A SCALING ANALYSIS OF A STAR NETWORK WITH LOGARITHMIC WEIGHTS

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ABSTRACT. The paper investigates the properties of a class of resource allocation algorithms for communication networks: if a node of this network has L requests to transmit and is idle, it tries to access the channel at a rate proportional to $\log(1+L)$. A stochastic model of such an algorithm is investigated in the case of the star network, in which J nodes can transmit simultaneously, but interfere with a central node 0 in such a way that node 0 cannot transmit while one of the other nodes does. One studies the impact of the log policy on these $J+1$ interacting communication nodes. A fluid scaling analysis of the network is derived with the scaling parameter N being the norm of the initial state. It is shown that the asymptotic fluid behavior of the system is a consequence of the evolution of the state of the network on a specific time scale $(N^t, t \in (0,1))$. The main result is that, on this time scale and under appropriate conditions, the state of a node with index $j \geq 1$ is of the order of $N^{a_j(t)}$, with $0 \leq a_j(t) < 1$, where $t \mapsto a_j(t)$ is a piecewise linear function. Convergence results on the fluid time scale and a stability property are derived as a consequence of this study.

CONTENTS

1. INTRODUCTION

This paper is an extension of the study of algorithms of resource allocation with logarithmic weights started in Robert and Véber [8]. For the architectures of communication networks considered (in a wireless context for example), if two nodes of the network are too close then, because of interference, they cannot use the local communication channel at the same time. For this reason, an algorithm has to be designed so that nodes can share the channel in a distributed way in order to transmit their messages. A natural class of algorithms in this setting are random access protocols, which work as follows. Each given node waits for some random duration of time before attempting to transmit. If the channel is free at that time, then it starts transmitting. Otherwise, if the channel is busy because a

communication is already under way in the neighborhood, then the node waits for another random amount of time. For the algorithms investigated in this paper, the waiting time is exponentially distributed with a rate proportional to the logarithm of the number L of pending requests at the node, i.e., of the form $K\log(1+L)$, where K is some large constant. These algorithms are now quite popular, see Shah and Wischik [9], Bouman et al. [1] and Ghaderi et al. [3]. They have nice properties in terms of fairness and efficiency. See [8] for a discussion of their use in communication networks.

Interaction of Communication Channels. The results obtained in our previous work [8] mainly deal with a network with two nodes. In this case, there is a single communication channel which can be used by only one of the two nodes at any given time. The impact of a log-policy was investigated in this case. Here one considers an additional important feature, with several communication channels which can be used at the same time provided that they do not interfere. The network analysed has a star topology with $J{+}1$ nodes: there are J nodes, numbered from 1 to J, which can transmit at the same time, i.e. their local communication channels do not interfere, see Figure 1), and a central node with index 0 which interferes with the communication channels of all the other nodes. Note that the star topology mentioned in the paper refers to the graph of interference relating the nodes, and not to flows of requests between nodes. As it will be seen, the fact that queues with index from 1 to J compete *as a group* for the communication resource with the central queue with index 0 complicates a lot the mathematical analysis compared to our previous work.

An extension of these results to the complete graph with J nodes, in which all nodes of the network use the same communication channel, is also formulated in [8] without proof. The proof of this statement can be easily derived from the proofs of the main results expounded in the present paper. Although the access to the communication medium is slightly different, our model is closely related to models of linear-bandwidth sharing for which a class of connections use several nodes at the same time and, at each node, there are "local" connections competing for the bandwidth. See Fayolle et al. [2].

Coming back to our description of the procedure, node 0 thus cannot transmit at the same time as any of the other nodes. One now gives a quick, rough, description of how the bandwidth is shared. Let L_i be the current number of pending messages at node i. In idle state, node 0 tries to transmit after an exponentially distributed random variable (the backoff time) with parameter $K\log(1+L_0)$, for some $K>0$. The attempt is successful only if all the channels are free at that time. Similarly, in idle state any node j, $1{\leq}j{\leq}J$, tries to transmit at rate $K\log(1+L_j)$ and the attempt is successful if node 0 is not transmitting at this moment. Hence, in idle state, node 0 transmits if its backoff time is smaller than the minimum of J exponential random variables, i.e. with probability

$$\log(1+L_0)/\big[\log(1+L_0)+\log(1+L_1)+\cdots+\log(1+L_J)\big],$$

provided that the values of L_j do not change too much meanwhile. This is the case if the constant K is large.

This situation will be represented formally as follows. Suppose the transmission times of requests at node j are exponentially distributed with rate μ_j and the state

of the network of $J+1$ queues is $L=(L_j, 0 \le j \le J)$. Then in our model, any non-empty node with index greater than or equal to 1 receives the instantaneous capacity $W(L)$ to transmit and node 0 receives $1-W(L)$ (the total available capacity is assumed to be 1), where

$$(1) \qquad W(L) \stackrel{\text{def.}}{=} \frac{\log(1+L_1) + \cdots + \log(1+L_J)}{\log(1+L_0) + \log(1+L_1) + \cdots + \log(1+L_J)}.$$

In particular, node $j \ge 1$ (resp., node 0) completes a transmission at rate $\mu_j W(L)$ (resp. $\mu_0(1-W(L))$). This model assumes in fact that the constant of proportionality K is sufficiently large so that the waiting times to try to access the channel are negligible and that a small fraction of a message is transmitted when transmission is successful. This is reminiscent of the usual representation of the round robin policy by the processor-sharing policy in queueing theory. See Kleinrock [5] for example.
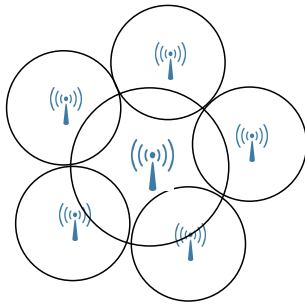

FIGURE 1. Star Network with $J=5$.

Assumptions and Notations. Requests arrive at node $0 \le j \le J$ according to a Poisson process with rate λ_j and their transmission times are exponentially distributed with parameter μ_j. The quantity ρ_j is the load of node j, $\rho_j = \lambda_j/\mu_j$. Throughout the paper, without loss of generality one assumes that $\rho_1 < \rho_2 < \cdots < \rho_J$. That is, excluding node 0, node J is the most loaded. One also defines

$$(2) \qquad \alpha_j^* = \frac{\rho_j}{1-\rho_j}, \quad 1 \le j \le J.$$

For $t \ge 0$, $L_j(t)$ denotes the number of requests at node j at time t. In what follows, the convergence of a sequence of processes on a time interval I is that associated to the topology of uniform convergence on compact subsets of I.

Scaling analysis. The purpose of this paper is to provide a fluid analysis of this network. This amounts to investigating the convergence properties of the following sequence of processes:

$$\left(\frac{L_j(Nt)}{N}, 0 \le j \le J \right),$$

where N is the norm of the initial state and tends to infinity. It was shown in Section 7 of [8] that such an analysis of the evolution of the state of the network with a fluid scaling also leads to results on the asymptotic behavior of the invariant distribution in a heavy traffic regime, and also on the transience properties of the overloaded network.

Among all possible large initial states, one will consider the most interesting (i.e. difficult) case in which the central node, with index 0, has N requests and all the other nodes are initially empty:

$$(3) \qquad\qquad L_0(0){=}N \text{ and } L_j(0){=}0 \text{ for } 1{\leq}j{\leq}J.$$

In the sequel, a superscript N will be used to recall the dependency on N and the process with initial condition (3) will thus be denoted by

$$L^N(t) = (L_0^N(t), L_1^N(t), \ldots, L_J^N(t)), \quad t \geq 0.$$

The other cases for the initial state can be treated in a similar (sometimes easier) way. See the discussion at the end of Section 4. The main problem is to describe how the numbers of requests at the initially empty nodes increase with time and the scaling parameter N.

To stress the differences with our previous analysis in [8], let us review the main results obtained on the time evolution of the network with two nodes, or $J = 1$. In all that follows, one uses the notation $x{\wedge}y = \min\{x, y\}$ and $x^+{=}\max\{x, 0\}$.

1.1. **Results for the network with two nodes.** This is the case $J{=}1$ with only one communication channel. It was shown in [8] that two other time scales have to be investigated to understand the convergence properties of the fluid scaling properly. It turns out that the most interesting case is when $\rho_1{<}1/2$, or equivalently when α_1^* defined by Equation (2) satisfies $\alpha_1^*{<}1$.

1) *The time scale $t{\to}N^t$ for $t{<}\alpha_1^*{\wedge}1$.*
   If the initial state is $(L_0^N(0), L_1^N(0)) = (N, 0)$, the convergence in distribution

$$(4) \qquad \lim_{N\to+\infty} \left(\frac{L_1^N(N^t)}{N^t}, 0 < t < \alpha_1^* \wedge 1 \right) = \left(\lambda_1 - \mu_1 \frac{t}{t+1}, 0 < t < \alpha_1^* \wedge 1 \right)$$

   holds and, at the first order in N, the state L_0^N of node 0 stays at N on this time scale.

2) *The time scale $t{\to}N^{\alpha_1^*}(\log N)\,t$.*
   If $(L_0^N(0), L_1^N(0)) = (N, \lfloor N^{\alpha_1^*} \rfloor)$, the convergence in distribution

$$\lim_{N\to+\infty} \left(\frac{L_1^N(N^{\alpha_1^*}(\log N)\,t) - N^{\alpha_1^*}}{\sqrt{N^{\alpha_1^*} \log N}} \right) = (Z(t))$$

   holds, where $(Z(t))$ is an Ornstein-Uhlenbeck process. On this time scale, L_1^N stabilizes around the value $N^{\alpha_1^*}$ and the process L_0^N still remains at N.

3) *The fluid time scale $t{\to}Nt$.*
   The relation

$$\lim_{N\to+\infty} \left(\left(\frac{L_0^N(Nt)}{N}, \frac{L_1^N(Nt)}{N^{\alpha_1^*}} \right), t > 0 \right) = \left((\gamma(t), \gamma(t)^{\alpha_1^*}), t > 0 \right)$$

   holds for the convergence in distribution, with

$$\gamma(t) = (1 + (\lambda_0 - \mu_0(1 - \rho_1))t)^+.$$

1.2. **Evolution of the state of the network with a star topology.** The results on the fluid time scale obtained in this paper are essentially done via a precise analysis of the network on the time scale $(N^t, t{\in}(0,1))$. In our previous work [8], the asymptotic analysis of the behavior on this time scale, case 1) of Section 1.1, was quite easy in fact. This is not at all the case for the network investigated below, for which most of the difficulties appear precisely on this time scale. Once the asymptotic behavior of the process (L_j^N) on this time scale is derived, the behavior of the full network on the fluid time scale $(Nt, t{\geq}0)$ can be obtained by using some of the results of [8].

The main convergence results are for processes associated to the exponents of the state vector, $(\log(1{+}L_j^N)/\log N, 1{\leq}j{\leq}J)$, on the time scale $(N^t, t{\in}(0,1))$. They turn out to be quite challenging to prove. If stochastic calculus and estimates for reflected random walks are, as in [8], important ingredients to prove our convergence results, a novel invariance relation has to be established, namely Proposition 5 of Section 3 in the case $J{=}2$ and Theorem 3 of Section 4 in the general case. In fact, this relation implies that the sum of the exponents $(\log(L_j(N^t))/\log N, \ j{\geq}1)$ is constant on specific time intervals. To prove it, one introduces a key technical tool, a family of space-time harmonic functions (10) in Section 3 which is used to derive L_2-estimates of the scaled processes $(L_j(N^t))$, see Relation (19). In spirit, it is connected to some perturbation techniques although it does not seem to be directly related to this framework, see Kurtz [6] for example. This is the main technical difficulty of the paper. The desired convergence results are then proved in Section 3 using stochastic calculus and several technical estimates related to the behavior of reflected random walks.

