GENERALIZING THE BLACK-SCHOLES FORMULA TO MULTIVARIATE CONTINGENT CLAIMS

RENÉ CARMONA AND VALDO DURRLEMAN

ABSTRACT. This paper provides approximate formulas that generalize the Black-Scholes formula in all dimensions. Pricing and hedging of multivariate contingent claims are achieved by computing lower and upper bounds. These bounds are given in closed form in the same spirit as the classical one-dimensional Black-Scholes formula. Lower bounds perform remarkably well. Like in the one-dimensional case, Greeks are also available in closed form. We discuss an extension to basket options with barrier.

1. INTRODUCTION

This paper provides approximate formulas that generalize the Black-Scholes formula in all dimensions. The classical Black-Scholes formula gives in closed form the price of a call or a put option on a single stock when the latter is modelled as a geometric Brownian motion. The use of the Black-Scholes formula has spread to fixed income markets to price caps and floors in Libor models or swaptions in Swap models when volatilities are deterministic.

Many options however have multivariate payoffs. Although the mathematical theory does not present any particular difficulties, actual computations of prices and hedges cannot be done in closed form any more. Financial practitioners have to resort to numerical integration, simulations, or approximations. In high dimensions, numerical integration and simulation methods may be too slow for practical purposes. Many areas of computational finance require robust and accurate algorithms to price these options.

Moreover, the classical Black-Scholes formula is still extremely popular despite the fact that the assumption that volatility is deterministic is not satisfied in reality. One can therefore be genuinely interested in obtaining multivariate equivalents of this one-dimensional formula.

In this paper we give approximate formulas that are fast, easy to implement and yet very accurate. These formulas are based on rigorous lower and upper bounds. These are derived under two assumptions. First, we restrict ourselves to a special class of multivariate payoffs. Throughout payoffs are of the European type (options can only be exercised at maturity) and when exercised these options pay a *linear combination* of asset prices. This wide class includes basket options (i.e., options on a basket of stocks), spread options (i.e., options on the difference between two stocks or indices) and more generally rainbow options but also discrete-time average Asian options and also combination of those like Asian spread options (i.e., options on the difference between time averages of two stocks or indices.) Second, we work in the so-called multidimensional Black-Scholes model: assets follow a

Date: August 18, 2005.

multidimensional geometric Brownian motion dynamics. In other words, all volatilities and correlations are constants. As usual, to extend the results to deterministic time dependent volatilities and/or correlations one just has to replace variances and covariances by their means over the option life.

To continue the discussion, let us fix some notations. In a multidimensional Black-Scholes model with n stocks $S_1, \ldots, S_n$, risk neutral dynamics are given by

$$\frac{dS_i(t)}{S_i(t)} = (r - q_i)dt + \sum_{j=1}^{n} \sigma_{ij} dB_j(t), \tag{1}$$

with some initial values $S_1(0), \ldots, S_n(0)$. $B_1, \ldots, B_n$ are independent standard Brownian motions. r is the short rate of interest and q_i is dividend yield on stock i. Correlations among different stocks are captured through the matrix (σ_{ij}). Given a vector of weights $(w_i)_{i=1,\ldots,n}$, we are interested, for instance, in valuing the following basket option struck at K whose payoff at maturity T is

$$\left(\sum_{i=1}^{n} w_i S_i(T) - K \right)^+,$$

as usual, $x^+ = \max\{0, x\}$. Risk neutral valuation gives the price at time 0 as the following expectation under risk neutral measure $\mathbb{P}$

$$p = e^{-rT} \mathbb{E} \left\{ \left(\sum_{i=1}^{n} w_i S_i(T) - K \right)^+ \right\}. \tag{2}$$

Deriving approximate formulas in closed form for such options with multivariate payoffs has already been tackled in the financial literature. For example, Jarrow and Rudd in [6] provide a general method based on Edgeworth (sometimes also called Charlier) expansions. Their idea is to replace the integration over the multidimensional log-normal distribution by an integration over another distribution with the same moments of low order and such that this last integration can be done in closed form. In the case where the new distribution is the Gaussian distribution, this approximation is often called the Bachelier approximation since it gives rise to formulas like those derived by Bachelier.

Another take on this problem (introduced by Vorst in [13] and by Ruttiens in [12]) is to replace arithmetic averages by their corresponding geometric averages. The latter have the nice property of being log-normally distributed; they therefore lead to formula like the Black-Scholes formula. See, for example, [9] pp. 218-25 for a presentation of these results. Their method assumes however that the weights $(w_i)_{i=1,\ldots,n}$ are all positive. Our method does not require such an assumption and will prove to be more accurate.

The trick of introducing the geometric average was later improved by Curran in [1]. By conditioning on the geometric average, he is able to split the price into two parts, one given in closed form and the second being approximated. Conditioning was also used by Rogers and Shi in [11] to compute explicit a lower bound on Asian option prices. They used the convexity of the function $x \mapsto x^+$ and Jensen's inequality to get their lower bound. More recently, these ideas were used and again improved in [3].

Let us close this review of existing results in the area by mentioning a closely related problem. It is when the multivariate distribution of the assets' terminal values is not specified. It is only constrained to be such that it gives back some known prices for other options (typically options on the single assets of the basket). The problem then consists in finding lower and upper bounds on basket option prices under these constraints. It can viewed as a semi-definite program as in [2].

There are two difficulties in computing (2): the lack of tractability of the multivariate log-normal distribution on the one hand and the non linearity of the function $x \mapsto x^+$ on the other. Whereas [6], [13], and [12] circumvent the first difficulty, our approach relies on finding optimal one-dimensional approximations thanks to properties of the function $x \mapsto x^+$. In one dimension, computations can then be carried out explicitly.

Approximations in closed form are given in Proposition 4 and 6 below. Various price sensitivities, the so-called Greeks, are given in Proposition 9, 10, 11 and 12. Section 6 shows actual numerical results. We close the paper with an extension to multivariate barrier options.

2. TWO OPTIMIZATION PROBLEMS

The dynamics giving the asset prices evolutions in (1) have the following well-known solutions

$$S_i(T) = S_i(0) \exp\left[\left(r - q_i - \frac{1}{2}\sum_{j=1}^{n}(\sigma_{ij})^2\right)T + \sum_{j=1}^{n}\sigma_{ij}B_j(T)\right]. \tag{3}$$

In order to work in a quite general framework, which will allow us to deal not only with basket options but with any other option paying a linear combination of these terminal prices, we let X be the random variable

$$X = \sum_{i=0}^{n} \varepsilon_i x_i e^{G_i\sqrt{T} - \text{Var}(G_i)T/2}.$$

$(G_i)_{i=0,\dots,n}$ is a mean zero Gaussian vector of size $n+1$ and covariance matrix Σ, $\varepsilon_i = \pm 1$, and $x_i > 0$ for all $i = 0, \dots, n$.

Our goal is to compute $\mathbb{E}\{X^+\}$. In the case of basket options, (see (2) and (3)) this notation just means $\varepsilon_i = \text{sgn}(w_i)$ and $x_i = |w_i|S_i(0)e^{-q_iT}$ for $i = 1, \dots, n$, $\Sigma_{ij} = (\sigma\sigma')_{ij}$ for $i, j = 1, \dots, n$. Here and throughout the rest of the paper, $'$ denotes transpose. The strike price K is also incorporated into X as 'stock 0', with zero volatility: $\varepsilon_0 = -1$, $x_0 = Ke^{-rT}$ and $\Sigma_{i0} = \Sigma_{0j} = 0$ for $i, j = 0, \dots, n$.

Without loss of generality, we suppose that not all of the ε_i have the same sign. If it were the case, $\mathbb{E}\{X^+\} = \mathbb{E}\{X\}^+$ and it does not present any difficulty. Note also that Σ is symmetric positive semi-definite but not necessarily definite. Before we explain our approximation method we need the following definition and proposition.

Definition 1. *For every $i, j, k = 0, \dots, n$, we let*

$$\Sigma_{ij}^k = \Sigma_{ij} - \Sigma_{ik} - \Sigma_{kj} + \Sigma_{kk}$$

and

$$\sigma_i = \sqrt{\Sigma_{ii}} \qquad \sigma_i^k = \sqrt{\Sigma_{ii}^k}. \tag{4}$$

The next proposition says that we have some freedom in choosing the covariance structure Σ when we compute $\mathbb{E}\{X^+\}$.

