right] \\ &= A + E\left[\xi^\varepsilon \int_{\theta^\varepsilon}^{\theta^{\varepsilon,n}} X(1-G)dF^{\varepsilon,n}\right] + E\left[\xi^\varepsilon \int_{\theta^\varepsilon}^{\theta^{\varepsilon,n}} Y(1-F^{\varepsilon,n})dG\right], \end{aligned}$$

where $\xi^\varepsilon = \mathbf{1}_{\{\theta^\varepsilon < T\}}$ and

$$A = E\left[\int_0^{\theta^\varepsilon} X(1-G)dF + Y(1-F)dG + \sum_{[0,\theta^\varepsilon]} Z\Delta F\Delta G\right].$$

Step 3. We now fix $\varepsilon > 0$ and let n go to infinity. As for the second expectation on the right-hand side of (5.4), observe that $Y_t\xi^\varepsilon(1-F_t^{\varepsilon,n})\mathbf{1}_{[\theta^\varepsilon,T]}(t)$ converges P-a.s. to zero for all $t \in (\theta^\varepsilon, T]$. Since G is right-continuous, this implies that $Y_t\xi^\varepsilon(1 - F_t^{\varepsilon,n})\mathbf{1}_{[\theta^\varepsilon,T]}(t)$ converges $m_G \otimes P$-a.s. to zero. Therefore, by dominated convergence (see Theorem I.4.31 in Jacod and Shiryaev (1987)), we have

$$\lim_{n\to\infty} E\left[\xi^\varepsilon \int_{\theta^\varepsilon}^{\theta^{\varepsilon,n}} Y(1-F^{\varepsilon,n})dG\right] = 0.$$

As for the first expectation on the right-hand side of (5.4), we have

$$\begin{aligned}
\limsup_{n\to\infty} E\left[\xi^\varepsilon \int_{\theta^\varepsilon}^{\theta^{\varepsilon,n}} X(1-G)dF^{\varepsilon,n}\right] &\geq \limsup_{n\to\infty} E\left[\xi^\varepsilon \inf_{[\theta^\varepsilon,\theta^{\varepsilon,n}]}(X(1-G)) \int_{\theta^\varepsilon}^{\theta^{\varepsilon,n}} dF^{\varepsilon,n}\right] \\
&= \limsup_{n\to\infty} E\left[\xi^\varepsilon \inf_{[\theta^\varepsilon,\theta^{\varepsilon,n}]}(X(1-G))(1-F_{\theta^\varepsilon})\right] \\
&= E\left[\xi^\varepsilon X_{\theta^\varepsilon}(1-G_{\theta^\varepsilon})(1-F_{\theta^\varepsilon})\right],
\end{aligned}$$

where the last equality follows by dominated convergence and right-continuity of $X(1-G)$. This yields

$$\begin{aligned}
\limsup_{n\to\infty} R(F^{\varepsilon,n}, G) - R(F,G) &\geq E\left[\xi^\varepsilon X_{\theta^\varepsilon}(1-G_{\theta^\varepsilon})(1-F_{\theta^\varepsilon})\right] \\
&\qquad - E\left[\xi^\varepsilon \int_{\theta^\varepsilon}^T X(1-G)dF + Y(1-F)dG\right] \\
&\qquad - E\left[\xi^\varepsilon \sum_{(\theta^\varepsilon,T]} Z\Delta F\Delta G\right] \\
&= E\left[\xi^\varepsilon X_{\theta^\varepsilon}(1-G_{\theta^\varepsilon})(1-F_{\theta^\varepsilon})\right] \\
&\qquad - E\left[\xi^\varepsilon \int_{\theta^\varepsilon}^T X(1-G)dF + Y(1-F_-)dG\right] \\
&\qquad + E\left[\xi^\varepsilon \sum_{(\theta^\varepsilon,T]} (Y-Z)\Delta F\Delta G\right] \\
&\geq E\left[\xi^\varepsilon X_{\theta^\varepsilon}(1-G_{\theta^\varepsilon})(1-F_{\theta^\varepsilon})\right] \\
&\qquad - E\left[\xi^\varepsilon \int_{\theta^\varepsilon}^T X(1-G)dF + Y(1-F_-)dG\right],
\end{aligned}$$

where we used the condition $Z \leq Y$ of Theorem 3.1. Set $\widetilde{F} := F - \Delta F\mathbf{1}_{\{T\}}$ and $\widetilde{G}$