The main results. One of the consequences of our work is that, under appropriate conditions, the fluid analysis of the network can be reduced to the study of a network with two nodes, or $J{=}1$, which is precisely the configuration studied in our previous work [8]. Hence, once the behavior of the star network on the time scale $(N^t, t \in (0,1))$ is understood, its fluid analysis simply follows from the results in [8]. In particular, one obtains the result conjectured in Wischik [10] that under the condition

$$\rho_0 + \max(\rho_j, 1 \leq j \leq J) = \rho_0 + \rho_J < 1,$$

the Markov process $(L_j(t), 0{\leq}j{\leq}J)$ is ergodic. It should be noted that Section 9 of [8] gives a presentation without proofs of the slightly different (but easier) case of a network with J nodes and the same communication channel, so that only one node can transmit at a time. As we already mentioned, the techniques which are developed in the present paper can in fact be used to establish these results.

Some transient behaviors of these networks are also obtained. Recall the quantities (α_j^*) defined by Relation (2) and set

$$\beta_j^* \stackrel{\text{def.}}{=} \frac{\alpha_j^*}{J-j}, \ 1{\leq}j{\leq}J, \ \text{and} \ \kappa \stackrel{\text{def.}}{=} \sup\left\{ j \geq 1 : \frac{\alpha_j^*}{J-j+1} < 1 \right\},$$

with $\sup(\emptyset) = 0$. It is shown that, on the fluid time scale, the states of the nodes with indices between 1 and $\kappa{-}1$ behave ultimately like $\kappa{-}1$ ergodic $M/M/1$ queues. In particular, if $\kappa = J$, nodes 0 and J are then the only nodes with a non-negligible number of pending requests (with respect to some power of N). The main results on the fluid behavior are gathered in Theorem 5 of Section 4. They are summarized more precisely as follows.

The following results of convergence in distribution hold on a time interval $(0, t_0)$, where $t_0 \in (0, \infty]$ depends on the parameters of the network. Recall that in all that follows, the convergence of a sequence of processes is that associated to the topology of uniform convergence on compact time intervals (of $(0, t_0)$ here).

1) If $\kappa = 0$,

$$\lim_{N \to \infty} \left(\frac{L_0^N(Nt)}{N}, \ldots, \frac{L_J^N(Nt)}{N} \right) = (\gamma_0(t), \gamma_1(t), \ldots, \gamma_J(t)).$$

2) In the case $1 \le \kappa < J$,
   (a) If $\beta_\kappa^* < 1$,

$$\lim_{N \to \infty} \left(\frac{L_0^N(Nt)}{N}, \frac{L_1^N(Nt)}{(\log N)^3}, \ldots, \frac{L_\kappa^N(Nt)}{(\log N)^3}, \frac{L_{\kappa+1}^N(Nt)}{N}, \ldots, \frac{L_J^N(Nt)}{N} \right)$$
$$= (\gamma_0(t), 0^{(\kappa)}, \gamma_{\kappa+1}(t), \ldots, \gamma_J(t)).$$

   (b) If $\beta_\kappa^* > 1$,

$$\lim_{N \to \infty} \left(\frac{L_0^N(Nt)}{N}, \frac{L_1^N(Nt)}{(\log N)^3}, \ldots, \frac{L_{\kappa-1}^N(Nt)}{(\log N)^3}, \frac{L_\kappa^N(Nt)}{N^{\alpha_\kappa^* - J + \kappa}}, \frac{L_{\kappa+1}^N(Nt)}{N}, \ldots, \frac{L_J^N(Nt)}{N} \right)$$
$$= \left(\gamma_0(t), 0^{(\kappa-1)}, \frac{1}{\gamma_{\kappa+1}(t)\gamma_{\kappa+2}(t) \cdots \gamma_J(t)}, \gamma_{\kappa+1}(t), \ldots, \gamma_J(t) \right).$$

3) If $\kappa = J$,

$$\lim_{N \to \infty} \left(\frac{L_0^N(Nt)}{N}, \frac{L_1^N(Nt)}{(\log N)^3}, \ldots, \frac{L_{J-1}^N(Nt)}{(\log N)^3}, \frac{L_J^N(Nt)}{N^{\alpha_J^*}} \right) = \left(\gamma_0(t), 0^{(J-1)}, \gamma_0(t)^{\alpha_J^*} \right).$$

The functions $(\gamma_j(t))$ are deterministic, non-trivial, affine functions. They are defined in Theorem 5 in Section 4. The constant t_0 is the first instant on the fluid time scale when the central node empties, i.e. $\gamma_0(t_0) = 0$. Its expression is also given in the statement of the theorem.

The expression given by Relation (1) of the capacity $W(L^N(N\cdot))$ allocated to the nodes with positive indices and the above results of convergence in distribution show the following property. If $\kappa > 0$, the states of the nodes whose indices are between 1 and $\kappa - 1$ do not have an impact on the quantity $W(L^N(N\cdot))$, and therefore on the asymptotic behaviour of the other nodes. The order of magnitude of the states of these nodes is negligible with respect to any power of N in the fluid regime. As one will see, they behave locally like ergodic $M/M/1$ queues. Hence, on the fluid time scale only a subset of the nodes remain non-negligible. For example, when $1 < \kappa < J$ and $\beta_\kappa^* > 1$, the state of the central node is of the order of N, the state of the node with index κ is of the order of $N^{\alpha_\kappa^* - (J-\kappa)}$, and all the nodes whose indices lie between $\kappa+1$ and J have a number of pending requests which is of the order of N. The other cases exhibit similar behaviors. Figure 2 corresponds to the case $\kappa = J$ for $J = 3$, but on the time scale $(N^t, t \in (0, 1))$.

Discontinuity on the fluid time scale. The convergence results for the fluid scaling are valid on an *open* interval $(0, t_0)$ excluding 0, i.e. on time intervals of the form $[aN, bN]$ with $0 < a < b$, hence "after" the time scale $(N^t, t \in (0, 1))$. In fact the process exhibits a kind of discontinuity at 0 on the fluid time scale, for example in the above case 3) where $\kappa = J$. Indeed, initially $L_J^N(0) = 0$ but $L_J^N(\varepsilon N) \sim N^{\alpha_J^*}$ for $\varepsilon > 0$ arbitrarily close to 0. In other words, the Jth coordinate jumps to $N^{\alpha_J^*}$ at

$t=0^+$. This phenomenon can be explained on the time scale $(N^t, t\in(0,1))$, which is one of the reasons why this time scale plays a major role in the analysis.

Time Varying Exponents in N with Piecewise Affine Behaviors. From the assumption (3) on the initial state, only the state of node 0 is not zero, and is equal to N initially. Under appropriate conditions on the α_j^* defined by Equation (2), the remarkable feature of the evolution of this network is as follows: on the time scale $(N^t, t\in(0,1))$, the state of a given node with index $1\leq j\leq J-1$ grows like a power of N until an instant after which it starts *decreasing* and finally stabilizes in a finite neighborhood of 0. This is in fact the most difficult technical point of the paper. See Figure 2. Section 3 is essentially devoted to the proof of this result.

Let us describe the phenomenon more precisely. Recall that the load ρ_1 of node 1 is such that $\rho_1<\rho_2<\cdots<\rho_J$. If $J\geq2$ and $\alpha_1^*/J<1$, then

$$\lim_{N\to+\infty}\left(\frac{L_1^N(N^t)}{N^t}, 0<t<\frac{\alpha_1^*}{J}\right) = \left(\lambda_1-\mu_1\frac{Jt}{Jt+1}, 0<t<\frac{\alpha_1^*}{J}\right),$$

which is a more or less straightforward analogue of Relation (4). More interesting and technically challenging is the behaviour of the process on the "next" time interval $(N^{\alpha_1^*/J}, N^{\alpha_1^*/(J-1)})$ on the time scale $(N^t, t\in(0,1))$. If $\alpha_1^*/(J-1)<1$, the convergence in distribution

$$(5)\qquad \lim_{N\to+\infty}\left(\frac{L_1^N(N^t)}{N^{\alpha_1^*-(J-1)t}}, \frac{\alpha_1^*}{J}<t<\frac{\alpha_1^*}{J-1}\right) = \left(\prod_{i=2}^{J}\frac{1}{\mu_i(\rho_i-\rho_1)}, \frac{\alpha_1^*}{J}<t<\frac{\alpha_1^*}{J-1}\right)$$

holds on this time interval. In contrast with the static equilibrium value α_1^* reached by the exponent in N of $L_1^N(N^t)$ in the case $J=1$ investigated in [8], this equilibrium value is now time-dependent and stabilizes at 0 after some time. Thus, at "time" $N^{\alpha_1^*/J}$, L_1^N is of the order of $N^{\alpha_1^*/J}$ but just after the exponent in N starts decreasing and is 0 at "time" $N^{\alpha_1^*/(J-1)}$. Furthermore, at that time the process L_1^N behaves like an ergodic $M/M/1$ queue and therefore does not scale with any power of N on the time scale $(N^t, t\in(\alpha_1^*/(J-1),1))$, while the states of the other nodes are still of the order of a power of N. Consequently, the component $\log(1+L_1)$ in the expression (1) of $W(L)$ can be discarded. In other words, after time $N^{\alpha_1^*/(J-1)}$, the system behaves like a network without node 1. See Figure 2 for a representation of the evolution of the queues on the time scale $(N^t, t\in(0,1))$.

Outline of the paper. The model and notation are introduced in Section 2, as well as a technical result, Proposition 1, which will be used repeatedly in the subsequent sections. The case of the network with three nodes, or $J=2$, is studied in detail in Section 3. It contains the main ingredients to extend the analysis to the general case $J\geq3$ in Section 4.

2. The Stochastic Model

In this section, one introduces the main stochastic processes and some notation. For $\xi\geq0$, $\mathcal{N}_\xi$ (resp. $\mathcal{N}_\xi^2$) denotes a Poisson process with rate ξ on $\mathbb{R}_+$ (resp. $\mathbb{R}_+^2$). For any $0\leq a\leq b$, the quantity $\mathcal{N}_\xi([a,b])$ denotes the number of points of $\mathcal{N}_\xi$ in the interval $[a,b]$. The Poisson processes on $\mathbb{R}_+^2$ are used as follows. If $x>0$, then $\mathcal{N}_\xi^2([0,x]\times\cdot)$ is a Poisson process on $\mathbb{R}_+$ with rate $x\xi$. Throughout the paper, the Poisson processes used are assumed to be independent. If f is an $\mathbb{R}^d$-valued
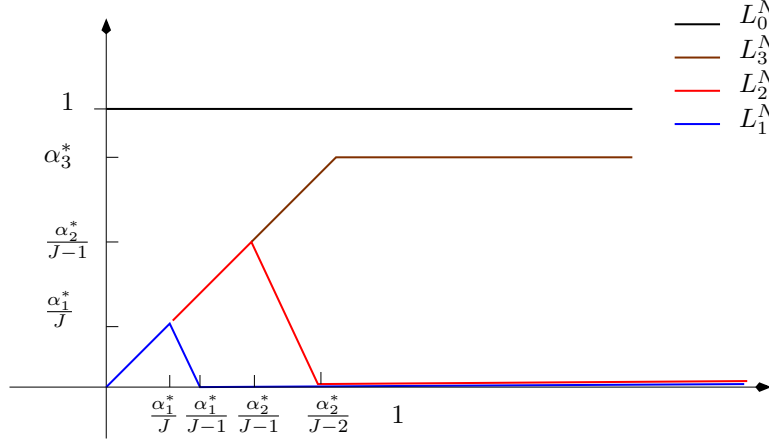
FIGURE 2. Evolution of $\log(L_j^N(N^t))/\log N$, the exponent in N of L_j^N on the time scale $(N^t, t \in (0,1))$. Here $J=3$, $\rho_1 < \rho_2 < \rho_3$ and the initial state is $(N, 0, 0, 0)$.

function on $\mathbb{R}_+$, $f(t-)$ denotes the limit of f to the left of $t > 0$, provided that it exists.