Proposition 1. *For every $k = 0, \dots, n$, let $(G_i^k)_{i=0,\dots,n}$ be a mean zero Gaussian vector with covariance Σ^k. Then,*

$$\mathbb{E}\{X^+\} = \mathbb{E}\left\{\left(\sum_{i=0}^n \varepsilon_i x_i e^{G_i^k\sqrt{T} - \mathrm{Var}\left(G_i^k\right)T/2}\right)^+\right\}. \tag{5}$$

Proof. It is an easy consequence of Girsanov's transform. Indeed,

$$\begin{aligned}\mathbb{E}\{X^+\} &= \mathbb{E}\left\{e^{G_k\sqrt{T} - \mathrm{Var}(G_k)T/2}\left(\sum_{i=0}^n \varepsilon_i x_i e^{(G_i - G_k)\sqrt{T} - (\mathrm{Var}(G_i) - \mathrm{Var}(G_k))T/2}\right)^+\right\}\\ &= \mathbb{E}_{\mathbb{Q}^k}\left\{\left(\sum_{i=0}^n \varepsilon_i x_i e^{(G_i - G_k)\sqrt{T} - (\mathrm{Var}(G_i) - \mathrm{Var}(G_k))T/2}\right)^+\right\},\end{aligned}$$

where probability measure $\mathbb{Q}^k$ is defined by its Radon-Nikodým derivative

$$\frac{d\mathbb{Q}^k}{d\mathbb{P}} = e^{G_k\sqrt{T} - \mathrm{Var}(G_k)T/2}.$$

Under $\mathbb{Q}^k$, $(G_i - G_k)_{0\le i\le n}$ is again a Gaussian vector. Its covariance matrix (which remains the same under both $\mathbb{P}$ and $\mathbb{Q}^k$) is Σ^k. ∎

Without loss of generality, we will also assume that for every $k = 0, \dots, n$, $\Sigma^k \neq 0$. Indeed if such were the case, Proposition 1 above would give us the price without any further computation.

The following simple observation is the key to the rest of the paper.

Proposition 2. *For any integrable random variable X,*

$$\sup_{0\le Y\le 1} \mathbb{E}\{XY\} = \mathbb{E}\{X^+\} = \inf_{\substack{X = Z_1 - Z_2\\ Z_1\ge 0\\ Z_2\ge 0}} \mathbb{E}\{Z_1\}, \tag{6}$$

where Y, Z_1, and Z_2 are random variables.

Proof. On the left-hand side, letting $0 \le Y \le 1$,

$$\mathbb{E}\{XY\} = \mathbb{E}\{X^+Y\} - \mathbb{E}\{X^-Y\} \le \mathbb{E}\{X^+\}$$

and taking $Y = \mathbf{1}_{\{X\ge 0\}}$ shows that the supremum is actually attained. On the right-hand side, it is well known that if $X = Z_1 - Z_2$ with both Z_1 and Z_2 non negative, then $Z_1 \ge X^+$. The decomposition $X = X^+ - X^-$ shows that the infimum is actually attained. ∎

These two optimization problems are dual of each other in the sense of linear programming.

3. Lower bound

Our lower bound is obtained by restricting the set over which the supremum in (6) is computed. The resulting problem will be solved exactly. We choose Y of the form $\mathbf{1}_{\{u\cdot G\le d\}}$ where $u \in \mathbb{R}^{n+1}$ and $d \in \mathbb{R}$ are arbitrary and $\cdot$ denotes the usual inner product of $\mathbb{R}^{n+1}$. Let us let

$$p_* = \sup_{u,d} \mathbb{E}\left\{X\mathbf{1}_{\{u\cdot G\le d\}}\right\}. \tag{7}$$

The next propositions give further information on p_*. First, we need the following definition.

Definition 2. *Let D be the $(n+1)\times(n+1)$ diagonal matrix whose ith diagonal element is $1/\sigma_i$ if $\sigma_i \neq 0$ and 0 otherwise. Define C to be*

$$C = D\Sigma D.$$

C is also a symmetric positive semi-definite matrix and we denote by $\sqrt{C}$ a square root of it (i.e., $C = \sqrt{C}\sqrt{C}'$.)

C is just the usual correlation matrix with the convention that the correlation is set to 0 in case one of the random variables has zero variance.

Proposition 3.

$$p_* = \sup_{d\in\mathbb{R}} \sup_{u\cdot\Sigma u=1} \sum_{i=0}^{n} \varepsilon_i x_i \Phi\left(d + (\Sigma u)_i \sqrt{T}\right) = \sup_{d\in\mathbb{R}} \sup_{\|v\|=1} \sum_{i=0}^{n} \varepsilon_i x_i \Phi\left(d + \sigma_i(\sqrt{C}v)_i \sqrt{T}\right). \tag{8}$$

Here and throughout the paper, we use the notation $\varphi(x)$ and $\Phi(x)$ for the density and the cumulative distribution function of the standard Gaussian distribution, i.e.,

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad \text{and} \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du.$$

Proof. By conditioning,

$$\begin{aligned}
p_* &= \sup_{d\in\mathbb{R}} \sup_{u\in\mathbb{R}^{n+1}} \mathbb{E}\left\{\mathbb{E}\left\{X|u\cdot G\right\} \mathbf{1}_{\{u\cdot G\le d\}}\right\} \\
&= \sup_{d\in\mathbb{R}} \sup_{u\in\mathbb{R}^{n+1}} \sum_{i=0}^{n} \varepsilon_i x_i \mathbb{E}\left\{ e^{\frac{\mathrm{Cov}(G_i, u\cdot G)}{u\cdot\Sigma u} u\cdot G\sqrt{T} - \frac{\mathrm{Cov}(G_i, u\cdot G)^2}{u\cdot \Sigma u} T/2} \mathbf{1}_{\{u\cdot G\le d\}} \right\} \\
&= \sup_{d\in\mathbb{R}} \sup_{u\cdot\Sigma u=1} \sum_{i=0}^{n} \varepsilon_i x_i \mathbb{E}\left\{ e^{\mathrm{Cov}(G_i, u\cdot G) u\cdot G\sqrt{T} - \mathrm{Cov}(G_i, u\cdot G)^2 T/2} \mathbf{1}_{\{u\cdot G\le d\}} \right\} \\
&= \sup_{d\in\mathbb{R}} \sup_{u\cdot\Sigma u=1} \sum_{i=0}^{n} \varepsilon_i x_i \Phi\left(d + (\Sigma u)_i \sqrt{T}\right).
\end{aligned}$$

Define D^{-1} to be the $(n+1)\times(n+1)$ diagonal matrix whose ith diagonal element is σ_i. We easily check that $\Sigma = D^{-1}\sqrt{C}\sqrt{C}'D^{-1}$. Therefore taking $v = D^{-1}\sqrt{C}'u$ gives the second equality of the proposition. ∎

To actually compute the supremum, it is interesting to look at the Lagrangian $\mathcal{L}$ of the second problem in (8)

$$\mathcal{L}(v,d) = \sum_{i=0}^{n} \varepsilon_i x_i \Phi\left(d + \sigma_i(\sqrt{C}v)_i\sqrt{T}\right) - \frac{\mu}{2}\left(\|v\|^2 - 1\right).$$

μ is the Lagrange multiplier associated with the constraint $\|v\| = 1$.

Proposition 4.

$$p_* = \sum_{i=0}^{n} \varepsilon_i x_i \Phi\left(d^* + \sigma_i(\sqrt{C}v^*)_i\sqrt{T}\right) \tag{9}$$

where d^* and v^* satisfy the following first order conditions

$$\sum_{i=0}^{n} \varepsilon_i x_i \sigma_i \sqrt{C}_{ij} \varphi\left(d^* + \sigma_i(\sqrt{C}v^*)_i\sqrt{T}\right)\sqrt{T} - \mu v_j^* = 0 \quad \textit{for } j = 0, \ldots, n \tag{10}$$

$$\sum_{i=0}^{n} \varepsilon_i x_i \varphi\left(d^* + \sigma_i(\sqrt{C}v^*)_i\sqrt{T}\right) = 0 \tag{11}$$

$$\|v^*\| = 1. \tag{12}$$

Note that (9) is as close to the classical Black-Scholes formula as one could hope for.

We now give a necessary condition for d^* to be finite. It is interesting when it comes to numerical computations but it also ensures us that lower bounds are not trivial. We need to make a non-degeneracy assumption. Recall that the matrix C was introduced in Definition 2. Through its definition, C may have columns and rows of zeros. We are now assuming that the square matrix $\tilde{C}$ obtained by removing these rows and columns of zeros is non-singular. $\tilde{C}$ is well defined because we assumed that none of the Σ^k (and therefore Σ) were actually the zero matrix.