$:= G - \Delta G\mathbf{1}_{\{T\}}$. Then

$$
\begin{aligned}
\limsup_{n\to\infty} R(F^{\varepsilon,n},G) - R(F,G) &\geq E\left[\xi^\varepsilon X_{\theta^\varepsilon}(1-G_{\theta^\varepsilon})(1-F_{\theta^\varepsilon}) - \xi^\varepsilon X_T(1-G_T)\Delta F_T\right] \\
&\quad - E\left[\xi^\varepsilon \int_{\theta^\varepsilon}^T X(1-G)d\widetilde{F} + Y(1-F_-)d\widetilde{G}\right] \\
&\quad - E\left[\xi^\varepsilon Y_T(1-F_T)\Delta G_T\right] \\
&\geq E\left[\xi^\varepsilon X_{\theta^\varepsilon}(1-G_{\theta^\varepsilon})(1-F_{\theta^\varepsilon}) - \xi^\varepsilon X_T(1-G_T)\Delta F_T\right] \\
&\quad - E\left[\xi^\varepsilon \int_{\theta^\varepsilon}^T X(1-G)d\widetilde{F} + Y(1-F_-)d\widetilde{G}\right] \qquad (5.4)
\end{aligned}
$$

since $Y_T\Delta G_T \leq 0$ by definition of $\mathcal{V}_2$.

Step 4. We now take limits as ε goes to zero. Since $\theta^\varepsilon \to T$, and both $\widetilde{F}$ and $\widetilde{G}$ are continuous at T, the second expectation on the right-hand side of (5.4) converges to zero. We now use the following claim, whose proof will be carried out later:

$$\xi^\varepsilon X_{\theta^\varepsilon} \longrightarrow \mathbf{1}_{\{0\leq X_T, Y_T\}} X_T, \qquad P\text{-a.s.} \tag{5.5}$$

Then, by dominated convergence and the fact that $G_{T-} \leq G_T$,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} E\left[\xi^\varepsilon X_{\theta^\varepsilon}(1-G_{\theta^\varepsilon})(1-F_{\theta^\varepsilon}) - \xi^\varepsilon X_T(1-G_T)\Delta F_T\right] \\
&\geq E\left[\mathbf{1}_{\{0\leq X_T,Y_T\}} X_T(1-G_T)(1-F_{T-}-\Delta F_T)\right] \\
&= E\left[\mathbf{1}_{\{0\leq X_T,Y_T\}} X_T(1-G_T)(1-F_T)\right] \\
&\geq 0
\end{aligned}
$$

by definition of F and G. Hence

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} R(F^{\varepsilon,n},G) - R(F,G) \geq 0.$$

It remains to prove (5.5). By definition of θ^ε, it is clear that θ^ε (hence also ξ^ε) increases as ε decreases to zero. Thus,

$$\xi^\varepsilon \longrightarrow \mathbf{1}_{\cap_{\varepsilon>0}\{\theta^\varepsilon<T\}}, \quad P\text{-a.s.}$$

Now, observe that $0 \leq X_T, 0 \leq Y_T$ on the event $\{\theta^\varepsilon < T \text{ for all } \varepsilon\}$ by continuity at T of the Snell envelopes U and V. Conversely, on the event $\{0 < X_T, 0 \leq Y_T\}$, it is clear that $\theta^\varepsilon < T$ for all ε, again by continuity of U and V. This provides claim (5.5). $\square$

Given the result of Lemma 5.1, the statement of Proposition 4.2 follows directly from the following reduction of strategies of Player 1 from $\mathcal{W}_1$ to $\mathcal{V}_1$.