The evolution of the Markov process $(L(s)) = (L_j(s), 0 \le j \le J)$ can be described as the solution to the following system of stochastic differential equations (SSDE)

$$(6) \quad \begin{cases} \mathrm{d}L_0(s) & = \mathcal{N}_{\lambda_0}(\mathrm{d}s) - \mathbb{1}_{\{L_0(s-)>0\}} \mathcal{N}_{\mu_0}^2([W(L(s-)),1],\mathrm{d}s), \\ \mathrm{d}L_j(s) & = \mathcal{N}_{\lambda_j}(\mathrm{d}s) - \mathbb{1}_{\{L_j(s-)>0\}} \mathcal{N}_{\mu_j}^2([0,W(L(s-))),\mathrm{d}s), \quad 1 \le j \le J, \end{cases}$$

where $L(s) = (L_0(s), L_1(s), \ldots, L_J(s))$ and $W(\cdot)$ is defined by Equation (1).

For $0 \le j \le J$, one denotes by $(L_j^N(s))$ the solution to the SSDE (6) with initial state $(N, 0, \ldots, 0)$, and $Y_j^N(s)$ describes the exponent in N of $L_j^N(s)$:

$$(7) \quad Y_j^N(s) = \frac{\log(1+L_j^N(s))}{\log N}.$$

Recall that $\rho_j = \lambda_j/\mu_j$. Without loss of generality, one assumes that nodes $1, \ldots, J$ are ordered so that $\rho_1 < \rho_2 < \cdots < \rho_J$.

The following result is a simple consequence of a result of Kingman [4] in the case of birth and death processes. It will be used repeatedly.

Proposition 1.

1) *If $(X(s))$ is a birth and death process on $\mathbb{Z}$ starting at 1 with birth rate λ and death rate $\mu > \lambda$, then for any integer $x \ge 0$,*

$$(8) \quad \mathbb{P}\left(\sup_{s \ge 0} X(s) \ge x\right) \le \left(\frac{\lambda}{\mu}\right)^{x-1}.$$

2) *If $(X_+(s))$ denotes the process with the same transitions as $(X(s))$ but with a reflection at 0, then for any $T > 0$,*

$$\mathbb{P}\left(\sup_{0 \le s \le T} X_+(s) \ge x\right) \le (\lambda T + 1)\left(\frac{\lambda}{\mu}\right)^{x-1}$$

and

$$\mathbb{E}\left(\sup_{0\leq s\leq T} X_+(s)^2\right) \leq \frac{2\mu^2}{\lambda^2}(\lambda T+1)\left(\frac{1}{\log(\mu/\lambda)}\right)^2$$

Proof. Relation (8) is Relation (3.3) in Theorem 3.5 of Robert [7] for example. The second and third relations follow by remarking that the sample paths of $(X_+(s))$ can be obtained as a concatenation of excursions of $(X(s))$ above 0. The estimate (8) is in fact an upper bound on the probability that the supremum of an excursion is greater than x. Two excursions are separated by at least an exponential random variable with parameter λ, so that the total number N_T of such excursions in the interval considered is stochastically bounded by 1 plus a Poisson random variable with parameter λT. Thus,

$$\mathbb{P}\left(\sup_{0\leq s\leq T} X_+(s) \geq x\right) \leq \mathbb{E}\left(\sum_{i=1}^{N_T} \mathbf{1}_{\{i-\text{th excursion exceeds } x\}}\right)$$

$$\leq (\lambda T+1)\mathbb{P}\left(\sup_{s\geq 0} X(s) \geq x\right).$$

Finally, since $\lambda < \mu$ one has for every $x \in \mathbb{R}_+$

$$\mathbb{P}\left(\sup_{0\leq s\leq T} X_+(s)^2 > x\right) = \mathbb{P}\left(\sup_{0\leq s\leq T} X_+(s) > \lfloor\sqrt{x}\rfloor\right) \leq (\lambda T+1)\left(\frac{\lambda}{\mu}\right)^{\sqrt{x}-2}.$$

Consequently,

$$\mathbb{E}\left(\sup_{0\leq s\leq T} X_+(s)^2\right) = \int_0^\infty \mathbb{P}\left(\sup_{0\leq s\leq T} X_+(s)^2 > x\right) \mathrm{d}x$$

$$\leq (\lambda T+1)\int_0^\infty \left(\frac{\lambda}{\mu}\right)^{\sqrt{x}-2} \mathrm{d}x$$

$$= \frac{2\mu^2}{\lambda^2}(\lambda T+1)\int_0^\infty \left(\frac{\lambda}{\mu}\right)^y y\mathrm{d}y = \frac{2\mu^2}{\lambda^2}(\lambda T+1)\frac{1}{\log(\mu/\lambda)^2}.$$

The proposition is proved. □

Let us begin with the case of a network with three nodes, or $J = 2$. The general case is analyzed in Section 4.

3. Three node network

In this section one assumes that $J = 2$ and $\rho_1 < \rho_2$. Recall that the initial state is given by $L_0^N(0) = N$ and $L_1^N(0) = 0 = L_2^N(0)$. The main results of this section can be summarized briefly as follows.

1) On the time interval $(0, (\alpha_1^*/2) \wedge 1)$, $L_1^N(N^t)$ and $L_2^N(N^t)$ grow like $C(t)N^t$, for some linear functions $C_1(t)$ and $C_2(t)$.
2) If $\alpha_1^*/2 < 1$, on the time interval $(\alpha_1^*/2, \alpha_1^* \wedge 1)$,
   (a) $(L_1^N(N^t))$ decreases like $c_1 N^{\alpha_1^*-t}$ for some constant $c_1 > 0$. If $\alpha_1^* < 1$, then it reaches a neighborhood of 0 in which it remains for the rest of the evolution.
   (b) $(L_2^N(N^t))$ still grows like $c_2 N^t$ for some constant c_2 such that $c_1 c_2 = 1$. If $\alpha_2^* < 1$, then it remains in a close neighborhood of $N^{\alpha_2^*}$ until $t = 1$ when the fluid time scale "begins".

In this section again, the convergence of a sequence of processes is that associated to the topology of uniform convergence on compact time intervals. First, the following proposition gives the behavior of the network up to time $N^{\alpha_1^*/2}$. Its proof is identical to that of Proposition 2 of Robert and Véber [8], and is therefore omitted.

Proposition 2. *The convergence of processes*

$$\lim_{N\to\infty}\left(\frac{L_1^N(N^t)}{N^t},\frac{L_2^N(N^t)}{N^t}\right)=\left(\lambda_1-\mu_1\frac{2t}{1+2t},\ \lambda_2-\mu_2\frac{2t}{1+2t}\right)$$

holds on the time interval $(0,\alpha_1^*/2\wedge 1)$.

If $\alpha_1^*/2>1$, for a suitably small $\varepsilon>0$ then one has that $L_i^N(\varepsilon N)$ is of the order of N for every $i\in\{0,1,2\}$, and the fluid analysis of this network is straightforward. See 1) of Theorem 2 below.

The first theorem describes the network on the time scale $t\mapsto N^t$ on the time interval $(\alpha_1^*/2,\alpha_1^*\wedge 1)$.

Theorem 1. *Under the assumption that $\alpha_1^*/2<1$, the convergence*

$$\lim_{N\to\infty}\left(\frac{L_1^N(N^t)}{N^{\alpha_1^*-t}},\frac{L_2^N(N^t)}{N^t}\right)=\left(\frac{1}{\mu_2(\rho_2-\rho_1)},\mu_2(\rho_2-\rho_1)\right)$$

holds on the time interval $(\alpha_1^*/2,\alpha_1^*\wedge 1)$.

In particular, $(L_1^N L_2^N(N^t)/N^{\alpha_1^*})$ converges uniformly to the process constant equal to one on this time interval.

When $\alpha_1^*<1$ one shows that after "time" $N^{\alpha_1^*}$, L_1^N remains of order $(\log N)^2$ at most, so that the system (L_0^N,L_2^N) is indeed equivalent to the 2-queue system analyzed in [8].

Proposition 3. *Under the condition $\alpha_1^*<1$, if*

$$\theta_0^N=\inf\{t>0:L_1^N(t)=0\},$$

then

1) *for $\varepsilon>0$, one has*

$$\lim_{N\to\infty}\mathbb{P}\big(\theta_0^N\le N^{\alpha_1^*}\log N+N^\varepsilon\log N\big)=1.$$

2) *For $\varepsilon>0$ sufficiently small, one has*

$$\lim_{N\to\infty}\mathbb{P}\left(\sup_{t\in[\theta_0^N,N^{(\alpha_2^*\wedge 1)-\varepsilon}]}L_1^N(t)\le(\log N)^2\right)=1.$$

3) *The convergence of processes*

$$\lim_{N\to\infty}\left(\frac{L_2^N(N^t)}{N^t}\right)=\left(\lambda_2-\mu_2\frac{t}{1+t}\right)$$

holds on the time interval $(\alpha_1^*,\alpha_2^*\wedge 1)$.

Finally, the second theorem gives the fluid limit of the network with three nodes.

Theorem 2 (Fluid Limits). *The following convergences of processes hold on the time interval $(0,t_0)$.*

1) If $\alpha_1^*/2>1$, then $t_0=3/(\mu_0-3\lambda_0)^+$ and

$$\lim_{N\to\infty}\left(\frac{L_0^N(Nt)}{N},\frac{L_1^N(Nt)}{N},\frac{L_2^N(Nt)}{N}\right)$$
$$=\left(1+\mu_0\left(\rho_0-\frac{1}{3}\right)t,\mu_1\left(\rho_1-\frac{2}{3}\right)t,\mu_2\left(\rho_2-\frac{2}{3}\right)t\right).$$

2) If $\alpha_1^*/2<1<\alpha_1^*$, then $t_0=1/(1-\rho_0-\rho_1)^+$ and

$$\lim_{N\to\infty}\left(\frac{L_0^N(Nt)}{N},\frac{L_1^N(Nt)}{N^{\alpha_1^*-1}},\frac{L_2^N(Nt)}{N}\right)$$
$$=\left(1+\mu_0(\rho_0+\rho_1-1)t,\frac{1}{\mu_2(\rho_2-\rho_1)t},\mu_2(\rho_2-\rho_1)t\right).$$

3) If $\alpha_1^*<1<\alpha_2^*$, then $t_0=1/(\mu_0(1/2-\rho_0))^+$ and

$$\lim_{N\to\infty}\left(\frac{L_0^N(Nt)}{N},\frac{L_1^N(Nt)}{(\log N)^3},\frac{L_2^N(Nt)}{N}\right)=\left(1+\mu_0\left(\rho_0-\frac{1}{2}\right)t,0,\mu_2\left(\rho_2-\frac{1}{2}\right)t\right).$$

4) If $\alpha_2^*<1$, then $t_0=+\infty$ and

$$\lim_{N\to\infty}\left(\frac{L_0^N(Nt)}{N},\frac{L_1^N(Nt)}{(\log N)^3},\frac{L_2^N(Nt)}{N^{\alpha_2^*}}\right)=((\gamma(t),0,\gamma(t)^{\alpha_2^*})),$$

with $\gamma(t)=(1+\mu_0(\rho_0+\rho_2-1)t)^+$.

The rest of this section is devoted to the proofs of these results. Theorem 1 is proved in Section 3.1, Proposition 3 in Section 3.2 and Theorem 2 in Section 3.3.