Condition 1.

$$\det(\tilde{C}) \neq 0$$

Proposition 5. *Under Condition 1,*

$$p_* > \mathbb{E}\{X\}^+, \quad \textit{or equivalently} \quad |d^*| < +\infty.$$

Proof. Assume for instance that $\mathbb{E}\{X\} \geq 0$. We want to show that $p_* > \mathbb{E}\{X\}$. Let us let $f_v(d) = \sum_{i=0}^n \varepsilon_i x_i \Phi\left(d + \sigma_i(\sqrt{C}v)_i\sqrt{T}\right)$. First note that for any v, $\lim_{d\to+\infty} f_v(d) = \mathbb{E}\{X\}$. We are going to show the claim by showing that there exists a unit vector v such that $f_v'(d) < 0$ when d is near $+\infty$. Under Condition 1, Range$(\sqrt{C}) = \bigoplus_{i=0}^n \sigma_i\mathbb{R} \neq \{0\}$ and we can pick a unit vector v such that $\sigma_i > 0 \Rightarrow \varepsilon_i(\sqrt{C}v)_i > 0$. For such a v, write f_v' as

$$f_v'(d) = \varphi(d)\left\{\sum_{i:\varepsilon_i=+1} x_i e^{-d\sigma_i(\sqrt{C}v)_i\sqrt{T} - \sigma_i^2(\sqrt{C}v)_i^2 T/2} - \sum_{i:\varepsilon_i=-1} x_i e^{-d\sigma_i(\sqrt{C}v)_i\sqrt{T} - \sigma_i^2(\sqrt{C}v)_i^2 T/2}\right\}.$$

By denoting $\underline{\sigma} = \min_{i:\varepsilon_i=+1} \sigma_i(\sqrt{C}v)_i \geq 0$ and $\overline{\sigma} = \max_{i:\varepsilon_i=-1} \sigma_i(\sqrt{C}v)_i \leq 0$, we get the following bound, valid for $d \geq 0$:

$$f_v'(d) \leq \varphi(d)\left\{\left(\sum_{i:\varepsilon_i=+1} x_i\right) e^{-d\underline{\sigma}\sqrt{T} - \underline{\sigma}^2 T/2} - \left(\sum_{i:\varepsilon_i=-1} x_i\right) e^{-d\overline{\sigma}\sqrt{T} - \frac{1}{2}\overline{\sigma}^2 T/2}\right\}.$$

Without loss of generality, we can assume that $\underline{\sigma}$ and $\overline{\sigma}$ are not simultaneously zero and the above upper bound is strictly negative for d large enough. The case where $\mathbb{E}\{X\} \leq 0$ is treated analogously by showing that $f_v' > 0$ around $-\infty$. ∎

Note that in case assets are perfectly correlated, it may happen that the true price is actually $\mathbb{E}\{X\}^+$. A condition such as Condition 1 is therefore necessary in the above statement.

4. UPPER BOUND

Our upper bound is obtained by restricting the set over which the infimum in (6) is computed. For every $k = 0, \dots, n$, let $\mathcal{E}_k = \{i : \sigma_i^k \neq 0\}$, which is never empty. Let us also let $\tilde{x}_k = |\sum_{i \notin \mathcal{E}_k} \varepsilon_i x_i|$ and $\tilde{\varepsilon}_k = \mathrm{sgn}(\sum_{i \notin \mathcal{E}_k} \varepsilon_i x_i)$. Without loss of generality, we can assume $\tilde{x}_k > 0$. Then, choose reals $(\lambda_i^k)_{i \in \mathcal{E}_k}$ such that $\sum_{i \in \mathcal{E}_k} \lambda_i^k = -\tilde{\varepsilon}_k$ and rewrite X as

$$\begin{aligned} X &= \sum_{i \in \mathcal{E}_k} \varepsilon_i x_i e^{G_i \sqrt{T} - \mathrm{Var}(G_i)T/2} - \lambda_i^k \tilde{x}_k e^{G_k \sqrt{T} - \mathrm{Var}(G_k)T/2} \\ &= \sum_{i \in \mathcal{E}_k} \left(\varepsilon_i x_i e^{G_i \sqrt{T} - \mathrm{Var}(G_i)T/2} - \lambda_i^k \tilde{x}_k e^{G_k \sqrt{T} - \mathrm{Var}(G_k)T/2} \right)^+ \\ &\quad - \sum_{i \in \mathcal{E}_k} \left(\varepsilon_i x_i e^{G_i \sqrt{T} - \mathrm{Var}(G_i)T/2} - \lambda_i^k \tilde{x}_k e^{G_k \sqrt{T} - \mathrm{Var}(G_k)T/2} \right)^- . \end{aligned}$$

The family of random variables Z_1 that we choose consists of those of the form

$$\sum_{i \in \mathcal{E}_k} \left(\varepsilon_i x_i e^{G_i \sqrt{T} - \mathrm{Var}(G_i)T/2} - \lambda_i^k \tilde{x}_k e^{G_k \sqrt{T} - \mathrm{Var}(G_k)T/2} \right)^+$$

where $k = 0, \dots, n$, $\sum_{i \in \mathcal{E}_k} \lambda_i^k = -\tilde{\varepsilon}_k$ and $\lambda_i^k \varepsilon_i > 0$ for all $i \in \mathcal{E}_k$. Because all the ε_i do not have the same sign, the set of such λ^k is nonempty for each k.

Proposition 6.

$$p^* = \min_{0 \le k \le n} \left\{ \sum_{i=0}^{n} \varepsilon_i x_i \Phi \left(d^k + \varepsilon_i \sigma_i^k \sqrt{T} \right) \right\} \tag{13}$$

where σ_i^k *is given in* (4) *and* d^k *is the unique solution of*

$$\sum_{i=0}^{n} \varepsilon_i x_i \varphi \left(d^k + \varepsilon_i \sigma_i^k \sqrt{T} \right) = 0.$$

Again, note that (13) is as close to the classical Black-Scholes formula as one could hope for.

Proof.

$$\begin{aligned} p^* &= \min_{0 \le k \le n} \inf_{\sum_{i \in \mathcal{E}_k} \lambda_i^k = -\tilde{\varepsilon}_k} \mathbb{E} \left\{ \sum_{i \in \mathcal{E}_k} \left(\varepsilon_i x_i e^{G_i \sqrt{T} - \mathrm{Var}(G_i)T/2} - \lambda_i^k \tilde{x}_k e^{G_k \sqrt{T} - \mathrm{Var}(G_k)T/2} \right)^+ \right\} \\ &= \min_{0 \le k \le n} \inf_{\sum_{i \in \mathcal{E}_k} \lambda_i^k = -\tilde{\varepsilon}_k} \sum_{i \in \mathcal{E}_k} \varepsilon_i x_i \Phi \left(\frac{\varepsilon_i}{\sigma_i^k \sqrt{T}} \ln \left(\frac{\varepsilon_i x_i}{\lambda_i^k \tilde{x}_k} \right) + \frac{\varepsilon_i \sigma_i^k \sqrt{T}}{2} \right) \\ &\qquad - \lambda_i^k \tilde{x}_k \Phi \left(\frac{\varepsilon_i}{\sigma_i^k \sqrt{T}} \ln \left(\frac{\varepsilon_i x_i}{\lambda_i^k \tilde{x}_k} \right) - \frac{\varepsilon_i \sigma_i^k \sqrt{T}}{2} \right) . \end{aligned}$$

Forming the Lagrangian $\mathcal{L}^k$

$$\begin{aligned}\mathcal{L}^k(\lambda^k) \quad = \quad & \sum_{i\in\mathcal{E}_k} \varepsilon_i x_i \Phi\left(\frac{\varepsilon_i}{\sigma_i^k\sqrt{T}}\ln\left(\frac{\varepsilon_i x_i}{\lambda_i^k \tilde{x}_k}\right)+\frac{\varepsilon_i\sigma_i^k\sqrt{T}}{2}\right) \\ & -\lambda_i^k\tilde{x}_k\Phi\left(\frac{\varepsilon_i}{\sigma_i^k\sqrt{T}}\ln\left(\frac{\varepsilon_i x_i}{\lambda_i^k \tilde{x}_k}\right)-\frac{\varepsilon_i\sigma_i^k\sqrt{T}}{2}\right)-\mu\left(\sum_{i\in\mathcal{E}_k}\lambda_i^k+\tilde{\varepsilon}_k\right),\end{aligned}$$

we find the first order conditions

$$\frac{\partial\mathcal{L}^k}{\partial\lambda_i^k} = -\tilde{x}_k\Phi\left(\frac{\varepsilon_i}{\sigma_i^k\sqrt{T}}\ln\left(\frac{\varepsilon_i x_i}{\lambda_i^k \tilde{x}_k}\right)-\frac{\varepsilon_i\sigma_i^k\sqrt{T}}{2}\right)-\mu = 0,$$

from which we deduce that the arguments of Φ must all equal each other. This leads to the following system:

$$\begin{aligned}\frac{\varepsilon_i}{\sigma_i^k\sqrt{T}}\ln\left(\frac{\varepsilon_i x_i}{\lambda_i^k \tilde{x}_k}\right)-\frac{\varepsilon_i\sigma_i^k\sqrt{T}}{2} \quad &= \quad \frac{\varepsilon_j}{\sigma_j^k\sqrt{T}}\ln\left(\frac{\varepsilon_j x_j}{\lambda_j^k \tilde{x}_k}\right)-\frac{\varepsilon_j\sigma_j^k\sqrt{T}}{2} = d^k \quad \text{for } i,j\in\mathcal{E}_k \\ \sum_{i\in\mathcal{E}_k}\lambda_i^k \quad &= \quad -\tilde{\varepsilon}_k \\ \lambda_i^k\varepsilon_i \quad &> \quad 0 \quad \text{for } i\in\mathcal{E}_k.\end{aligned}$$

Solving for the λ_i^k's yields

$$\sum_{i=0}^{n}\varepsilon_i x_i \exp\left(-\varepsilon_i\sigma_i^k\sqrt{T}d^k-(\sigma_i^k)^2T/2\right)=0.$$

It is a decreasing function of d^k. Moreover, since not all the ε_i have the same sign, its limits at $\pm\infty$ are $\gtrless 0$. ∎

Cases of equality. It is easily seen that when $n = 1$, lower and upper bounds both reduce to the Black-Scholes formula and therefore give the true value. Let us stress that $n = 1$ not only contains the classical call and put options but also the exchange option of Margrabe [8].