LEMMA 5.2. *Let (X, Y, Z) be a triple of processes satisfying* (2.1). *Assume further that X is a semimartingale and $Z \leq Y$. Then, for any $G \in \mathcal{V}_2$ and $F \in \mathcal{W}_1$, there exists a sequence (F^n) in $\mathcal{V}_1$ such that*

$$\limsup_{n\to\infty} R(F^n, G) \geq R(F,G).$$

Proof. For each integer n, define $\widetilde{F}^n \in \mathcal{V}^+$ by

$$\widetilde{F}^n_t = F_t - \sum_{s\leq t} \Delta F_s \mathbf{1}_{\{\Delta F_s \leq n^{-1}\}}$$

so that the jumps of $\widetilde{F}^n$ are of size greater than n^{-1}, and therefore $\widetilde{F}^n$ has a finite number of jumps. Clearly, we have the pointwise convergence

$$\widetilde{F}^n_t \longrightarrow F_t, \quad 0 \le t \le T, \quad P\text{-a.s.} \tag{5.6}$$

Since $\widetilde{F}^n$ has a finite number of jumps, it follows from a diagonal extraction argument that there exists a sequence of *continuous* processes $F^n \in \mathcal{V}^+$ such that $F^n - \widetilde{F}^n_- \to 0$ pointwise, P-a.s. From the pointwise convergence (5.6), this provides

$$F^n \;\longrightarrow\; F_-, \quad P\text{-a.s.}$$

In order to obtain the required result, we shall prove that

$$\lim_{n\to\infty} R(F^n, G) \ge R(F, G). \tag{5.7}$$

First, observe that by Itô's lemma (see, e.g., Theorem I.4.57 in Jacod and Shiryaev (1987)), we have

$$\begin{aligned} R(F,G) &= E\left[\int_0^T Y(1-F_-)dG\right] - E\left[\int_0^T F_- d(X(1-G))\right] \\ &\quad + E\left[\sum_{[0,T]} (Z-Y)\Delta F \Delta G\right] \\ &\quad + E\left[X_T(1-G_T)F_{T-}\right] + E\left[X_T(1-G_T)\Delta F_T\right] \\ &\le E\left[\int_0^T Y(1-F_-)dG\right] - E\left[\int_0^T F_- d(X(1-G))\right] \\ &\quad + E\left[X_T(1-G_T)F_{T-}\right] + E\left[X_T(1-G_T)\Delta F_T\right], \end{aligned} \tag{5.8}$$

where we used the condition $Z \le Y$ of Theorem 3.1. Since $F^n \to F_-$, m_G and $m_{X(1-G)}$-a.s., it follows from dominated convergence that

$$\lim_{n\to\infty} E\left[\int_0^T Y(1-F^n)dG\right] = E\left[\int_0^T Y(1-F_-)dG\right],$$

$$\lim_{n\to\infty} E\left[\int_0^T F^n d(X(1-G))\right] = E\left[\int_0^T F_- d(X(1-G))\right],$$

$$\lim_{n\to\infty} E\left[X_T(1-G_T)F^n_T\right] = E\left[X_T(1-G_T)F_{T-}\right].$$

In view of (5.8), and since F^n is continuous, this proves that

$$\lim_{n\to\infty} R(F^n, G) \ge R(F,G) - E\left[X_T(1-G_T)\Delta F_T\right].$$

Finally, observe that

$$\begin{aligned} X_T(1-G_T)\Delta F_T &\le X_T(1-G_T)\Delta F_T \mathbf{1}_{\{X_T>0\}} \mathbf{1}_{\{G_T<1\}} \\ &= X_T(1-G_T)\Delta F_T \mathbf{1}_{\{0<X_T, 0\le Y_T\}} \mathbf{1}_{\{G_T<1\}} \\ &= 0, \end{aligned}$$

where we used the fact that $G \in \mathcal{V}_2$ and $F \in \mathcal{W}_1$. This ends the proof of (5.7), and the proof of Lemma 5.2 is complete. ∎

Remark 5.1. In the last proof, we used for the first time the fact that X is a semimartingale. The reason is that we needed to apply integration by parts in the integral $\int_0^T X(1-G)dF$, and therefore we needed the stochastic integral with respect to process X to be well defined. Similar integration by parts are involved in the proofs of Lemmas 6.3 and 6.4, which then require the assumption that X and Y are semimartingales.