3.1. **Proof of Theorem 1.** This is done in several steps. Since $L_0^N(0)=N$ and one is concerned with the time scale $(N^t,\ 0<t<1)$, the fluctuations of the process $(L_0^N(t))$ around N are negligible for our purpose. It will be implicitly assumed that $L_0^N\equiv N$ on this time scale. To make this rigorous, one can proceed as in the proofs of related results in [8] (see the proof of Proposition 1 in this reference for example) and use a coupling of L_0^N with its arrival process and with its departure process to establish that the results below hold in these worst-case scenarios.

First, from Proposition 2 one sees that at time $N^{\alpha_1^*/2}$, the drift term of node 1 cancels while that of node 2 is still positive. This suggests that, at least for a small amount of time after $N^{\alpha_1^*/2}$, $(L_2^N(t))$ keeps on increasing. The following lemma establishes a preliminary result in this direction.

Lemma 1. For $\gamma\in\left(\alpha_1^*/2,\alpha_2^*/2\wedge1\right)$, the relation

$$\lim_{N\to\infty}\mathbb{P}\left(\frac{1}{2}\left(\lambda_2-\mu_2\frac{2\gamma}{1+2\gamma}\right)<\inf_{s\in[N^{\alpha_1^*/2},N^\gamma]}\frac{L_2^N(s)}{s}\le\sup_{s\in[N^{\alpha_1^*/2},N^\gamma]}\frac{L_2^N(s)}{s}<2\lambda_2\right)=1,$$

holds and there exists some $A_\gamma>0$ such that

$$\lim_{N\to\infty}\mathbb{P}\left(\sup_{s\in[N^{\alpha_1^*/2},N^\gamma]}L_1^N(s)<A_\gamma N^{\alpha_1^*/2}\right)=1.$$

To clarify the exposition, Lemma 1 is proved in Appendix A.1.

Theorem 1 states in particular that the sequence of processes $(L_1^N(s)L_2^N(s)/N^{\alpha_1^*})$ converges to the constant process (1), uniformly over compact sub-intervals of the time interval $(\alpha_1^*/2,\alpha_2^*/2\wedge\alpha_1^*\wedge1)$. The following result gives a weaker version of

that. It shows that it is true for $(Y_1^N(s))$ and $(Y_2^N(s))$, the exponents in N of $(L_1^N(s))$ and $(L_2^N(s))$.

Lemma 2. *For every $\gamma \in \left(\alpha_1^*/2, \alpha_2^*/2 \wedge \alpha_1^* \wedge 1\right)$ and every $\varepsilon > 0$, one has*

$$\lim_{N\to\infty} \mathbb{P}\left(\sup_{t\in[\alpha_1^*/2,\gamma]} \left|Y_1^N(N^t) + Y_2^N(N^t) - \alpha_1^*\right| > \varepsilon\right) = 0.$$

Again, the proof of Lemma 2 is postponed until Appendix A.2 .

Lemmas 1 and 2 show that, for $t \in (\alpha_1^*/2, \alpha_2^*/2 \wedge \alpha_1^* \wedge 1)$, $L_2^N(N^t)$ is of the order of N^t while $L_1^N(N^t) \sim N^{\alpha_1^*-t}$. The following proposition gives a more precise result, on the (larger) interval $(\alpha_1^*/2, \alpha_1^* \wedge 1)$.

Proposition 4. *For the convergence of processes, the relation*

$$\lim_{N\to\infty} \left(Y_1^N(N^t), \frac{L_2^N(N^t)}{N^t}\right) = \left(\alpha_1^* - t,\, \mu_2(\rho_2 - \rho_1)\right)$$

holds on the time interval $(\alpha_1^*/2, \alpha_1^* \wedge 1)$.

Proof. One shows instead the equivalent statement

$$\lim_{N\to\infty} \left(Y_1^N(N^t) + Y_2^N(N^t), \frac{L_2^N(N^t)}{N^t}\right) = \left(\alpha_1^*,\, \mu_2(\rho_2 - \rho_1)\right).$$

Let us start by fixing a constant γ such that

$$\frac{\alpha_1^*}{2} < \gamma < \frac{\alpha_2^*}{2} \wedge \alpha_1^* \wedge 1,$$

and show the desired convergence on the interval $(\alpha_1^*/2, \gamma]$. The convergence of the first coordinate is then a direct consequence of Lemma 2.

As a first step, one shows that for any $t \in (\alpha_1^*/2, \gamma]$, the convergence

(9) $$\lim_{N\to+\infty} \frac{L_2^N(N^t)}{N^t} = \mu_2(\rho_2 - \rho_1)$$

holds for the L_2-norm.

Define the function F by

(10) $$F(l_1, l_2, t) \stackrel{\text{def.}}{=} \frac{1}{2}\left(\frac{l_2}{N^t} - \mu_2(\rho_2 - \rho_1)\right)^2 - \frac{\mu_2}{\mu_1}\frac{l_1}{N^t}\left(\frac{l_2}{N^t} - \mu_2(\rho_2 - \rho_1)\right).$$

Using the SSDE's (6), trite calculations give that the infinitesimal generator G^N associated to the process $(L_1^N(N^t), L_2^N(N^t), t)$, applied to the function F is given by

$$G^N F(l_1, l_2, t) = -(\log N)\left(\frac{l_2}{N^t} - \mu_2(\rho_2 - \rho_1)\right)^2 + C_1^N(l_1, l_2, t)\frac{\log N}{N^t}$$

$$+ C_2^N(l_1, l_2, t)\frac{l_1}{N^t}\log N + C_3^N(l_1, l_2, t)\frac{l_1}{N^t}\frac{l_2}{N^t}\log N,$$

and there exists $K > 0$ such that, for $i \in \{1, 2, 3\}$, the relation $|C_i^N(l_1, l_2, t)| \leq K$ holds for any $l_1, l_2 \in \mathbb{N}$, $t \in (\alpha_1^*/2, \gamma]$ and $N \in \mathbb{N}$. Define

$$\psi^N(l_1, l_2, t) = C_1^N(l_1, l_2, t)\frac{\log N}{N^t} + C_2^N(l_1, l_2, t)\frac{l_1}{N^t}\log N + C_3^N(l_1, l_2, t)\frac{l_1}{N^t}\frac{l_2}{N^t}\log N.$$

Then

(11) $\quad (M^N(t)) \overset{\text{def.}}{=} \Bigg(F\left(L_1^N(N^t), L_2^N(N^t), t\right) - F\left(L_1^N(N^{\alpha_1^*/2}), L_2^N(N^{\alpha_1^*/2}), \alpha_1^*/2\right)$

$$+ \int_{\alpha_1^*/2}^t \left[\log N \left(\frac{L_2^N(N^u)}{N^u} - \mu_2(\rho_2 - \rho_1) \right)^2 - \psi^N\left(L_1^N(N^u), L_2^N(N^u), u\right) \right] du \Bigg)$$

is a zero-mean martingale on the time interval $(\alpha_1^*/2, \gamma]$. Taking the expectation and reordering the terms conveniently, one obtains that

$$\mathbb{E}\left(\left(\frac{L_2^N(N^t)}{N^t} - \mu_2(\rho_2 - \rho_1) \right)^2 \right) = -2(\log N) \int_{\alpha_1^*/2}^t \mathbb{E}\left(\left(\frac{L_2^N(N^u)}{N^u} - \mu_2(\rho_2 - \rho_1) \right)^2 \right) du$$

$$+ 2\mathbb{E}\left(F\left(L_1^N(N^{\alpha_1^*/2}), L_2^N(N^{\alpha_1^*/2}), \alpha_1^*/2\right) \right) + \frac{2\mu_2}{\mu_1} \mathbb{E}\left(\frac{L_1^N(N^t)}{N^t} \left(\frac{L_2^N(N^t)}{N^t} - \mu_2(\rho_2 - \rho_1) \right) \right)$$

$$+ 2 \int_{\alpha_1^*/2}^t \mathbb{E}\left(\psi^N(L_1^N(N^u), L_2^N(N^u), u) \right) du.$$

Now, let $(X_+(s))$ be the process introduced in the proof of Lemma 1. A slight adaptation of the arguments given there shows that one can couple $(L_1^N(s))$ and $(2A\lambda_1 N^{\alpha_1^*/2} + X_+(s))$ in such a way that $L_1^N(s) \le 2A\lambda_1 N^{\alpha_1^*/2} + X_+(s - N^{\alpha_1^*/2})$ for every $s \in [N^{\alpha_1^*/2}, N^\gamma]$. Using this fact together with the Cauchy-Schwartz Inequality, one can write

(12)

$$\left| \mathbb{E}\left(\frac{L_1^N(N^t)}{N^t} \left(\frac{L_2^N(N^t)}{N^t} - \mu_2(\rho_2 - \rho_1) \right) \right) \right|$$

$$\le \left[\mathbb{E}\left(\left(\frac{L_1^N(N^t)}{N^t} \right)^2 \right) \mathbb{E}\left(\left(\frac{L_2^N(N^t)}{N^t} - \mu_2(\rho_2 - \rho_1) \right)^2 \right) \right]^{1/2}$$

$$\le \left[\mathbb{E}\left(\left(\frac{2A\lambda_1 N^{\alpha_1^*/2} + X_+(N^t - N^{\alpha_1^*/2})}{N^t} \right)^2 \right) \mathbb{E}\left(\left(\frac{\mathcal{N}_{\lambda_2}[0, N^t]}{N^t} \right)^2 + \mu_2^2(\rho_2 - \rho_1)^2 \right) \right]^{1/2}$$

$$\le C_4 \left[\mathbb{E}\left(\left[\frac{1}{N^t} \left(2A\lambda_1 N^{\alpha_1^*/2} + \sup_{\alpha_1^*/2 \le u \le \gamma} X_+(N^u - N^{\alpha_1^*/2}) \right) \right]^2 \right) \right]^{1/2}$$

$$\le C_4 \frac{2A\lambda_1 N^{\alpha_1^*/2}}{N^t} + \frac{C_4}{N^t} \mathbb{E}\left(\sup_{\alpha_1^*/2 \le u \le \gamma} X_+(N^u - N^{\alpha_1^*/2})^2 \right)^{1/2}.$$

Using the second part of Proposition 1 with $\lambda = \lambda_1$ and $\mu = \lambda_1 + C/\log N$, one can then write that

$$\mathbb{E}\left(\sup_{\alpha_1^*/2 \le u \le \gamma} X_+(N^u - N^{\alpha_1^*/2})^2 \right)^{1/2} \le C' \sqrt{(\lambda_1 N^\gamma + 1)(\log N)^2},$$

and since $\gamma < \alpha_1^*$ by construction, one can conclude that the quantity in the left hand side of (12) is bounded by

$$C_5 N^{\alpha_1^*/2 - t},$$

where the constant C_5 is independent of t and N. Likewise, there exist some constants C_6 and C_7 such that for any $u \in [\alpha_1^*/2, \gamma)$,

$$\mathbb{E}\big(|\psi^N(L_1^N(N^u), L_2^N(N^u), u)|\big) \leq C_6 \frac{\log N}{N^{\alpha_1^*/2}} + C_7 \frac{\log N}{N^{u-\alpha_1^*/2}}.$$

Since

$$(\log N) \int_{\alpha_1^*/2}^t \frac{\mathrm{d}u}{N^{u-\alpha_1^*/2}} = 1 - \frac{1}{N^{t-\alpha_1^*/2}},$$

the process

$$\left(2 \int_{\alpha_1^*/2}^t \mathbb{E}\left(|\psi^N\left(L_1^N(N^u), L_2^N(N^u), u\right)|\right) \mathrm{d}u \right)$$

is bounded by a constant uniformly in $t \in [\alpha_1^*/2, \gamma]$ and N. Finally, similar arguments give the existence of a constant C_8 such that

$$\mathbb{E}\Big(|F(L_1^N(N^{\alpha_1^*/2}), L_2^N(N^{\alpha_1^*/2}), \alpha_1^*/2)|\Big) \leq C_8.$$

One can now use Gronwall's Lemma to conclude that there exists $C_9 > 0$ independent of N such that for every $t \in [\alpha_1^*/2, \gamma]$,

$$(13) \qquad \mathbb{E}\left(\left(\frac{L_2^N(N^t)}{N^t} - \mu_2(\rho_2 - \rho_1) \right)^2 \right) \leq C_9\, e^{-2(\log N)(t-\alpha_1^*/2)} = \frac{C_9}{N^{2t-\alpha_1^*}},$$

which proves (9).