The following proposition confirms this property and gives other cases where lower and upper bounds are in fact equal to the true value. These correspond to cases when stocks are perfectly correlated or uncorrelated depending on the signs ε_i in the payoff.

Proposition 7. *If for all* $i, j = 0, \ldots, n$,

$$\Sigma_{ij} = \varepsilon_i\varepsilon_j\sigma_i\sigma_j,$$

then

$$p_* = p^*.$$

Proof. Exactly as in Proposition 1 we have the freedom to choose among the different covariance matrices Σ^k. Therefore, for any k

$$p_* = \sup_{d\in\mathbb{R}}\sup_{u\cdot\Sigma^k u=1}\sum_{i=0}^{n}\varepsilon_i x_i\Phi\left(d+(\Sigma^k u)_i\sqrt{T}\right).$$

Also, for any k

$$\sup_{u\cdot\Sigma^k u=1}\sum_{i=0}^{n}\varepsilon_i x_i\Phi\left(d^k+(\Sigma^k u)_i\sqrt{T}\right)\le p_*\le p^*\le\sum_{i=0}^{n}\varepsilon_i x_i\Phi\left(d^k+\varepsilon_i\sigma_i^k\sqrt{T}\right). \tag{14}$$

Let us choose k such that $\sigma_k=\min_{0\le i\le n}\sigma_i$. Note that under the hypothesis, $\Sigma^k_{ij}=(\varepsilon_i\sigma_i-\varepsilon_k\sigma_k)(\varepsilon_j\sigma_j-\varepsilon_k\sigma_k)$. Notice further that since all the ε_i do not have the same sign, we can define the following vector u:

$$u_i=\frac{\mathrm{sgn}(\varepsilon_i\sigma_i-\varepsilon_k\sigma_k)}{\sum_{j=0}^{n}|\varepsilon_j\sigma_j-\varepsilon_k\sigma_k|}.$$

One trivially checks that $u\cdot\Sigma^k u=1$ and that $(\Sigma^k u)_i=\varepsilon_i\sigma_i-\varepsilon_k\sigma_k$. Because of the way we chose k,

$$(\Sigma^k u)_i=\varepsilon_i\sigma_i-\varepsilon_k\sigma_k=\varepsilon_i|\varepsilon_i\sigma_i-\varepsilon_k\sigma_k|=\varepsilon_i\sigma_i^k.$$

This proves that the inequalities in (14) are in fact equalities. ∎

Bound on the gap. Although an estimate on the gap is readily available as soon as lower and upper bounds are computed, it is interesting to have an *a priori* bound on the gap p^*-p_*.

Proposition 8.

$$0\le p^*-p_*\le\sqrt{\frac{2T}{\pi}}\min_{0\le k\le n}\left\{\sum_{i=0}^{n}x_i\sigma_i^k\right\}. \tag{15}$$

Proof.

$$\begin{aligned}p^*-p_* &\le \min_{0\le k\le n}\sum_{i=0}^{n}\varepsilon_i x_i\Phi\left(d^k+\varepsilon_i\sigma_i^k\sqrt{T}\right)-\max_{0\le k\le n}\sum_{i=0}^{n}\varepsilon_i x_i\Phi\left(d^k+(\Sigma^k u^*)_i\sqrt{T}\right)\\ &\le \min_{0\le k\le n}\sum_{i=0}^{n}\varepsilon_i x_i\left\{\Phi\left(d^k+\varepsilon_i\sigma_i^k\sqrt{T}\right)-\Phi\left(d^k+(\Sigma^k u^*)_i\sqrt{T}\right)\right\}.\end{aligned}$$

By Cauchy-Schwarz inequality,

$$\left|(\Sigma^k u^*)_i\right|=\left|\sum_{j=0}^{n}\sum_{l=0}^{n}\Sigma^k_{lj}u_j^*\delta_{il}\right|\le\sqrt{\sum_{j=0}^{n}\sum_{l=0}^{n}\Sigma^k_{lj}u_j^*u_l^*}\sqrt{\sum_{j=0}^{n}\sum_{l=0}^{n}\Sigma^k_{lj}\delta_{il}\delta_{ij}}=\sigma_i^k$$

where $\delta_{ij}=1$ if $i=j$ and 0 otherwise. It follows that

$$p^*-p_*\le\min_{0\le k\le n}\sum_{i=0}^{n}x_i\|\varphi\|_\infty\left(\sigma_i^k-(\Sigma^k u^*)_i\right)\sqrt{T}\le 2\sqrt{T}\|\varphi\|_\infty\min_{0\le k\le n}\sum_{i=0}^{n}x_i\sigma_i^k.$$

Since $\|\varphi\|_\infty=\sup_{x\in\mathbb{R}}|\varphi(x)|=\frac{1}{\sqrt{2\pi}}$, it gives the desired upper bound on the gap. ∎

5. Greeks Computations

Lower bound. We now compute partial derivatives with respect to the various parameters: the initial asset prices, their volatilities, their correlation coefficients, and the maturity of the option.

Proposition 9.

$$\frac{\partial p_*}{\partial x_i} = \varepsilon_i \Phi\left(d^* + \sigma_i(\sqrt{C}v^*)_i\sqrt{T}\right) \tag{16}$$

$$\frac{\partial p_*}{\partial \sigma_i} = \varepsilon_i x_i (\sqrt{C}v^*)_i \varphi\left(d^* + \sigma_i(\sqrt{C}v^*)_i\sqrt{T}\right)\sqrt{T} \tag{17}$$

$$\frac{\partial p_*}{\partial \rho_{ij}} = \varepsilon_i\varepsilon_j x_i x_j \sigma_i \sigma_j \frac{\varphi\left(d^* + \sigma_i(\sqrt{C}v^*)_i\sqrt{T}\right)\varphi\left(d^* + \sigma_j(\sqrt{C}v^*)_j\sqrt{T}\right)}{\sum_{k=0}^{n}\varepsilon_k x_k \sigma_k (\sqrt{C}v^*)_k \varphi\left(d^* + \sigma_k(\sqrt{C}v^*)_k\sqrt{T}\right)}\sqrt{T} \tag{18}$$

$$\frac{\partial p_*}{\partial T} = \frac{1}{2\sqrt{T}}\sum_{k=0}^{n}\varepsilon_k x_k \sigma_k (\sqrt{C}v^*)_k \varphi\left(d^* + \sigma_k(\sqrt{C}v^*)_k\sqrt{T}\right). \tag{19}$$

Recall that in our general formulation x_i can be $|w_i|S_i(0)e^{-q_iT}$, or Ke^{-rT}.

Proof. First order derivatives are easily computable thanks to the following observation.

$$\frac{dp_*}{dx_i} = \frac{\partial p_*}{\partial x_i} + \frac{\partial p_*}{\partial d}\frac{\partial d}{\partial x_i} + \nabla_v p_* \cdot \frac{\partial v}{\partial x_i} = \frac{\partial p_*}{\partial x_i} + \frac{\mu}{2}\frac{\partial \|v^*\|^2}{\partial x_i} = \frac{\partial p_*}{\partial x_i}$$

because p_* satisfies the first order conditions (10-12) at (v^*, d^*). The same reasoning applies to any first order derivative with respect to any parameter. This leads to the formulas above, except for $\frac{\partial p_*}{\partial \rho_{ij}}$, where an additional computation is needed.

We first make the computation when C is the symmetric square root of C (see Definition 2). We further assume that it is non singular. The final formula will not depend on these choices and that will prove the claim. Under these assumptions

$$\frac{\partial \sqrt{C}_{kl}}{\partial \rho_{ij}} = \frac{1}{2}(\delta_{ik}\sqrt{C}^{-1}_{jl} + \delta_{jk}\sqrt{C}^{-1}_{il})$$

and therefore

$$\frac{\partial p_*}{\partial \rho_{ij}} = \frac{\sqrt{T}}{2}\left(\varepsilon_i x_i \sigma_i \varphi_i (\sqrt{C}^{-1}v^*)_j + \varepsilon_j x_j \sigma_j \varphi_j (\sqrt{C}^{-1}v^*)_i\right)$$

where we used the short-hand notation $\varphi_i = \varphi\left(d^* + \sigma_i(\sqrt{C}v^*)_i\sqrt{T}\right)$. On the other hand because of (10),

$$\varepsilon_i x_i \sigma_i \varphi_i = \frac{\mu}{\sqrt{T}}(\sqrt{C}^{-1}v^*)_i$$

and also because of (10) and (12),

$$\mu = \sum_{i=0}^{n}\varepsilon_i \sigma_i x_i (\sqrt{C}v^*)_i \varphi_i \sqrt{T}.$$

This gives the formula of the proposition. When C is singular, the claim is proved by a density argument. Finally, the formula only depends on $\sqrt{C}v^*$, which does not depend on the choice of the square root made in Definition 2. ∎

Second order derivatives are more difficult to obtain since the previous trick is no longer possible. There exist however simple and natural approximations that satisfy the multidimensional Black-Scholes equation.