6. The value on restricted strategy spaces. This section is devoted to the proof of Proposition 4.3. As argued earlier, we shall apply Sion's theorem to the sets $S = \mathcal{V}_1$ and $T = \mathcal{V}_2$. We first define a suitable topology on $\mathcal{V}_1$ and $\mathcal{V}_2$.

Let $\mathcal{S}$ be the set of all $\mathbb{F}$-adapted processes Z satisfying $Z_{0-} = 0$ and

$$E\left[\int_0^T Z_t^2 dt + (\Delta Z_T)^2\right] < +\infty, \quad \text{where } \Delta Z_T = Z_T - \liminf_{t\nearrow T} Z_t.$$

The space $\mathcal{S}$ is a separable Hilbert space when endowed with the scalar product

$$\frac{1}{T+1} E\left[\int_0^T W_t Z_t dt + \Delta W_T \Delta Z_T\right].$$

Notice that $\mathcal{V}_1$ and $\mathcal{V}_2$ are convex subsets of $B_{\mathcal{S}}$, the unit ball of $\mathcal{S}$.

LEMMA 6.1. *The set $\mathcal{V}_2$ is compact for the weak topology $\sigma(\mathcal{S}, \mathcal{S})$.*

Proof. Since $B_{\mathcal{S}}$ is compact for the weak topology $\sigma(\mathcal{S}, \mathcal{S})$, it suffices to prove that $\mathcal{V}_2$ is closed for the weak topology or, equivalently, for the strong topology, by convexity.

Let (Z^n) be a sequence in $\mathcal{V}_2$, which converges strongly to some $Z \in \mathcal{S}$. Then, possibly along some subsequence,

$$Z^n \longrightarrow Z, \quad \lambda \otimes P\text{-a.s.,} \tag{6.1}$$

and

$$Z_T^n \longrightarrow Z_T, \quad P\text{-a.s.} \tag{6.2}$$

Clearly, this shows that Z inherits the nondecrease of (Z^n), $Z_{0-} = 0$, and $Z_T \leq 1$. We now check that $\Delta Z_T^n \to \Delta Z_T$, P-a.s. By Fubini's theorem, it follows from (6.1) that, P-a.s., $Z_t^n \to Z_t$ for λ-a.e. $t \in [0,T]$. Since Z^n and Z are nondecreasing, we see that, P-a.s., $Z_{t-}^n \to Z_{t-}$ for every $t \in [0,T]$. Thus, from (6.2), this yields $\Delta Z_T^n \to \Delta Z_T$, P-a.s. The required result follows from the fact that $\Delta Z_T^n = 0$ on the event $\{Y_T > 0\}$. Observe finally that $Z_T^n = 1$ for every n implies $Z_T = 1$. ∎

LEMMA 6.2. *Let $(F^n)_n$ be a sequence in $\mathcal{V}_1$ converging to some $F \in \mathcal{V}_1$ in the sense of the strong topology of $\mathcal{S}$. Then*

$$\lim_{n\to\infty} F_t^n = F_t \qquad \textit{for all } \ t \in [0,T], \qquad P\textit{-a.s.}$$

after possibly passing to a subsequence.

Proof. Let (F^n) be as in the statement of the lemma. Then, by possibly passing to a subsequence, $F^n \longrightarrow F$, $\lambda \otimes P$-a.s., and $F_T^n \longrightarrow F_T$, P-a.s. By the same argument as in the previous proof, we use Fubini's theorem and the nondecrease of F_n and F

to see that $F^n_{t-} \longrightarrow F_{t-}$ for all $t \in [0,T]$, P-a.s. The required result follows from the continuity of F_n and F. ∎

LEMMA 6.3. *Let (X,Y,Z) be a triple of processes satisfying* (2.1). *Assume further that X is a semimartingale. Then, for all $G \in \mathcal{V}_2$, the function $R(.,G)$ is continuous on $\mathcal{V}_1$ in the sense of the strong topology of $\mathcal{S}$.*