As a second step, one now shows the uniform convergence of $(L_2^N(N^t)/N^t)$ towards the constant process $\mu_2(\rho_2 - \rho_1)$, over any time interval of the form $[\alpha_1^*/2 + \varepsilon, \gamma]$. By Doob's Maximal Inequality applied to the martingale $(M^N(t))$ defined by Relation (11), one has for every $\eta > 0$

$$\mathbb{P}\left(\sup_{\alpha_1^*/2+\varepsilon \leq t \leq \gamma} |M^N(t)| > \eta \right) \leq \frac{1}{\eta} \mathbb{E}\left(|M^N(\gamma)|\right)$$

$$\leq \frac{1}{\eta} \mathbb{E}\left(\left(\frac{L_2^N(N^\gamma)}{N^\gamma} - \mu_2(\rho_2 - \rho_1) \right)^2 \right) + \frac{1}{\eta} \mathbb{E}\left(\left(\frac{L_2^N(N^{\alpha_1^*/2+\varepsilon})}{N^{\alpha_1^*/2+\varepsilon}} - \mu_2(\rho_2 - \rho_1) \right)^2 \right)$$

$$+ \frac{\log N}{\eta} \int_{\alpha_1^*/2+\varepsilon}^\gamma \mathbb{E}\left(\left(\frac{L_2^N(N^u)}{N^u} - \mu_2(\rho_2 - \rho_1) \right)^2 \right) \mathrm{d}u$$

$$+ \frac{1}{\eta} \int_{\alpha_1^*/2+\varepsilon}^\gamma \mathbb{E}\big(|\psi^N(L_1^N(N^u), L_2^N(N^u), u)|\big) \mathrm{d}u.$$

The quantity in the right hand side above converges to 0 as N tends to infinity, by all the estimates obtained so far and by using Lebesgue's Convergence Theorem.

Consequently, using Relations (10) and (11), one obtains that

$$\mathbb{P}\left(\sup_{\alpha_1^*/2+\varepsilon \le t \le \gamma} \left(\frac{L_2^N(N^t)}{N^t} - \mu_2(\rho_2 - \rho_1)\right)^2 > 6\eta\right)$$

$$\le \mathbb{P}\left(\sup_{\alpha_1^*/2+\varepsilon \le t \le \gamma} |M^N(t)| > \eta\right) + \mathbb{P}\left(\left(\frac{L_2^N(N^{\alpha_1^*/2+\varepsilon})}{N^{\alpha_1^*/2+\varepsilon}} - \mu_2(\rho_2 - \rho_1)\right)^2 > \eta\right)$$

$$+ \mathbb{P}\left(\sup_{\alpha_1^*/2+\varepsilon \le t \le \gamma} \frac{\mu_2}{\mu_1} \frac{L_1^N(N^t)}{N^t} \left|\frac{L_2^N(N^t)}{N^t} - \mu_2(\rho_2 - \rho_1)\right| > \eta\right)$$

$$+ \mathbb{P}\left(\log N \int_{\alpha_1^*/2+\varepsilon}^{\gamma} \left(\frac{L_2^N(N^u)}{N^u} - \mu_2(\rho_2 - \rho_1)\right)^2 \mathrm{d}u > \eta\right)$$

$$+ \mathbb{P}\left(\int_{\alpha_1^*/2+\varepsilon}^{\gamma} |\psi^N(L_1^N(N^u), L_2^N(N^u), u)| \, \mathrm{d}u > \eta\right).$$

Again, by the Markov Inequality and the estimates obtained before, each of the six terms in the right hand side of the last relation converges to 0 as N becomes large. This shows the desired uniform convergence on the time interval $(\alpha_1^*/2, \gamma]$.

The third and last step extends the convergence result to the whole interval $(\alpha_1^*/2, \alpha_1^* \wedge 1)$. Let $\varepsilon \in (0, \alpha_2^* - \alpha_1^*)$ and $\eta \in (0, \varepsilon)$. Let ν_0^N be defined by

$$\nu_0^N \overset{\text{def.}}{=} \inf\{s \ge \gamma : L_2^N(N^s) \le N^{s-\eta}\}.$$

The results of the first part of this proof show that, for some constant C_{10},

$$\lim_{N \to +\infty} \mathbb{P}\left(L_2^N(N^\gamma) \ge C_{10} N^\gamma, L_1^N(N^\gamma) \in [N^{\alpha_1^* - \gamma - \varepsilon}, N^{\alpha_1^* - \gamma + \varepsilon}]\right) = 1.$$

Thus, $\nu_0^N > \gamma$ and the process $(L_1^N(s), N^\gamma \le s \le N^{\nu_0^N})$ is stochastically bounded from above by $\lceil N^{\alpha_1^* - \gamma + \varepsilon}\rceil + X_{3,+}(\cdot - N^\gamma)$, where $(X_{3,+}(s))$ is a birth and death process reflected at 0 for which the transition $x \mapsto x+1$ occurs at rate λ_1 and $x \mapsto x-1$ at rate

$$\mu_1 \frac{(\alpha_1^* - \gamma + \varepsilon) + (\gamma - \eta)}{1 + \alpha_1^* - \gamma + \varepsilon + \gamma - \eta} = \lambda_1 + \mu_1 \frac{\varepsilon - \eta}{(1 + \alpha_1^*)(1 + \alpha_1^* + \varepsilon - \eta)},$$

since $\mu_1 \alpha_1^*/(1 + \alpha_1^*) = \lambda_1$. Since $\varepsilon < \eta$, the drift of X is negative and one can conclude that

$$\lim_{N \to \infty} \mathbb{P}\left(\sup_{t \in [\gamma, \nu_0^N]} L_1^N(N^t) \ge 2N^{\alpha_1^* - \gamma + \varepsilon}\right) = 0.$$

As a consequence, the process $(L_2^N(N^\gamma + s), 0 \le s \le N^{\nu_0^N} - N^\gamma)$ is stochastically bounded from below by the birth and death process $(X_4(s))$, with $X_4(0) = C_{10} N^\gamma$, birth rate λ_2 and transitions $x \mapsto x-1$ occurring at rate

$$\mu_2 \frac{(\alpha_1^* - \gamma + \varepsilon) + t}{1 + \alpha_1^* - \gamma + \varepsilon + t}.$$

Let $\Delta \overset{\text{def.}}{=} \alpha_2^* - (\alpha_1^* + \varepsilon)$. By our choice of ε, $\Delta > 0$ and

$$\lambda_2 > \mu_2 \frac{(\alpha_1^* - \gamma + \varepsilon) + t}{1 + \alpha_1^* - \gamma + \varepsilon + t} \quad \text{as long as} \quad t - \gamma \le \Delta.$$

Hence, there exists $C_{11} > 0$ such that

$$\lim_{N \to \infty} \mathbb{P}\left(\inf_{t \in [\gamma, \gamma + \Delta]} \frac{L_2^N(N^t)}{N^t} < C_{11} \right) = 0,$$

and in particular $\nu_0^N > \gamma + \Delta$. Now that it has been proved that, with probability tending to 1, $L_2^N(N^t) \geq C_{11} N^t$ for any $t \in [\gamma, \gamma + \Delta]$, one can adapt Lemmas 1 and 2 and the first part of the proof of Theorem 1 to show that the uniform convergence holds on the time interval $(\alpha_1^*/2, (\gamma + \Delta) \wedge \alpha_1^* \wedge 1)$ too.

Finally, since the definition of Δ does not depend on γ, one can proceed by induction (in finitely many steps) and conclude that

$$\lim_{N \to \infty} \left(\left(Y_1^N(N^t) + Y_2^N(N^t), \frac{L_2^N(N^t)}{N^t} \right) \right) = (\alpha_1^*, \mu_2(\rho_2 - \rho_1))$$

for the convergence in distribution of processes on $(\alpha_1^*/2, \alpha_1^* \wedge 1)$. The proposition is proved. □

Proposition 4 establishes the behavior of L_2^N stated in Theorem 1. It remains to show that, on this interval of time, the convergence

$$\lim_{N \to +\infty} \left(\frac{L_1^N(N^t)}{N^{\alpha_1^* - t}} \right) = \left(\frac{1}{\mu_2(\rho_2 - \rho_1)} \right)$$

holds. Equivalently, one shows the following important result.

Proposition 5. *The convergence in distribution of processes*

$$\lim_{N \to \infty} \left(\frac{L_1^N(N^t) L_2^N(N^t)}{N^{\alpha_1^*}} \right) = (1)$$

holds on the time interval $(\alpha_1^*/2, \alpha_1^* \wedge 1)$.

The proof uses several technical lemmas, whose proofs are given in the Appendix. It is based on the idea that, for a given value of L_2^N, if L_1^N moves too far apart from its equilibrium value $N^{\alpha_1^*}/L_2^N$ (corresponding to the point where the drift of L_1^N cancels), then it is driven back to this value in much less time than L_2^N needs to change.

More precisely, for $\eta \in (0, \alpha_1^*/2)$, the time interval $[N^{\alpha_1^*/2 + \eta}, N^{(\alpha_1^* \wedge 1) - \eta}]$ can be covered by $N^{\alpha_1^*/2}$ (at most) sub-intervals of length $N^{\alpha_1^*/2}$. One will first consider an interval of the form $[T_N, T_N + N^{\alpha_1^*/2}]$ and $\varepsilon > 0$ such that $L_1^N L_2^N(T_N) < (1 + 3\varepsilon/2) N^{\alpha_1^*}$. As one will see, with probability tending to 1 the process $(L_1^N L_2^N(s))$ does not exceed the value $(1 + 2\varepsilon) N^{\alpha_1^*}$ on the time interval $[T_N, T_N + N^{\alpha_1^*/2}]$.

In a second step, one will show that $L_1^N L_2^N(T_N + N^{\alpha_1^*/2}) \leq (1 + 3\varepsilon/2) N^{\alpha_1^*}$, so that the same result can be applied to the next time interval of width $N^{\alpha_1^*/2}$.

First, let $\tau^N(T_N)$ be the stopping time defined by

$$(14) \quad \tau^N(T_N) \stackrel{\text{def.}}{=} \inf \left\{ s \geq T_N : L_1^N(s) L_2^N(s) \leq (1 + \varepsilon) N^{\alpha_1^*} \right\} \wedge (T_N + N^{\alpha_1^*/2}).$$

Of course, when $L_1^N L_2^N(T_N) \leq (1 + \varepsilon) N^{\alpha_1^*}$ one has $\tau^N(T_N) = T_N$. On the other hand, when $L_1^N L_2^N(T_N)/N^{\alpha_1^*} \in (1 + \varepsilon, 1 + 3\varepsilon/2)$, the following lemma controls the probability that $L_1^N L_2^N$ exceeds $(1 + 2\varepsilon) N^{\alpha_1^*}$ on the time interval $[T_N, \tau^N(T_N)]$.