Proposition 10. *Let*

$$\frac{\partial^2 p_*}{\partial x_i \partial x_j} = \varepsilon_i \varepsilon_j \frac{\varphi\left(d^* + \sigma_i(\sqrt{C}v^*)_i\sqrt{T}\right)\varphi\left(d^* + \sigma_j(\sqrt{C}v^*)_j\sqrt{T}\right)}{\sum_{k=0}^n \varepsilon_k x_k \sigma_k (\sqrt{C}v^*)_k \varphi\left(d^* + \sigma_k(\sqrt{C}v^*)_k\sqrt{T}\right)\sqrt{T}}, \tag{20}$$

then

$$-\frac{\partial p_*}{\partial T} + \frac{1}{2}\sum_{i=0}^n \sum_{j=0}^n \Sigma_{ij} x_i x_j \frac{\partial^2 p_*}{\partial x_i \partial x_j} = 0.$$

Another interesting feature of these formulas is that $\frac{\partial^2 p_*}{\partial x_i \partial x_j}$ and $\frac{\partial p_*}{\partial \rho_{ij}}$ are proportional to each other when $i \neq j$. A property that is shared with the true price.

Note that there are no first order derivative terms in the Black-Scholes equation. It comes from the fact that in our framework, interest rate and dividends are hidden in the x_i's. They do not show up explicitly as long as we are working with the x_i's as main variables.

Proof. It suffices to show that:

$$-\left(\sum_{k=0}^n \varepsilon_k x_k \sigma_k (\sqrt{C}v^*)_k \varphi_k\right)^2 + \sum_{i=0}^n \sum_{j=0}^n \varepsilon_i \varepsilon_j \Sigma_{ij} x_i x_j \varphi_i \varphi_j = 0.$$

Simply note that because of (10),

$$\begin{aligned}\sum_{i=0}^n \sum_{j=0}^n \varepsilon_i \varepsilon_j \Sigma_{ij} x_i x_j \varphi_i \varphi_j &= \sum_{i=0}^n \sum_{j=0}^n \sum_{k=0}^n \varepsilon_i \varepsilon_j \sigma_i \sigma_j \sqrt{C}_{ik} \sqrt{C}_{kj} x_i x_j \varphi_i \varphi_j \\ &= \frac{\mu}{\sqrt{T}} \sum_{j=0}^n \varepsilon_j \sigma_j x_j (\sqrt{C}v^*)_j \varphi_j.\end{aligned}$$

Again because of (10) and (12), we have:

$$\mu = \sum_{i=0}^n \varepsilon_i \sigma_i x_i (\sqrt{C}v^*)_i \varphi_i \sqrt{T}.$$

This completes the proof. ∎

Upper bound. We now turn ourselves to the case of upper bounds. The function $\min$ is only almost everywhere differentiable. Therefore the next two propositions have to be understood in an almost sure sense. k^* denotes the value for which the minimum is achieved in (13).

Proposition 11.

$$\frac{\partial p^*}{\partial x_i} = \varepsilon_i \Phi\left(d^{k^*} + \varepsilon_i \sigma_i^{k^*}\sqrt{T}\right) \tag{21}$$

$$\frac{\partial p^*}{\partial \sigma_i} = x_i \frac{\sigma_i - \rho_{ik^*}\sigma_{k^*}}{\sigma_i^{k^*}} \varphi\left(d^{k^*} + \varepsilon_i \sigma_i^{k^*}\sqrt{T}\right)\sqrt{T} \tag{22}$$

$$\frac{\partial p^*}{\partial \rho_{ij}} = -\delta_{jk^*} x_i \frac{\rho_{ik^*}\sigma_i\sigma_{k^*}}{\sigma_i^{k^*}} \varphi\left(d^{k^*} + \varepsilon_i \sigma_i^{k^*}\sqrt{T}\right)\sqrt{T} \tag{23}$$

$$\frac{\partial p^*}{\partial T} = \frac{1}{2\sqrt{T}} \sum_{l=0}^{n} x_l \sigma_l^{k^*} \varphi\left(d^{k^*} + \varepsilon_l \sigma_l^{k^*}\sqrt{T}\right) \tag{24}$$

with the convention $0/0 = 0$.

Proof. We proceed in the same way as for the lower bound. ∎

As in the case of the lower bound approximation, there are formulas for the Gamma's that satisfy the Black-Scholes equation.

Proposition 12. *Let*

$$\frac{\partial^2 p^*}{\partial x_i \partial x_j} = \begin{cases} \frac{\varphi\left(d^{k^*} + \varepsilon_i \sigma_i^{k^*}\sqrt{T}\right)}{x_i \sigma_i^{k^*}\sqrt{T}} \delta_{ij} & \textit{if } i \in \mathcal{E}_{k^*} \\ 0 \quad \textit{for all } j & \textit{if } i \notin \mathcal{E}_{k^*}, \end{cases} \tag{25}$$

then

$$-\frac{\partial p^*}{\partial T} + \frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n} \Sigma_{ij}^{k^*} x_i x_j \frac{\partial^2 p^*}{\partial x_i \partial x_j} = 0.$$

6. Numerical examples and performance

We implemented the lower and upper bound formulas in C++. The optimization procedures are simple conjugate gradient methods. On a standard PC, lower and upper bounds are given in a fraction of a second.

Basket options. As a first example, we shall consider the case of a basket option. For simplicity, let us suppose that there are n stocks whose initial values are all \$100 and whose volatilities are also all the same, equal to σ. Correlation between any two distinct stocks is ρ. This amounts to the following:

$$\varepsilon = \begin{bmatrix} -1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad x = \begin{bmatrix} K \\ 100/n \\ \vdots \\ 100/n \end{bmatrix} \quad \text{and} \quad \Sigma = \sigma^2 \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \rho & \cdots & \rho \\ 0 & \rho & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \rho \\ 0 & \rho & \cdots & \rho & 1 \end{bmatrix}.$$

Interest rate and dividend are set to zero. The option has maturity 1 year. We present the results in Figure 1 when $n = 50$ and for different volatilities ($\sigma = 10\%, 20\%, 30\%$) and different correlation parameters ($\rho = 30\%, 50\%, 70\%$.) We plot lower and upper bounds against strike K. For the sake of comparison we also plot results of brute force Monte Carlo simulations.

In view of the plots in Figure 1 one can make two comments. The gap between lower and upper bound tends to decrease as correlation increases, which is in total agreement with Proposition 7. On the other hand the gap increases with the volatility, this fact, in turn, could be suspected from Proposition 8.