Proof. By Itô's lemma,

$$\begin{aligned} X_T(1-G_T)F_T &= \int_0^T X(1-G_-)dF + \int_0^T F(1-G_-)dX - \int_0^T FXdG \\ &= \int_0^T X(1-G)dF + \int_0^T F(1-G_-)dX - \int_0^T FXdG \end{aligned}$$

since F is a continuous process. Then

$$\begin{aligned} R(F,G) = E\left[\int_0^T YdG\right] - E\left[\int_0^T F(1-G_-)dX\right] + E\left[X_T(1-G_T)F_T\right] \\ +E\left[\int_0^T (X-Y)FdG\right]. \end{aligned}$$

Let $(F^n)_n$ be a sequence in $\mathcal{V}_1$ converging to $F \in \mathcal{V}_1$. We intend to prove that

$$\lim_{n\to\infty} R(F^n,G) = R(F,G).$$

Consider any subsequence (F^{n_k}) such that $\lim_k R(F^{n_k},G)$ exists. It suffices to prove that this limit is independent of the subsequence and equal to $R(F,G)$. For ease of notation, rename the subsequence (F^n). From Lemma 6.2, by possibly passing to a subsequence, we can assume that, P-a.s.,

$$\lim_n F^n_t = F_t \quad \text{for all } t \in [0,T].$$

Then, $F^n \to F$, $m_X \otimes P$-a.s., and $m_G \otimes P$-a.s. and the result follows by dominated convergence. ∎

LEMMA 6.4. *Let (X,Y,Z) be a triple of processes satisfying* (2.1). *Assume further that Y is a semimartingale. Then, for all $F \in \mathcal{V}_1$, the function $R(F,.)$ is continuous on $\mathcal{V}_2$ in the sense of the strong topology of $\mathcal{S}$.*

Proof. As in the previous proof, let (G^n) be a sequence in $\mathcal{V}_2$ converging to $G \in \mathcal{V}_2$. We intend to prove that

$$\lim_{n\to\infty} R(F,G^n) = R(F,G).$$

Consider any subsequence (G^{n_k}) such that $\lim_k R(F,G^{n_k})$ exists. It suffices to prove that this limit is independent of the subsequence and equal to $R(F,G)$. For ease of notation, rename the subsequence (G^n). Recall that G is nondecreasing. Then, applying the same argument as in the proof of Lemma 6.1, we see that by possibly passing to a subsequence, we can assume that

$$G^n \longrightarrow G, \quad \lambda \otimes P\text{-a.s.}, \qquad G^n_- \longrightarrow G_-, \quad P\text{-a.s.}$$

and

$$G^n_T \longrightarrow G_T, \quad P\text{-a.s.}$$

Set $\widehat{Y} := Y(1-F)$. By Itô's formula and the continuity of F,

$$\begin{aligned}\int_0^T Y(1-F)dF^n &= \widehat{Y}_T G^n_T - \int_0^T G^n d\hat{Y} + \sum_{0\le t\le T} \Delta Y_t(1-F_{t-})\Delta G^n_t \\ &= \widehat{Y}_T G^n_T - \int_0^T G^n d\hat{Y}^c + \sum_{0\le t\le T} \Delta Y_t(1-F_t) G^n_{t-}.\end{aligned}$$

Since F and $\widehat{Y}^c$ are continuous, $G^n \to G$, $m_F \otimes P$-a.s. and $m_{\widehat{Y}^c} \otimes P$-a.s, and the result follows by dominated convergence. ▯

Proof of Proposition 4.3. The strategy sets $S = \mathcal{V}_1$ and $T = \mathcal{V}_2$ are convex topological spaces when endowed with the weak topology $\sigma(\mathcal{S}, \mathcal{S})$. From Lemma 6.1, $\mathcal{V}_2$ is compact for $\sigma(\mathcal{S}, \mathcal{S})$.