Lemma 3. *Assume that $L_1^N L_2^N(T_N)/N^{\alpha_1^*} \in (1+\varepsilon, 1+3\varepsilon/2)$. Then, there exists a constant $C_1(\varepsilon) > 0$ which is independent of the value of $L_1^N L_2^N(T_N)$ and such that*

$$\mathbb{P}\left(\sup_{s \in [T_N, \tau^N(T_N)]} L_1^N L_2^N(s) \geq (1+2\varepsilon)N^{\alpha_1^*}\right) \leq \exp\left(-C_1 \frac{N^{\alpha_1^*}}{T_N \log N}\right).$$

The proof of this lemma is given in Appendix A.3.

The next lemma shows that $\tau^N(T_N)$ is negligible compared to $N^{\alpha_1^*/2}$.

Lemma 4. *Suppose that $L_1^N L_2^N(T_N)/N^{\alpha_1^*} \in (1+\varepsilon, 1+3\varepsilon/2)$. Then there exists $C_2(\varepsilon) > 0$ independent of the value of $L_1^N L_2^N(T_N)$ and such that*

$$\mathbb{P}\left(\tau^N(T_N) - T_N > \frac{N^{\alpha_1^*/2}}{\log N}\right) \leq \exp\left(-C_2 \frac{N^{\alpha_1^*/2}}{(\log N)^2}\right).$$

Lemma 4 is proven in Appendix A.4.

The third lemma controls the probability that $(L_1^N L_2^N(s)/N^{\alpha_1^*})$ reaches again the value $(1+3\varepsilon/2)$ when it starts below $1+\varepsilon$.

Lemma 5. *Suppose that $L_1^N L_2^N(T_N) \leq (1+\varepsilon)N^{\alpha_1^*}$. Then, there exists $C_3(\varepsilon) > 0$ independent of the initial value of $L_1^N L_2^N$ and such that*

$$\mathbb{P}\left(\sup_{s \in [T_N, T_N + N^{\alpha_1^*/2}]} L_1^N L_2^N(s) \geq \left(1 + \frac{3}{2}\varepsilon\right)N^{\alpha_1^*}\right) \leq \exp\left(-C_3 \frac{N^{\alpha_1^*}}{T_N \log N}\right).$$

The proof of this lemma is given in Appendix A.3.

These lemmas are used as follows. If

$$\mathcal{E}_N \stackrel{\text{def.}}{=} \left\{L_1^N L_2^N(T_N) \leq (1+3\varepsilon/2)N^{\alpha_1^*}\right\},$$

and if one defines $I_N = [T_N, T_N + N^{\alpha_1^*/2}]$ and $J_N = [\tau^N(T_N), \tau^N(T_N) + N^{\alpha_1^*/2}]$, then

$$\mathbb{P}\left(\left\{\sup_{s \in I_N} L_1^N L_2^N(s) > (1+2\varepsilon)N^{\alpha_1^*}\right\} \bigcup \left\{L_1^N L_2^N(T_N + N^{\alpha_1^*/2}) \geq \left(1 + \frac{3}{2}\varepsilon\right)\right\} \Big| \mathcal{E}_N\right)$$

$$\leq \mathbb{P}\left(\sup_{s \in [T_N, \tau^N(T_N)]} L_1^N L_2^N(s) > (1+2\varepsilon)N^{\alpha_1^*} \Big| \mathcal{E}_N\right) + \mathbb{P}\left(\tau^N(T_N) - T_N > \frac{N^{\alpha_1^*/2}}{\log N} \Big| \mathcal{E}_N\right)$$

$$+ \mathbb{P}\left(\sup_{s \in J_N} L_1^N L_2^N(s) > \left(1 + \frac{3}{2}\varepsilon\right)N^{\alpha_1^*} \Big| \mathcal{E}_N\right).$$

The first term in the right hand side of the above relation is controlled by the inequality of Lemma 3, the second term by Lemma 4 and the third one by Lemma 5 with T_N replaced by $\tau^N(T_N)$. One finally obtains the existence of a constant $C_4(\varepsilon) > 0$ such that

$$\mathbb{P}\left(\left\{\sup_{s \in I_N} L_1^N L_2^N(s) > (1+2\varepsilon)N^{\alpha_1^*}\right\}\right.$$

$$\left.\bigcup \left\{L_1^N L_2^N(T_N + N^{\alpha_1^*/2}) \geq \left(1 + 3/2\,\varepsilon\right)\right\} \Big| \mathcal{E}_N\right) \leq \exp\left(-C_4 \frac{N^\eta}{\log N}\right)$$

holds for N sufficiently large. Since $[N^{\alpha_1^* + \eta}, N^{(\alpha_1^* \wedge 1) - \eta}]$ can be covered by at most $N^{\alpha_1^*/2}$ intervals of length $N^{\alpha_1^*/2}$, it follows that

$$\mathbb{P}\left(\sup_{s \in [N^{\alpha_1^* + \eta}, N^{(\alpha_1^* \wedge 1) - \eta}]} L_1^N L_2^N(s) > (1 + 2\varepsilon)N^{\alpha_1^*} \right)$$

$$\leq \mathbb{P}\left(L_1^N L_2^N(N^{\alpha_1^*/2 + \eta}) \geq \left(1 + \frac{3}{2}\varepsilon\right)N^{\alpha_1^*} \right) + N^{\alpha_1^*/2} e^{-C_4 N^\eta / \log N}.$$

It remains to show that the first term in the right hand side of the inequality just above converges to 0 as N tends to infinity. This corresponds to the following result.

Lemma 6. *The convergence*

$$\lim_{N \to \infty} \mathbb{P}\left(L_1^N L_2^N(N^{\alpha_1^*/2 + \eta}) \geq \left(1 + \frac{3}{2}\varepsilon\right)N^{\alpha_1^*} \right) = 0$$

holds.

See the proof of this lemma in Appendix A.5.

One can therefore conclude that

(15) $$\lim_{N \to \infty} \mathbb{P}\left(\sup_{s \in [N^{\alpha_1^* + \eta}, N^{(\alpha_1^* \wedge 1) - \eta}]} L_1^N L_2^N(s) > (1 + 2\varepsilon)N^{\alpha_1^*} \right) = 0.$$

Similar arguments give an estimation for the lower bound:

$$\lim_{N \to \infty} \mathbb{P}\left(\inf_{s \in [N^{\alpha_1^* + \eta}, N^{(\alpha_1^* \wedge 1) - \eta}]} L_1^N L_2^N(s) < (1 - 2\varepsilon)N^{\alpha_1^*} \right) = 0,$$

and since this conclusion holds for any $\varepsilon > 0$, Proposition 5 is proved. Combining Propositions 4 and 5, Theorem 1 is proved.

3.2. **Proof of Proposition 3.** Again, one starts by establishing some crude bounds on the number of pending requests in nodes 1 and 2 over the time interval of interest. Recall the notation $\kappa = \mu_2(\rho_2 - \rho_1)$.

Lemma 7. *For $\varepsilon > 0$ sufficiently small there exists $C_\varepsilon > 0$ such that*

$$\lim_{N \to \infty} \mathbb{P}\left(\sup_{s \in I_N} L_1(s) < N^\varepsilon, \inf_{s \in I_N} \frac{L_2(s)}{s} \geq C_\varepsilon \right) = 1,$$

with $I_N = [N^{\alpha_1^* - \varepsilon/2}, N^{(\alpha_2^* \wedge 1) - 2\varepsilon}]$.

As before, for the sake of clarity the proof of this lemma is postponed until Appendix A.5.

One can now complete the proof of Proposition 3.

Proof of a) of Proposition 3. Let $\varepsilon > 0$. From Lemma 7, one knows that with probability tending to 1, $L_1^N(s)$ remains below N^ε and $L_2^N(s)/s$ remains above C_ε on the time interval $[N^{\alpha_1^* - \varepsilon/4}, N^{(\alpha_2^* \wedge 1) - 2\varepsilon}]$.

Hence, on the sub-interval $[N^{\alpha_1^*} \log N, N^{(\alpha_2^* \wedge 1) - 2\varepsilon}]$, $(L_1^N(t))$ is stochastically bounded from above by $N^\varepsilon + X_+(\cdot - N^{\alpha_1^*} \log N)$, where X_+ is a birth and death process starting at 0, reflected at $-N^\varepsilon$, with birth rate λ_1 and death rate

$$\mu_1 \frac{\alpha_1^* + \log(C_\varepsilon \log N)/\log N}{1 + \alpha_1^* + \log(C_\varepsilon \log N)/\log N} = \lambda_1 + C \frac{\log \log N}{\log N}$$

for some $C > 0$. Standard estimates on random walks thus yield

$$\lim_{N \to \infty} \mathbb{P}\big((X_+(t)) \text{ does not hit } (-N^\varepsilon) \text{ before } N^\varepsilon \log N\big) = 0,$$

from which the result follows. $\qquad\square$

Proof of b) of Proposition 3. The same coupling as in the proof of a) of Proposition 3 still holds on the interval $[\theta_0^N, N^{(\alpha_2^* \wedge 1) - 2\varepsilon}]$ (replacing the initial value N^ε by 0 and reflecting X_+ at 0 instead of $-N^\varepsilon$). In particular, by Proposition 1b)

$$\lim_{N \to \infty} \mathbb{P}\left((X_+(t)) \text{ reaches } \frac{(\log N)^2}{\sqrt{\log \log N}} \text{ before time } N^{\alpha_1^* + \varepsilon}\right) = 0.$$

Next, since $(L_2^N(s))$ increases linearly on the time interval $[N^{\alpha_1^* + \varepsilon}, N^{(\alpha_2^* \wedge 1) - 2\varepsilon}]$, another coupling in which $(X_+(t))$ has infinitesimal drift $-C\varepsilon$ (due to the fact that $s \geq N^{\alpha_1^* + \varepsilon}$) and the initial value is $(\log N)^2/\sqrt{\log \log N}$) shows that

$$\lim_{N \to \infty} \mathbb{P}\left(\sup_{s \in [N^{\alpha_1^* + \varepsilon}, N^{(\alpha_2^* \wedge 1) - 2\varepsilon}]} L_1^N(s) > (\log N)^2\right) = 0,$$

by Proposition 1. These two facts combined give the result. $\qquad\square$

Proof of c) of Proposition 3. Fix $\varepsilon > 0$ small. Since $\theta_0^N < N^{\alpha_1^* + \varepsilon}$ with probability tending to 1 by a) of Proposition 3, by b) of Proposition 3 one has that $(L_1^N(t))$ is bounded by $(\log N)^2$ on the time interval $[N^{\alpha_1^* + \varepsilon}, N^{(\alpha_2^* \wedge 1) - \varepsilon}]$. A proof similar to that of Proposition 2 in [8] then gives the result. $\qquad\square$

3.3. **Proof of Theorem 2.** Fix $\varepsilon > 0$ small. Since $(L_0^N(t))$ is stochastically bounded from above by a Poisson process with rate λ_0, and from below by N minus a Poisson process with rate μ_0, if $\eta \leq \min\{\varepsilon/(2\lambda_0), \varepsilon/(2\mu_0)\}$ one has

$$(16) \qquad \lim_{N \to \infty} \mathbb{P}\left(\sup_{s \in [0, \eta N]} \left|\frac{L_0^N(s)}{N} - 1\right| \leq \varepsilon\right) = 1.$$

Suppose the conditions of case 1) Theorem 2 are satisfied. It is easy to see that Proposition 2 holds on the interval $(0, 1 + (\log \eta)/\log N]$. Hence,

$$\lim_{N \to \infty} \mathbb{P}\left(\left|\frac{L_0^N(\eta N)}{N} - 1\right| \leq \varepsilon, \left|\frac{L_1^N(\eta N)}{N} - \mu_1\left(\rho_1 - \frac{2}{3}\right)\eta\right| \leq \varepsilon, \right.$$
$$\left. \left|\frac{L_2^N(\eta N)}{N} - \mu_2\left(\rho_2 - \frac{2}{3}\right)\eta\right| \leq \varepsilon\right) = 1.$$