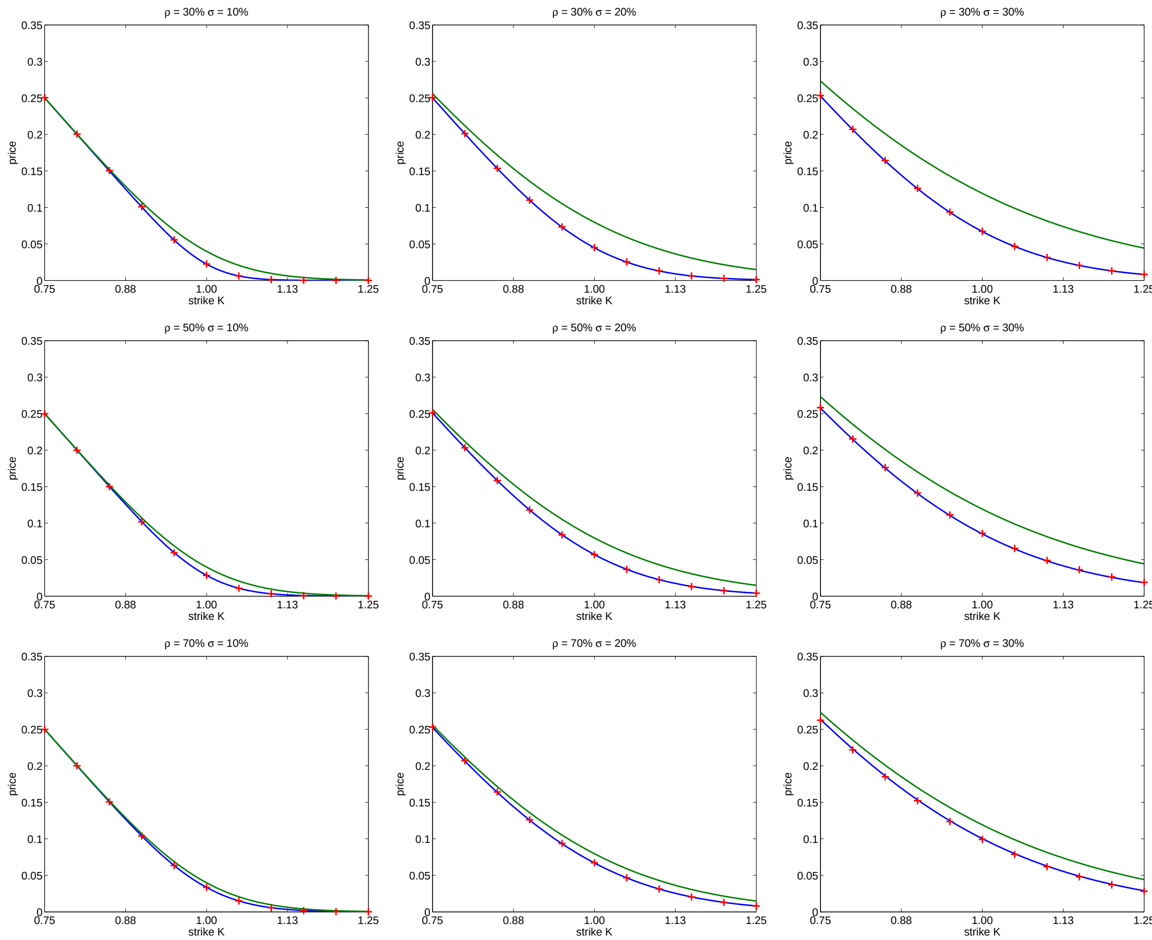

FIGURE 1. Lower and upper bound on the price for a basket option on 50 stocks (each one having a weight of $1/50$) as a function of K. "+" denote Monte Carlo results. Prices and strikes have been divided by 100.

Table 1 gives numerical values for the same experiment. It is striking to see that lower bounds are always within the standard errors of Monte Carlo estimates. On the other hand, upper bounds are really not as good as lower bounds. It is unfortunate that we cannot achieve the same degree of accuracy with upper bounds.

The absolute precision on lower and upper bounds is of the order of 10^{-8}. There are two approximations in the implementation: one comes from the optimization algorithm, the other from the standard Gaussian cumulative distribution function. The optimization procedure uses one-dimensional search with first derivative as explained in [10] pp. 405-8. We set the tolerance parameter to 10^{-8}. On the other hand, for the cumulative distribution function, we use the Marsaglia-Zaman-Marsaglia algorithm as described in [4] p.70. The relative error is of the order of 10^{-15}.

6.1. **Greeks computation for basket options.** In the framework of (2), the different sensitivities are given as follows for $i = 1, \dots, n$:

$$\begin{aligned}
\Delta_i = \frac{\partial p}{\partial S_i(0)} &= |w_i|\, e^{-q_i T}\, \frac{\partial p}{\partial x_i} \\
\kappa = \frac{\partial p}{\partial K} &= e^{-rT}\, \frac{\partial p}{\partial x_0} \\
\text{Vega}_i &= \frac{\partial p}{\partial \sigma_i} \\
\Gamma_{ij} = \frac{\partial^2 p}{\partial S_i(0) \partial S_j(0)} &= |w_i|\,|w_j|\, e^{-(q_i+q_j)T}\, \frac{\partial^2 p}{\partial x_i \partial x_j} \\
\chi_{ij} &= \frac{\partial p}{\partial \rho_{ij}}.
\end{aligned}$$

where p is either p_* or p^*.

These are compared with Monte Carlo estimates in Table 2. Estimates for the Γ's are computed from those for the χ's in the following way. When $i \neq j$,

$$\widehat{\Gamma}_{ij} = \frac{\widehat{\chi}_{ij}}{x_i x_j \sigma_i \sigma_j T},$$

and

$$\widehat{\Gamma}_{ii} = \frac{1}{x_i^2 \sigma_i^2 T} \left(\sigma_i \widehat{\text{Vega}}_i - \sum_{j \neq i} \rho_{ij} \widehat{\chi}_{ij} \right),$$

when $i = j$. Doing so is fully justified in the present Black-Scholes model. These formulas indeed follow from the following relationship between derivatives of the multivariate Gaussian density:

$$\frac{\partial^2}{\partial z_i \partial z_j} \left(\frac{1}{\sqrt{\det(\Sigma)}} e^{-\frac{1}{2} z \cdot \Sigma^{-1} z} \right) = (1 + \delta_{ij}) \frac{\partial}{\partial \Sigma_{ij}} \left(\frac{1}{\sqrt{\det(\Sigma)}} e^{-\frac{1}{2} z \cdot \Sigma^{-1} z} \right),$$

where Σ is of course symmetric positive definite. From this we deduce that when $i \neq j$,

$$\Gamma_{ij} = \frac{1}{x_i x_j} \frac{\partial p}{\partial \Sigma_{ij}} = \frac{1}{x_i x_j \sigma_i \sigma_j T} \frac{\partial p}{\partial \rho_{ij}}$$

		$\sigma = 10\%$			$\sigma = 20\%$			$\sigma = 30\%$		
ρ	K	LB	MC	UB	LB	MC	UB	LB	MC	UB
	90	10.0618	10.0619 (0.0034)	10.7124	10.9906	10.9914 (0.0061)	13.5891	12.5699	12.5760 (0.0086)	17.0129
	95	5.5331	5.5340 (0.0030)	6.8881	7.3049	7.3090 (0.0053)	10.5195	9.3320	9.3338 (0.0077)	14.2936
0.3	100	2.2352	2.2357 (0.0021)	3.9878	4.4687	4.4696 (0.0043)	7.9656	6.6986	6.7020 (0.0067)	11.9235
	105	0.6082	0.6089 (0.0011)	2.0640	2.5072	2.5103 (0.0033)	5.9056	4.6525	4.6589 (0.0057)	9.8817
	110	0.1072	0.1072 (0.0004)	0.9539	1.2904	1.2919 (0.0024)	4.2920	3.1310	3.1343 (0.0047)	8.1410
	90	10.2099	10.2095 (0.0042)	10.7124	11.8128	11.8127 (0.0075)	13.5891	14.0601	14.0585 (0.0107)	17.0129
	95	5.9638	5.9652 (0.0036)	6.8881	8.4043	8.4048 (0.0066)	10.5195	11.0537	11.0570 (0.0098)	14.2936
0.5	100	2.8484	2.8499 (0.0027)	3.9878	5.6932	5.6956 (0.0056)	7.9656	8.5307	8.5379 (0.0088)	11.9235
	105	1.0752	1.0750 (0.0017)	2.0640	3.6721	3.6718 (0.0046)	5.9056	6.4689	6.4674 (0.0078)	9.8817
	110	0.3165	0.3163 (0.0009)	0.9539	2.2582	2.2597 (0.0036)	4.2920	4.8252	4.8266 (0.0068)	8.1410
	90	10.3997	10.4047 (0.0048)	10.7124	12.5681	12.5675 (0.0086)	13.5891	15.3433	15.3413 (0.0125)	17.0129
	95	6.3574	6.3573 (0.0042)	6.8881	9.3298	9.3254 (0.0077)	10.5195	12.4795	12.4832 (0.0116)	14.2936
0.7	100	3.3511	3.3503 (0.0032)	3.9878	6.6962	6.6988 (0.0067)	7.9656	10.0296	10.0361 (0.0106)	11.9235
	105	1.4977	1.4980 (0.0022)	2.0640	4.6502	4.6530 (0.0057)	5.9056	7.9713	7.9749 (0.0096)	9.8817
	110	0.5645	0.5646 (0.0013)	0.9539	3.1289	3.1318 (0.0047)	4.2920	6.2704	6.2650 (0.0086)	8.1410

TABLE 1. Numerical results for a basket of 50 stocks. LB and UB are lower and upper bound values. MC are Monte Carlo estimates with 10,000,000 replications, numbers in parenthesis are the standard errors computed with the Gaussian quantile at 95%.

and when $i = j$,

$$\Gamma_{ii} = \frac{2}{x_i^2}\frac{\partial p}{\partial \Sigma_{ii}} = \frac{1}{x_i^2 \sigma_i T}\left(\frac{\partial p}{\partial \sigma_i} - \sum_{j \neq i} \frac{\rho_{ij}}{\sigma_i}\frac{\partial p}{\partial \rho_{ij}}\right)$$

This trick allows us to estimate the second order derivatives from first order ones. As is well known, these are much easier to obtain. We used the pathwise derivatives estimates (see, for instance [4] pp.386–92).

	price	Δ	Vega	Γ	χ	Θ	κ
Lower Bound	3.3511	0.0103	0.6698	$1.9\ 10^{-5}$	0.0019	1.6746	–0.4832
Upper Bound	3.9878	0.0104	0.7969	$8.0\ 10^{-4}$	0	1.9922	–0.4801
Monte Carlo	3.3508 (0.0101)	0.0103 ($2.1\ 10^{-5}$)	0.6700 (0.0027)	$4.3\ 10^{-6}$ ($1.3\ 10^{-4}$)	0.0019 (0.0004)	1.6746 (0.0053)	-0.4857 (0.0010)

TABLE 2. Greeks for a basket option on 50 stocks, each having initial value $100 and weight $1/50$; $K = 100$, $T = 1$, $\sigma = 10\%$, $\rho = 0.7$. Monte Carlo estimates were computed with 1,000,000 replications and the standard errors (below in parenthesis) with the Gaussian quantile at 95%. By symmetry, the Δ's, vega's, Γ's, and χ's are the same for every stock or pair of stocks.