Since $R(F, G)$ is bilinear, the sets $\{G \in \mathcal{V}_2 : R(F^0, G) \le c\}$ and $\{F \in \mathcal{V}_1 : R(F, G^0) \ge c\}$ are convex for all $F^0 \in \mathcal{V}_1$, $G^0 \in \mathcal{V}_2$, and $c \in \mathbb{R}$. Then in order to prove that they are closed for the weak topology $\sigma(\mathcal{S}, \mathcal{S})$, it suffices to prove that they are closed for the strong topology of $\mathcal{S}$. The latter is a direct consequence of Lemmas 6.3 and 6.4. We are then in the context of Sion's theorem, and the proof is complete. ▯

7. Extended problem and randomized stopping times. In this section, we first provide a justification of $\mathcal{V}^+$ as being the natural mixed strategy set, which has been described heuristically in section 3. Then we derive rigorously the payoff function $R(F, G)$ defined in the extended strategy set $\mathcal{V}^+ \times \mathcal{V}^+$.

For ease of exposition, we shall discuss the case $Z = Y$ only. The general case follows immediately by adding up the jump term induced by Z.

In game theory, mixed strategies are defined as probability distributions over pure strategies. In the context of Dynkin games, pure strategies are stopping times. At this stage, the main problem is to define a measurable structure on the set of stopping times. There are two ways to avoid this difficulty. Following Aumann (1964), one may define mixed strategies by enlarging the probability space; this viewpoint is discussed in section 7.1. An alternative approach consists of defining the notion of randomized stopping time by means of functional analysis arguments; this is discussed in section 7.2. We shall (essentially) show that $\mathcal{V}^+$ is in one-to-one correspondence with the set of mixed strategies and with the set of randomized stopping times. Therefore, both approaches are equivalent.

7.1. Mixed strategies. We enlarge the probability space from (Ω, P) to $([0,1] \times \Omega, \lambda_1 \otimes P)$, where λ_1 is the Lebesgue measure. A mixed strategy (for Player 1) is then defined as a $\lambda_1 \otimes P$ measurable function ϕ mapping $[0,1] \times \Omega$ into $[0, T]$ such that

$$\text{for } \lambda_1\text{-a.e.}, \quad r \in [0,1], \qquad \sigma_r := \phi(r, \cdot) \text{ is a stopping time.}$$

We denote by Φ the space of mixed strategies. Loosely speaking, $([0,1], \lambda_1)$ is a randomizing device for Player 1. In order to introduce an independent randomizing device for Player 2, we need to have an independent copy $([0,1], \lambda_2)$ of the probability space $([0,1], \lambda_1)$. The corresponding set of mixed strategies is denoted by Ψ; a generic element of Ψ will be denoted by ψ, and, for $r \in [0,1]$, we set $\tau_r := \psi(r, \cdot)$.

Hence, the underlying probability space for the extended Dynkin game is $([0,1] \times [0,1] \times \Omega, \lambda_1 \otimes \lambda_2 \otimes P)$.

Recall that the payoff function on the stopping times is denoted by $\widetilde{R}$, and its extension to $\mathcal{V}^+$ is denoted by R. The following result provides a justification of the definition of $\mathcal{V}^+$ as the set of mixed strategies, and R as the payoff function on the extended strategy sets.

PROPOSITION 7.1. (i) *There exists a mapping H from Φ (or Ψ) onto $\mathcal{V}^+$.*

(ii) *For every $(\phi, \psi) \in \Phi \times \Psi$, we have*

$$E_{\lambda_1 \otimes \lambda_2}\left[\widetilde{R}(\sigma, \tau)\right] = R(H(\phi), H(\psi)).$$

Proof. We only prove (i) for the set Φ. For $\phi \in \Phi$, define the process $H(\phi)$ by

$$H(\phi)_t = \int \mathbf{1}_{\{\sigma_r \le t\}} \lambda_1(dr) \;=\; E_{\lambda_1}[\mathbf{1}_{\{\sigma \le t\}}] \quad \text{for } t \in [0, T].$$