From ηN on, the processes $(L_0^N(t))$, $(L_1^N(t))$ and $(L_2^N(t))$ are all of the order of N. Recalling the definition (1) of the quantity $W(L)$, one can thus conclude that the processes of the number of requests $(L_1^N(t))$ and $(L_2^N(t))$ receive a fraction $2/3$ of the available capacity, while the number of requests in the central node $(L_0^N(t))$ receive $1/3$ of the available capacity. By coupling $(L_0^N(\eta N + Nt), L_1^N(\eta N + Nt), L_2^N(\eta N + Nt))$ with the solutions to the system (6) starting from the extremal values

$$(1 \pm \varepsilon, \mu_1(\rho_1 - 2/3)\eta \pm \varepsilon, \mu_2(\rho_2 - 2/3)\eta \pm \varepsilon),$$

one obtains that for any $T \in [\eta, t_0 - \eta)$,

$$\lim_{N \to +\infty} \mathbb{P}\left(\sup_{t \in [\eta, T]} \left| \frac{L_0^N(Nt)}{N} - 1 - \mu_0 \left(\rho_0 - \frac{1}{3} \right) t \right| \leq 2\varepsilon, \right.$$

$$\sup_{t \in [\eta, T]} \left| \frac{L_1^N(Nt)}{N} - \mu_1 \left(\rho_1 - \frac{2}{3} \right) t \right| \leq 2\varepsilon,$$

$$\left. \sup_{t \in [\eta, T]} \left| \frac{L_2^N(Nt)}{N} - \mu_2 \left(\rho_2 - \frac{2}{3} \right) t \right| \leq 2\varepsilon \right) = 1.$$

This result shows in particular that $L_0^N(Nt)$ becomes negligible compared to N when t approaches t_0, hence the bound on the interval of time considered. Since η can be chosen as small as one wants, this proves the desired uniform convergence on $(0, t_0)$.

Suppose now that the conditions of case 2) of Theorem 2 are satisfied. Using Relation (16), Theorem 1 can be extended to the time interval $[\alpha_1^*/2, 1 + (\log \eta)/\log N]$. Consequently, with probability tending to 1, $L_0^N(\eta N)$ and $L_2^N(\eta N)$ are both of the order of N while $L_1^N(\eta N)$ is of the order of $N^{\alpha_1^* - 1}$. Then a close look at the proof of Proposition 5 reveals that $L_1^N L_2^N / N^{\alpha_1^*}$ converges to 1 as long as L_2^N is of the order of N. Consequently, one obtains that, on the time interval of interest, node 0 receives a fraction $1/(1 + \alpha_1^*) = 1 - \rho_1$ of the available capacity, and nodes 1 and 2 receive a fraction $\alpha_1^*/(1 + \alpha_1^*) = \rho_1$. Using the coupling with the system (6) starting at the extremal values mentioned in the previous paragraph, one can then conclude.

Assuming that the conditions of case 3) of Theorem 2 are satisfied, Proposition 3 can be extended to the time interval $(\alpha_1^*, 1 + (\log \eta)/\log N]$, showing that this time $L_0^N(\eta N)$ and $L_2^N(\eta N)$ are of order N while $L_1^N(\eta N) \leq (\log N)^2$ is negligible compared to any power of N. Thus, as long as $(L_0^N(t))$ and $(L_2^N(t))$ remain of order N, the same proof as that of b) of Proposition 3 guarantees that with probability tending to 1, $(L_1^N(t))$ remains below $(\log N)^2$. In particular, by the definition (1) of $W(L)$, this means that each of the nodes 0 and 2 receives a fraction $1/2$ of the available capacity. The conclusion follows from the same arguments as above (see the proofs of Theorem 4 and Proposition 8 in [8] for more details).

Finally, the same reasoning together with Theorem 3 in [8] prove the result of case 4) of Theorem 2.

4. GENERAL CASE

In this section, one extends the results of Section 3 to the case $J \geq 3$. As before, jobs arrive at each queue i according to a Poisson process with rate λ_i and each job has an exponential service time with parameter μ_i. Recall that when the Markov process is in state $L = (L_j)$, for every $i \in \{1, \ldots, J\}$, queue i receives the fraction of available capacity

$$W(L) = \sum_{j=1}^{J} \log(1 + L_j) \left/ \sum_{j=0}^{J} \log(1 + L_j) \right.$$

while queue 0 receives

$$1 - W(L) = \log(1 + L_0) \bigg/ \sum_{j=0}^{J} \log(1 + L_j) \ .$$

The initial state is $(L_0^N(0), \ldots, L_J^N(0)) = (N, 0, \ldots, 0)$ and α_i^* denotes $\rho_i/(1 - \rho_i)$, where $\rho_i = \lambda_i/\mu_i$. The nodes with indices greater than or equal to 1 are numbered so that $\rho_1 < \rho_2 < \cdots < \rho_J$.

Theorem 5 at the end of this section summarizes the results obtained on the fluid time scale. Because most of its proof consists in using or slightly adapting the arguments presented in Section 3, below one only details the reasoning for the particularly interesting case when only queues 0 and J are non trivial in the fluid regime. One will first analyze the network on the time interval $[0, N^{\alpha_1^*/(J-1)}]$, and then on $[N^{\alpha_1^*/(J-1)}, +\infty)$. Concerning the first interval, the results are analogous to those obtained in Section 3 and their proofs are very similar (if not identical). For this reason, only the non-obvious modifications will be given. Concerning the second time interval, assuming that $\alpha_1^*/(J-1) < 1$, one will show that after time $N^{\alpha_1^*/(J-1)}$, the process $(L_1^N(t))$ remains negligible compared to the sizes of the other queues and therefore does not contribute to $W(L^N)$. As a consequence, the impact of queue 1 on the other queues can be ignored, one is left with a system of J queues, and a simple recurrence then concludes the study.

The results concerning the first phase on the time interval $[0, N^{\alpha_1^*/(J-1)}]$ are the following. Their proofs are sketched towards the end of this section.

Proposition 6. *The convergence in distribution*

$$\lim_{N \to \infty} \left(\frac{L_1^N(N^t)}{N^t}, \ldots, \frac{L_J^N(N^t)}{N^t} \right) = \left(\lambda_1 - \mu_1 \frac{Jt}{1 + Jt}, \ldots, \lambda_J - \mu_J \frac{Jt}{1 + Jt} \right)$$

holds on the time interval $\left(0, \alpha_1^*/J \wedge 1 \right)$.

Next, assuming that $\alpha_1^*/J < 1$, once again there exists $\gamma \in (\alpha_1^*/J, (\alpha_2^*/J) \wedge 1)$ such that the infinitesimal drift of each of the queues with index between 2 and J remains positive up to time N^γ. As in Section 3, this leads to the following theorem.

Theorem 3. *The convergence in distribution*

$$\lim_{N \to \infty} \left(\frac{L_1^N(N^t)}{N^{\alpha_1^* - (J-1)t}}, \frac{L_2^N(N^t)}{N^t}, \ldots, \frac{L_J^N(N^t)}{N^t} \right)$$

$$= \left(\prod_{j=2}^{J} \frac{1}{\mu_j(\rho_j - \rho_1)}, \mu_2(\rho_2 - \rho_1), \ldots, \mu_J(\rho_J - \rho_1) \right)$$

holds on the time interval $(\alpha_1^*/J, \alpha_1^*/(J-1) \wedge 1)$.

Finally, concerning the second phase $[N^{\alpha_1^*/(J-1)}, +\infty)$ one has the following analogue of Proposition 3.

Proposition 7. *Under the condition $\alpha_1^*/(J-1) < 1$, if*

$$\theta_0^N = \inf\{t > 0 : L_1^N(t) = 0\},$$

then

1) *for $\varepsilon > 0$, one has*

$$\lim_{N \to \infty} \mathbb{P}\left(\theta_0^N \leq N^{\alpha_1^*/(J-1)} \log N + N^\varepsilon \log N\right) = 1.$$

2) *For $\varepsilon \in (0, \alpha_2^*/(J-1) \wedge 1)$, one has*

(17)
$$\lim_{N \to \infty} \mathbb{P}\left(\sup_{s \in [\theta_0^N, N^{(\alpha_2^*/(J-1)\wedge 1)-2\varepsilon}]} L_1^N(s) \leq (\log N)^2\right) = 1.$$

3) *The convergence of processes*

(18) $$\lim_{N \to \infty} \left(\frac{L_2^N(N^t)}{N^t}, \ldots, \frac{L_J^N(N^t)}{N^t}\right)$$

$$= \left(\lambda_2 - \mu_2 \frac{(J-1)t}{1+(J-1)t}, \ldots, \lambda_J - \mu_J \frac{(J-1)t}{1+(J-1)t}\right)$$

holds on the time interval $(\alpha_1^*/(J-1), \alpha_2^*/(J-1) \wedge 1)$.

Consequently, Relations (17) and (18) tell us that from time $N^{\alpha_1^*/(J-1)}$ on, queue 1 does not contribute to the quantity $W(L^N)$. One is thus left with a network with J nodes indexed by $0, 2, \ldots, J$ and starting from the state

$$L_0^N \sim N \text{ and } L_j^N \sim \mu_j(\rho_j - \rho_1)N^{\alpha_1^*/(J-1)}, \quad 2 \leq j \leq J.$$

Under the condition $\alpha_2^*/(J-1) < 1$, at time $N^{\alpha_2^*/(J-1)}$ the infinitesimal drift of L_2^N cancels while the infinitesimal drifts of the processes $(L_3^N(t)), \ldots, (L_J^N(t))$ remain positive for some time. Consequently, the processes $(L_3^N(N^t)), \ldots, (L_J^N(N^t))$ grow proportionally to N^t. At the same time, the product $(L_2^N \cdots L_J^N(N^t))$ remains close to $N^{\alpha_2^*}$, and so $(L_2^N(N^t))$ decreases like $N^{\alpha_2^*-(J-2)t}$. As before, once $(L_2^N(t))$ has reached 0, it remains below $(\log N)^2$ with probability tending to 1. From time $N^{\alpha_2^*/(J-2)}$ on, one is left with a system of $J-1$ queues, and so on.

As mentioned earlier, the following theorem describes the most interesting case, in which only queues 0 and J are non trivial on the fluid time scale. Its proof is similar to that of Theorem 2 and is therefore omitted.

Theorem 4. *Under the condition $\alpha_J^* < 1$, the convergence in distribution*

$$\lim_{N \to \infty} \left(\frac{L_0^N(Nt)}{N}, \frac{L_1^N(Nt)}{(\log N)^3}, \ldots, \frac{L_{J-1}^N(Nt)}{(\log N)^3}, \frac{L_J^N(Nt)}{N^{\alpha_J^*}}\right) = \left(\gamma_0(t), 0, \ldots, 0, \gamma_0(t)^{\alpha_J^*}\right)$$

holds on the time interval $(0, +\infty)$ with $\gamma_0(t) = (1+\mu_0(\rho_0+\rho_J-1)t)^+$.

Before formulating the most general result that can be obtained in the case $J \geq 2$, let us give the main modifications to the proof of Theorem 1 required to prove Theorem 3.