6.2. **Discrete-time average Asian options.** In the case of Asian option, we compare the lower bound with another often used approximative lower bound for Asian option. The other lower bound is obtained by replacing an arithmetic average by a geometric one (see, for example, [13].) Results are reported in Table 3. Again, we take an option with 1 year to expiry and an initial value for the stock of $100. Interest rate and dividend are set to zero. Averaging is performed over 50 equally spaced dates. Results are given for different stock volatilities ($\sigma = 10\%, 20\%, 30\%$.) The lower bound is uniformly better than the geometric average approximation.

	$\sigma = 10\%$			$\sigma = 20\%$			$\sigma = 30\%$		
K	GA	LB	UB	GA	LB	UB	GA	LB	UB
90	9.9928	10.0754	10.1595	10.7979	11.0867	11.5724	12.1687	12.7492	13.6343
95	5.5089	5.5922	5.8451	7.1954	7.4631	8.1119	9.0419	9.5786	10.5953
100	2.2606	2.3372	2.6964	4.4319	4.6708	5.3880	6.5083	6.9971	8.0699
105	0.6391	0.6886	0.9595	2.5165	2.7164	3.3926	4.5423	4.9778	6.0320
110	0.1204	0.1399	0.2600	1.3181	1.4717	2.0292	3.0784	3.4557	4.4312

TABLE 3. Asian prices using the geometric average approximation (GA), lower bound (LB) and upper bound (UB) approximations.

6.3. **Another example.** In the example of Section 6, the optimization procedure for the lower bound can be somewhat simplified because we can compute v^* explicitly (namely, $v_i^* = 1/\sqrt{n}$ for every i) because of the problem's symmetries.

For the sake of completeness, we give here another example of an option on a weighted sum of five arbitrary stocks whose characteristics are given in Table 4. In this case, we do not know a closed form expression for v^* and we have compute it by performing the optimization.

Price and Greeks are given in Table 5.

Assets	$S(0)$	weights	dividend yields	volatilities	correlations				
1	110	0.1	5%	20%	1	0.5	0.4	0.3	0.1
2	105	0.4	4%	25%		1	0.8	0.2	0.3
3	100	0.1	7%	30%			1	0.5	0.6
4	95	0.2	6%	35%				1	0.5
5	90	0.2	9%	40%					1

TABLE 4. Data for Section 6.3. $r = 3\%$, $K = 100$, and $T = 1$.

7. BASKET OPTIONS WITH BARRIER

To our knowledge, the first paper dealing with barrier options with more than one underlying is [5]. The options they consider are usual barrier options on a single stock except that the barrier event is related to another asset. Later [7] and [14] tackled the same problem when the option is written on more than one asset, in fact, they consider options written on the maximum of several assets. In this subsection, we look at different but related problem where the option is written on a basket and the barrier event is based on a particular stock.

More precisely, we show how to extend the previous results to the case of a basket option with a down-and-out barrier condition on the first stock of the basket. We will only focus on lower bounds since they are much more accurate than upper bounds. More specifically, the option payoff is

$$\left(\sum_{i=1}^{n} w_i S_i(T) - K\right)^+ \mathbf{1}_{\left\{\inf_{t\le T} S_1(t)\ge H\right\}}.$$

We assume, without loss of generality, that $H < S_1(0)$. With the notation of the paper, the price of the barrier option is

$$\mathbb{E}\left\{\left(\sum_{i=0}^{n} \varepsilon_i x_i e^{G_i(T)-\sigma_i^2 T/2} \mathbf{1}_{\left\{\inf_{t\le T} S_1(0) e^{(r-q_1-\sigma_1^2/2)t+G_1(t)}\ge H\right\}}\right)^+\right\},$$

where $\{G(t); t \le T\}$ is $(n+1)$-dimensional Brownian motion starting from 0 with covariance Σ. Although we have the particular case of basket options in mind, it should be stressed that this formulation is valid for any payoff with a linear structure. We simply assumed $\varepsilon_1 = +1$ and $\sigma_1 > 0$. We propose to approximate the option's price and its replicating strategy with an optimal lower bound.

$$p_* = \sup_{u,d} \mathbb{E}\left\{\sum_{i=0}^{n} \varepsilon_i x_i e^{G_i(T)-\sigma_i^2 T/2} \mathbf{1}_{\left\{\inf_{t\le T} S_1(0) e^{(r-q_1-\sigma_1^2/2)t+G_1(t)}\ge H;\ u\cdot G(T)\le d\right\}}\right\}.$$

	Price	Δ	Vega	$\Gamma \times 10^4$					χ			
	7.0931	0.0425	2.1665	1.72	7.02	1.70	3.44	3.34	0.4056	0.1125	0.2520	0.2644
LB	Θ	0.1866	12.9675		28.6	6.94	14.0	13.6		0.5466	1.2246	1.2853
	2.6299	0.0480	3.3914			1.68	3.40	3.30			0.3395	0.3564
	κ	0.0937	4.7147				6.88	6.67				0.7984
	–0.3941	0.0948	4.6577					6.47				
	Price	Δ	Vega	$\Gamma \times 10^4$					χ			
	7.1218 (0.0078)	0.0426 ($3.4\ 10^{-5}$)	2.1836 (0.0049)	1.78 (0.078)	7.08 (0.049)	1.71 (0.022)	3.44 (0.026)	3.25 (0.030)	0.4090 (0.0028)	0.1126 (0.0015)	0.2518 (0.0019)	0.2575 (0.0024)
MC	Θ	0.1868 ($1.4\ 10^{-4}$)	13.0868 (0.0200)		29.3 (0.187)	6.98 (0.060)	13.6 (0.062)	13.1 (0.058)		0.5494 (0.0047)	1.1879 (0.0054)	1.2410 (0.0055)
	2.6779 (0.0039)	0.0480 ($3.7\ 10^{-5}$)	3.3854 (0.0048)			1.70 (0.110)	3.36 (0.035)	3.25 (0.036)			0.3352 (0.0035)	0.3511 (0.0039)
	κ	0.0939 ($7.6\ 10^{-5}$)	4.8182 (0.0101)				7.31 (0.077)	6.63 (0.026)				0.7939 (0.0031)
	−0.3907 (0.0003)	0.0949 ($7.9\ 10^{-5}$)	4.7413 (0.0098)					6.87 (0.075)				
	Price	Δ	Vega	$\Gamma \times 10^4$					χ			
	9.9699	0.0458	4.1700	17.2	0	0	0	0	0	0	0	0
UB	Θ	0.1929	16.0984		58.4	0	0	0		0	0	0
	3.8838	0.0487	3.7142			12.4	0	0			0	0
	κ	0.1020	7.0995				22.5	0				0
	–0.3912	0.1026	6.4849					20.0				

TABLE 5. Greeks obtained from Lower Bound, Upper Bound, and Monte Carlo methods. The parameters are given in Table 4. Monte Carlo estimates were computed with 10,000,000 replications and the standard errors (below in parenthesis) with the Gaussian quantile at 95%.

Using Girsanov's theorem, it rewrites

$$p_* = \sup_{u,d} \sum_{i=0}^{n} \varepsilon_i x_i \mathbb{P}\left\{\inf_{t\le T} G_1(t) + \left(\Sigma_{i1} + r - q_1 - \sigma_1^2/2\right) t \ge \ln\left(\frac{H}{x_1}\right); u \cdot G(T) \le d - (\Sigma u)_i T\right\}.$$

Let us define a new standard Brownian motion $\{W(t); t \le T\}$ independent of $\{G_1(t); t \le T\}$ by

$$u \cdot G(t) = \sqrt{u \cdot \Sigma u - \frac{(\Sigma u)_1^2}{\sigma_1^2}} W(t) + \frac{(\Sigma u)_1}{\sigma_1^2} G_1(t).$$

Like in Proposition 3, we use homogeneity to normalize u by setting $u \cdot \Sigma u = 1$. Suppose first that we restrict ourselves to the half ellipsoid $(\Sigma u)_1 < 0$. Letting $\lambda_i = \Sigma_{i1} + r - q_1 - \sigma_1^2/2$ and

$$Y = \frac{\sigma_1^2}{(\Sigma u)_1}\left(d - (\Sigma u)_i T - \sqrt{1 - \frac{(\Sigma u)_1^2}{\sigma_1^2}} W(T)\right),$$

we get for the above expression:

$$\sup_{d\in\mathbb{R}} \sup_{\substack{u\cdot\Sigma u = 1\\ (\Sigma u)_1 < 0}} \sum_{i=0}^{n} \varepsilon_i x_i \mathbb{P}\left\{\inf_{t\le T} G_1(t) + \lambda_i t \ge \ln\left(\frac{H}{x_1}\right); G_1(T) \ge Y\right\}.$$

In fact, since we assumed that $\varepsilon_1 = +1$, this last expression is equal to p_* even if we restricted the sup over $(\Sigma u)_1 < 0$. It remains to compute these probabilities; we use the following classical result (see, for example, [9] p. 470.)