Clearly, $H(\phi)_{0-} = 0$, $H(\phi)$ is nondecreasing, right-continuous and $H(\phi)_T \le 1$. Since σ_r is a stopping time for λ_1-a.e., $r \in [0, 1]$, the process $H(\phi)$ is $\mathbb{F}$-adapted. This proves that $H(\phi) \in \mathcal{V}^+$. To see that H is onto, define

$$\phi^F(r, \omega) := \inf\{s \ge 0 : \; F_s(\omega) > r\} \quad \text{for } F \in \mathcal{V}^+.$$

Observe that $\phi^F \in \Phi$, since F is $\mathbb{F}$-adapted and right-continuous. Set $\sigma_r := \phi^F(r, \cdot)$. For $t \in [0, T]$, we compute

$$H(\phi^F)_t = \int \mathbf{1}_{\{\sigma_r \le t\}} \lambda_1(dr) \;=\; \int \mathbf{1}_{\{F_t \ge r\}} \lambda_1(dr) \;=\; F_t,$$

which concludes the proof of (i).

Let $(\phi, \psi) \in \Phi \times \Psi$, and set $F_t = \mathbf{1}_{\{\sigma \le t\}}$ and $G_t = \mathbf{1}_{\{\tau \le t\}}$. By Fubini's theorem,

$$E_{\lambda_1 \otimes \lambda_2 \otimes P}\left[\int_0^T X(1 - G)dF\right] = E_{\lambda_1 \otimes P}\left[\int_0^T X(1 - H(\psi))dF\right].$$

By Itô's lemma, this provides

$$\begin{aligned} E_{\lambda_1 \otimes \lambda_2 \otimes P}\left[\int_0^T X(1 - G)dF\right] &= E_{\lambda_1 \otimes P}\left[X_T(1 - H(\psi)_T)F_T - \int_0^T F_- d(X(1 - H(\psi))\right] \\ &= E_P\left[X_T(1 - H(\psi)_T)H(\phi)_T - \int_0^T H(\phi)_- d(X(1 - H(\psi))\right], \end{aligned}$$

where we again used Fubini's theorem. By another application of Itô's lemma, we get

$$E_{\lambda_1 \otimes \lambda_2 \otimes P}\left[\int_0^T X(1 - G)dF\right] = E_P\left[\int_0^T X(1 - H(\psi))dH(\phi)\right].$$

The same argument applies to the second integral $\int_0^T Y(1 - F_-)dG$. Hence,

$$\begin{aligned} E_{\lambda_1 \otimes \lambda_2}\left[\widetilde{R}(\sigma, \tau)\right] &= E_P\left[\int_0^T X(1 - H(\psi))dH(\phi) + Y(1 - H(\phi)_-)dH(\psi)\right] \\ &= R(H(\phi), H(\psi)). \qquad \square \end{aligned}$$

7.2. Randomized stopping times. In this section, we describe briefly the functional analysis approach in order to define the notion of randomized stopping times introduced by Bismut (1979). We shall recall a representation theorem which connects randomized stopping times to our set $\mathcal{V}^+$.

Let $\mathcal{Y}$ be the space of càdlàg optional processes Y defined on $[0,T]$ such that

$$E\left[\sup_{t\in[0,T]}|Y_t|\right] < +\infty. \tag{7.1}$$

Observe that $\mathcal{Y}$ is a Banach space when endowed with the norm defined by (7.1). We denote by $\mathcal{Y}'$ the dual space of $\mathcal{Y}$. Then we have the following representation result of elements of $\mathcal{Y}'$.

PROPOSITION 7.2 (see Bismut (1979)). *For any $\mu \in \mathcal{Y}'$, there exist two right-continuous adapted processes with finite variation A and B valued in $\mathbb{R}\cup\{+\infty\}$ such that*

$$\langle \mu, Y\rangle \;=\; E\left[\int_0^T Y dA + Y_- dB\right] \quad \textit{for all } Y \in \mathcal{Y}.$$

Proposition 1.3 in Bismut (1979) provides a uniqueness result for such a representation under further restrictions on A and B.

DEFINITION 7.1. *A randomized stopping time is an element $\mu \in \mathcal{Y}'$, for which there exists a representation with $B = 0$, A nondecreasing and $A_T \le 1$.*

The following easy consequence establishes the connection between our set of extended strategies $\mathcal{V}^+$ and the set of randomized stopping times.

COROLLARY 7.1. *There is a bijection between $\mathcal{V}^+$ and the set of randomized stopping times.*

Proof. To every randomized stopping time μ, we can associate $A \in \mathcal{V}^+$ by the above representation. Conversely, given $A \in \mathcal{V}^+$, it is easy to check that $Y \mapsto E\int_0^T Y dA$ belongs to $\mathcal{Y}'$. ∎

Acknowledgments. The authors wish to thank Eran Shmaya and Eilon Solan for helpful comments.

REFERENCES

M. ALARIO-NAZARET, J.P. LEPELTIER, AND B. MARCHAL (1982), *Dynkin games*, in Stochastic Differential Systems (Bad Honnef, 1982), Lecture Notes in Control Inform. Sci. 43, Springer-Verlag, Berlin, New York, pp. 23–32.

R.J. AUMANN (1964), *Mixed and behavior strategies in infinite extensive games*, in Advances in Game Theory, Ann. Math. Stud. 52, M. Dresher, L.S. Shapley, and A.W. Tucker, eds., Princeton University Press, Princeton, NJ.

A. BENSOUSSAN AND A. FRIEDMAN (1974), *Non-linear variational inequalities and differential games with stopping times*, J. Funct. Anal., 16, pp. 305–352.

J.-M. BISMUT (1977), *Sur un problème de Dynkin*, Z. Warsch. V. Geb., 39, pp. 31–53.

J.-M. BISMUT (1979), *Temps d'arrêt optimal, quasi-temps d'arrêt et retournement du temps*, Ann. Probab., 6, pp. 933–964.

J. CVITANIĆ AND I. KARATZAS (1996), *Backward stochastic differential equation with reflection and Dynkin games*, Ann. Probab., 24, pp. 2024–2056.

C. DELLACHERIE AND P.A. MEYER (1975), *Probabilités et potentiel*, Hermann, Paris.

E.B. DYNKIN (1967), *Game variant of a problem on optimal stopping*, Soviet Math. Dokl., 10, pp. 270–274.

E.B. DYNKIN AND A.A. YUSHKEVICH (1968), *Theorems and Problems in Markov Processes*, Plenum Press, New York.

J. Jacod and A.N. Shiryaev (1987), *Limit Theorems for Stochastic Processes*, Springer-Verlag, Berlin.

I. Karatzas and S. Shreve (1998), *Methods of Mathematical Finance*, Springer-Verlag, New York.

I. Karatzas and H. Wang (2001), *Connections between bounded variation control and Dynkin games*, in Optimal Control and Partial Differential Equations (Volume in honor of A. Bensoussan), J.L. Menaldi, E. Rofman, and A. Sulem, eds., IOS Press, Amsterdam, pp. 363–373.

H.W. Kuhn (1953), *Extensive games and the problem of information*, in Contributions to the Theory of Games, Ann. Math. Stud. 28, H.W. Kuhn and A.W. Tucker, eds., Princeton University Press, Princeton, NJ.

J.P. Lepeltier and M.A. Maingueneau (1984), *Le jeu de Dynkin en théorie générale sans l'hypothèse de Mokobodsky*, Stochastics, 13, pp. 25–44.

J.-F. Mertens, S. Sorin, and S. Zamir (1994), *Repeated Games, Part A*, Core Discussion Papers 9420, Université Catholique de Louvain.

H. Morimoto (1984), *Dynkin games and martingale methods*, Stochastics, 13, pp. 213–228.

J. Neveu (1975), *Discrete Parameter Martingales*, North Holland, Amsterdam.

M. Sion (1958), *On general minimax theorems*, Pacific J. Math., 8, pp. 171–176.

L. Stettner (1982), *Zero-sum Markov games with stopping and impulsive strategies*, Appl. Math. Optim., 9, pp. 1–24.