Lemma 1. *Let B be standard Brownian motion and $X(t) = \sigma B(t) + \lambda t$, then for $y \le 0$,*

$$\mathbb{P}\left\{\inf_{t\le T} X(t) \ge y; X(T) \ge x\right\} = \begin{cases} \Phi\left(\frac{-x+\lambda T}{\sigma\sqrt{T}}\right) - e^{\frac{2\lambda y}{\sigma^2}} \Phi\left(\frac{-x+\lambda T+2y}{\sigma\sqrt{T}}\right) & \text{if } y \le x \\ \Phi\left(\frac{-y+\lambda T}{\sigma\sqrt{T}}\right) - e^{\frac{2\lambda y}{\sigma^2}} \Phi\left(\frac{y+\lambda T}{\sigma\sqrt{T}}\right) & \text{otherwise.} \end{cases}$$

The probabilities above are expressible in terms of Φ_2, the cumulative distribution function of the standard bivariate Gaussian distribution:

$$\Phi_2(x, y, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{x} \int_{-\infty}^{y} \exp\left(-\frac{1}{2}\frac{u^2 - 2\rho uv + v^2}{1-\rho^2}\right) dudv.$$

Recall that Y is independent of G_1, therefore, by first conditioning on Y we get

$$
\begin{aligned}
&\mathbb{P}\left\{\inf_{t\le T} G_1(t)+\lambda_i t \ge \ln\left(\frac{H}{x_1}\right); G_1(T)\ge Y\right\} \\
&= \mathbb{E}\left\{\left[\Phi\left(-\frac{Y}{\sigma_1\sqrt{T}}\right)-\left(\frac{H}{x_1}\right)^{\frac{2\lambda_i}{\sigma_1^2}}\Phi\left(\frac{2\ln(H/x_1)-Y}{\sigma_1\sqrt{T}}\right)\right]\mathbf{1}_{\{\ln(H/x_1)\le Y+\lambda_i T\}}\right\} \\
&\quad +\mathbb{E}\left\{\left[\Phi\left(\frac{\lambda_i T-\ln(H/x_1)}{\sigma_1\sqrt{T}}\right)-\left(\frac{H}{x_1}\right)^{\frac{2\lambda_i}{\sigma_1^2}}\Phi\left(\frac{\lambda_i T+\ln(H/x_1)}{\sigma_1\sqrt{T}}\right)\right]\mathbf{1}_{\{\ln(H/x_1)\ge Y+\lambda_i T\}}\right\} \\
&= \Phi_2\left(d-(\Sigma u)_i\sqrt{T},-a_i,-\gamma\right)-\left(\frac{H}{x_1}\right)^{\frac{2\lambda_i}{\sigma_1^2}}\Phi_2\left(d-(\Sigma u)_i\sqrt{T}-2\ln(H/x_1),-a_i,-\gamma\right) \\
&\quad +\left[\Phi\left(\frac{\lambda_i T-\ln(H/x_1)}{\sigma_1\sqrt{T}}\right)-\left(\frac{H}{x_1}\right)^{\frac{2\lambda_i}{\sigma_1^2}}\Phi\left(\frac{\lambda_i T+\ln(H/x_1)}{\sigma_1\sqrt{T}}\right)\right]\Phi(a_i)
\end{aligned}
$$

with

$$
a_i=\frac{1}{\gamma}\left(d-(\Sigma u)_i\sqrt{T}+\frac{(\Sigma u)_1}{\sigma_1^2\sqrt{T}}(\lambda_i T-\ln(H/x_1))\right)\quad\text{and}\quad \gamma=\sqrt{1-\frac{(\Sigma u)_1^2}{\sigma_1^2}}.
$$

Deltas could be readily computed from these expressions but we shall refrain to do so. Instead, we report in Table 6 price estimates for such options for various sets of parameters. Framework and notation are the same as in Section 6. There are n stocks whose initial values are \$100 and whose volatilities are all equal to σ. Correlation between any two distinct stocks is ρ and options are at-the-money, i.e., $K = 100$. The numerical aspect of this optimization problem is of course more challenging than in the case without barrier but it seems to perform quite well and to converge very quickly. We note that option's prices tend to decrease to a constant as the number of stocks increase. We observed the same phenomenon for European basket options.

8. CONCLUSION

This paper shows how to efficiently compute approximate prices and hedges of options on any linear combination of assets. Our general method allowed us to treat all these options in a common framework. This methodology was applied to the pricing of basket, discrete-time average Asian options and basket options with a barrier. As an important by-product of this method, first and second order sensitivities are given in closed form at no extra cost.

We compared our method with Monte Carlo estimates and lower bounds proved to be extremely accurate. In terms of computational time, our method clearly outperforms Monte Carlo methods. Our examples show that we essentially get the same degree of accuracy as with 1,000,000 replications in a fraction of a second.

Incidentally, in the Black-Scholes model, we also showed how the Gamma's relate to Vega's and first order derivatives with respect to correlation parameters. This allowed us to estimate Gamma's by Monte Carlo methods using pathwise derivative estimates.

σ	ρ	H	$n = 10$	$n = 20$	$n = 30$
20%	0.5	70	5.8897	5.7514	5.7045
		80	5.6065	5.4579	5.4076
		90	4.0862	3.9477	3.9013
	0.7	70	6.8025	6.7318	6.7080
		80	6.6127	6.5356	6.5097
		90	5.0713	4.9918	4.9653
	0.9	70	7.5981	7.5772	7.5702
		80	7.4838	7.4613	7.4537
		90	6.0030	5.9777	5.9692
30%	0.5	70	8.4844	8.2647	8.1905
		80	7.3050	7.0793	7.0035
		90	4.5820	4.4101	4.3527
	0.7	70	9.9782	9.8648	9.8267
		80	8.9029	8.7784	8.7368
		90	5.8227	5.7202	5.6861
	0.9	70	11.2602	11.2272	11.2161
		80	10.3586	10.3207	10.3080
		90	7.0638	7.0296	7.0182

TABLE 6. Lower bounds for a down-and-out call option on a basket of n stocks.

Acknowledgments. The authors thank Robert Vanderbei for helping them out with AMPL and LOQO in the early stages of the project.

REFERENCES

[1] M. Curran. Valuing Asian and portfolio options by conditioning on the geometric mean price. *Management Science*, 40(12):1705–1711, 1994.

[2] A. d'Aspremont and L. El Ghaoui. Static arbitrage bounds on basket option prices. Technical report, 2003.

[3] G. Deelstra, J. Liinev, and M. Vanmaele. Pricing of arithmetic basket and Asian basket options by conditioning. Technical report, Ghent University, 2003.

[4] P. Glasserman. *Monte Carlo Methods in Financial Engineering*. Springer-Verlag, New York, 2003.

[5] R. Heynen and H. Kat. Crossing the barrier. *Risk*, (7):46–51, June 1994.

[6] R. Jarrow and A. Rudd. Approximate option valuation for arbitrary stochastic processes. *J. Finan. Econ.*, 10:347–369, 1982.

[7] Y.-K. Kwok, L. Wu, and H. Yu. Pricing multi-asset options with an external barrier. *Int. J. Theoretical Appl. Finance*, 1(4):523–541, 1998.

[8] W. Margrabe. The value of an option to exchange one asset for another. *J. Finance*, 33:177–186, 1978.

[9] M. Musiela and M. Rutkowski. *Martingale Methods in Financial Modelling*. Springer Verlag, 1997.

[10] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling. *Numerical Recipes in C : The Art of Scientific Computing*. Cambridge University Press, 2nd edition, 1992.

[11] L. C. G. Rogers and Z. Shi. The value of an Asian option. *J. Appl. Probab.*, 32:1077–1088, 1995.

[12] A. Ruttiens. Currency options on average exchange rates pricing and exposure management. In *20th annual meeting of the Decision Science Institute*, New Orleans, 1990.

[13] T. Vorst. Price and hedge ratios of average exchange options. *Internat. Rev. Finan. Anal.*, 1:179–193, 1992.

[14] H. Y. Wong and Y.-K. Kwok. Multi-asset barrier options and occupation time derivatives. *Appl. Math. Finance*, 10(3):245–266, 2003.

DEPARTMENT OF OPERATIONS RESEARCH AND FINANCIAL ENGINEERING, PRINCETON UNIVERSITY, PRINCETON, NJ 08544, ALSO WITH THE BENDHEIM CENTER FOR FINANCE AND THE APPLIED AND COMPUTATIONAL MATHEMATICS PROGRAM.
E-mail address: rcarmona@princeton.edu

DEPARTMENT OF OPERATIONS RESEARCH AND FINANCIAL ENGINEERING, PRINCETON UNIVERSITY, PRINCETON, NJ 08544.
E-mail address: vdurrlem@princeton.edu