Sketch of the proof of Theorem 3. The analogue of the function $F(\cdot)$ of the proof of Proposition 4 is given by the functions

(19) $$F_j(\ell, t) = \frac{1}{2}\left(\frac{l_j}{N^t} - \mu_j(\rho_j - \rho_1)\right)^2 - \frac{\mu_j}{\mu_1}\frac{l_1}{N^t}\left(\frac{l_j}{N^t} - \mu_j(\rho_j - \rho_1)\right),$$

for $j \geq 2$, $\ell = (l_1, \ldots, l_J) \in \mathbb{N}^J$ and $t \geq 0$. The infinitesimal generator G^N of the Markov process $(L_1(N^t), \ldots, L_J(N^t), t)$ applied to this function F_j yields

$$G^N(F_j)(\ell, t) = -(\log N) \left(\frac{l_j}{N^t} - \mu_j(\rho_j - \rho_1) \right)^2 + C_{j,1}^N(\ell, t) \frac{l_1 l_j}{N^{2t}} \log N$$

$$+ C_{j,2}^N(\ell, t) \frac{l_1}{N^t} \log N + C_{j,3}^N(\ell, t) \frac{\log N}{N^t},$$

where the functions $C_{j,1}^N(\cdot)$, $C_{j,2}^N(\cdot)$ and $C_{j,3}^N(\cdot)$ are bounded by some constant $K > 0$, uniformly in their arguments and in $N \geq 1$. Using the corresponding martingale problem for each $j \in \{2, \ldots, J\}$ separately, the same arguments as in the proof of Proposition 4 carry over and lead to the uniform convergence of each coordinate. From this, it is straightforward to conclude. □

One concludes by gathering some of the results of the paper into the following theorem. It is restricted to the time interval where the central node is still in the fluid scale regime, i.e. of the order of N. The quantity κ defined in this theorem is related to the number of nodes which can be removed without changing the behavior of the other nodes on the fluid time scale.

If the central node 0 becomes empty, the formulation of the results after that instant is not difficult. It corresponds to the case where the central node and a subset of the other nodes are at equilibrium, in the sense that their numbers of requests are $o((\log N)^3)$ on a finite time interval on the fluid time scale. Analogous results can be stated when the initial state $L^N(0)$ given by Relation (3) is changed in the following way:

$$L^N(0) = (L_j^N(0)) = N \cdot (\ell_0, \ldots, \ell_J) + o(N),$$

where $(\ell_j) \in \mathbb{R}_+^{J+1}$ and $\ell_0 + \cdots + \ell_J = 1$.

Theorem 5 (Convergence on the Fluid Time Scale). *Suppose that $\rho_1 < \rho_2 < \cdots < \rho_J$, recall that*

$$\alpha_j^* = \frac{\rho_j}{1-\rho_j},$$

and let, for $1 \leq j \leq J$,

$$\beta_j^* \stackrel{\text{def.}}{=} \frac{\alpha_j^*}{J-j}, \qquad \kappa \stackrel{\text{def.}}{=} \sup\left\{k : \frac{\alpha_k^*}{J-k+1} < 1\right\}$$

with the convention that $\sup(\emptyset)=0$. Condition (C) is that either $\kappa=0$ or that $1 \leq \kappa < J$ and $\beta_\kappa^* < 1$.

For $j \geq 1$ and $t \geq 0$, define

$$(\gamma_0(t), \gamma_j(t)) = \begin{cases} \left(1 + \mu_0\left(\rho_0 - \dfrac{1}{J-\kappa+1}\right)t, \mu_j\left(\rho_j - \dfrac{J-\kappa}{J-\kappa+1}\right)t\right) & \text{if (C) holds} \\ \left(1 + \mu_0\left(\rho_0 + \rho_\kappa - 1\right)t, \mu_j\left(\rho_j - \rho_\kappa\right)t\right), & \text{otherwise,} \end{cases}$$

and

$$t_0 \stackrel{\text{def.}}{=} \begin{cases} \dfrac{J-\kappa+1}{\mu_0(1-\rho_0(J-\kappa+1))^+} & \text{if (C) holds,} \\ \dfrac{1}{\mu_0(1-\rho_0-\rho_\kappa)^+} & \text{otherwise.} \end{cases}$$

The following convergences in distribution of processes hold on the time interval $(0, t_0)$.

1) *If $\kappa=0$,*

$$\lim_{N\to\infty} \left(\frac{L_0^N(Nt)}{N}, \ldots, \frac{L_J^N(Nt)}{N}\right) = (\gamma_0(t), \gamma_1(t), \ldots, \gamma_J(t));$$

2) *In the case $1 \leq \kappa < J$, there are two possible behaviors depending on β_κ^*,*

   (a) *If $\beta_\kappa^* < 1$,*

$$\lim_{N\to\infty} \left(\frac{L_0^N(Nt)}{N}, \frac{L_1^N(Nt)}{(\log N)^3}, \ldots, \frac{L_\kappa^N(Nt)}{(\log N)^3}, \frac{L_{\kappa+1}^N(Nt)}{N}, \ldots, \frac{L_J^N(Nt)}{N}\right)$$
$$= (\gamma_0(t), 0^{(\kappa)}, \gamma_{\kappa+1}(t), \ldots, \gamma_J(t)),$$

   where $0^{(\kappa)}$ is the κth dimensional zero vector;

   (b) *If $\beta_\kappa^* > 1$,*

$$\lim_{N\to\infty} \left(\frac{L_0^N(Nt)}{N}, \frac{L_1^N(Nt)}{(\log N)^3}, \ldots, \frac{L_{\kappa-1}^N(Nt)}{(\log N)^3}, \frac{L_\kappa^N(Nt)}{N^{\alpha_\kappa^*-(J-\kappa)}}, \frac{L_{\kappa+1}^N(Nt)}{N}, \ldots, \frac{L_J^N(Nt)}{N}\right)$$
$$= \left(\gamma_0(t), 0^{(\kappa-1)}, \frac{1}{\gamma_{\kappa+1}(t)\gamma_{\kappa+2}(t)\cdots\gamma_J(t)}, \gamma_{\kappa+1}(t), \ldots, \gamma_J(t)\right),$$

3) *If $\kappa = J$,*

$$\lim_{N\to\infty} \left(\frac{L_0^N(Nt)}{N}, \frac{L_1^N(Nt)}{(\log N)^3}, \ldots, \frac{L_{J-1}^N(Nt)}{(\log N)^3}, \frac{L_J^N(Nt)}{N^{\alpha_J^*}}\right) = \left(\gamma_0(t), 0^{(J-1)}, \gamma_0(t)^{\alpha_J^*}\right).$$

Note that, by definition of κ, we have

$$\frac{\alpha_\kappa^*}{J-\kappa+1} < 1 \leq \frac{\alpha_{\kappa+1}^*}{J-\kappa} \quad \text{and} \quad \beta_\kappa^* \in \left(\frac{\alpha_\kappa^*}{J-\kappa+1}, \frac{\alpha_{\kappa+1}^*}{J-\kappa}\right).$$

Cases 2a) or 2b) depend on β_κ^* being before or after 1 in the last time interval. Either the queue with index κ has the time to come back to 0 on the time scale $(N^t, t \in (0,1))$, corresponding to case 2)a), or it does not, and this is case 2)b). All other results are direct consequences of Theorems 2 and 4.

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Appendix A. Technical Results

A.1. **Proof of Lemma 1.** The processes $(L_1^N(s))$ and $(L_2^N(s))$ are stochastically bounded from above by their arrival processes, which are Poisson processes with respective rates λ_1 and λ_2. Hence, by the Ergodic Theorem for Poisson processes, for each $i \in \{1,2\}$ one has $\limsup_{s \to \infty} L_i^N(s)/s \le \lambda_i$ and so

$$(20) \quad \lim_{N \to \infty} \mathbb{P}\left(\sup_{s \in [N^{\alpha_1^*/4}, N^\gamma]} \frac{L_2^N(s)}{s} < 2\lambda_2 \right)$$
$$= \lim_{N \to \infty} \mathbb{P}\left(\sup_{s \in [N^{\alpha_1^*/4}, N^\gamma]} \frac{L_1^N(s)}{s} < 2\lambda_1 \right) = 1.$$

Recall the notation Y_j^N introduced in (7). Thus, for every $s \in [N^{\alpha_1^*/4}, N^\gamma]$ and some appropriate $C > 0$, one has

$$\frac{Y_1^N(s) + Y_2^N(s)}{1 + Y_1^N(s) + Y_2^N(s)} \le \frac{\log(2\lambda_1 N^\gamma) + \log(2\lambda_2 N^\gamma)}{\log N + \log(2\lambda_1 N^\gamma) + \log(2\lambda_2 N^\gamma)} \le \frac{2\gamma}{1 + 2\gamma} + \frac{C}{\log N}$$

with probability tending to 1. As a consequence, on the time interval $[N^{\alpha_1^*/4}, N^\gamma]$ the process $(L_2^N(s))$ is stochastically bounded from below by the process

$$\left(\mathcal{N}_{\lambda_2}[N^{\alpha_1^*/4}, s] - \mathcal{N}_{2\gamma\mu_2/(1+2\gamma)+C\mu_2/\log N}[N^{\alpha_1^*/4}, s]\right).$$

Since $\gamma < \rho_2/(2(1 - \rho_2))$ by assumption, one has $\lambda_2 > 2\mu_2\gamma/(1 + 2\gamma)$, which enables one to conclude that

$$(21) \qquad \lim_{N\to\infty} \mathbb{P}\left(\inf_{s\in[N^{\alpha_1^*/2}, N^\gamma]} \frac{L_2^N(s)}{s} > \frac{1}{2}\left(\lambda_2 - \mu_2\frac{2\gamma}{1 + 2\gamma}\right)\right) = 1.$$

Together with (20), this proves the first statement of the lemma. Furthermore, the event

$$\mathcal{E}_N \overset{\text{def.}}{=} \left\{\sup_{s\in[N^{\alpha_1^*/4}, N^\gamma]} \frac{L_1^N(s)}{s} < 2\lambda_1\right\} \bigcap \left\{\inf_{s\in[N^{\alpha_1^*/2}, N^\gamma]} \frac{L_2^N(s)}{s} > \frac{1}{2}\left(\lambda_2 - \mu_2\frac{2\gamma}{1 + 2\gamma}\right)\right\}$$

has a probability arbitrarily close to 1 for N large enough.

Set $\eta = (\lambda_2 - 2\gamma\mu_2/(1 + 2\gamma))/2$ and fix $A > 1$ such that $A\eta > 1$. On the event $\mathcal{E}_N$, if at some instant $t_0 \geq AN^{\alpha_1^*/2}$ one has $L_1^N(t_0) \geq N^{\alpha_1^*/2}$, then the total service rate of class 1 jobs at that time is bounded from below by

$$\mu_1 \frac{\log(N^{\alpha_1^*/2}) + \log(\eta AN^{\alpha_1^*/2})}{\log N + \log(N^{\alpha_1^*/2}) + \log(A\eta N^{\alpha_1^*/2})}$$

$$= \mu_1\frac{\alpha_1^*}{1 + \alpha_1^*} + \mu_1\frac{\log(\eta A)}{(\alpha_1^* + 1)\log N}\frac{\log N}{\log(\eta A) + \log(N)} \geq \lambda_1 + \frac{C}{\log N},$$

for all $N \geq 2$ and some constant C. Since $A\eta > 1$, one can choose $C > 0$ here. Hence, on the time interval $[AN^{\alpha_1^*/2}, N^\gamma]$, when the process $(L_1^N(s))$ is above the level $N^{\alpha_1^*/2}$, it is stochastically bounded (from above) by the process $N^{\alpha_1^*/2} + X_+(s)$, where $(X_+(s))$ is a birth and death process reflected at 0 with birth rate λ_1 and death rate $\beta_1 = \lambda_1 + C/\log N$. From the second part of Proposition 1 one obtains

$$(22) \quad \mathbb{P}\left(\sup_{s\in[AN^{\alpha_1^*/2}, N^\gamma]} X_+(s) \geq N^{\alpha_1^*/2}\right) \leq (\lambda_1 N^\gamma + 1)\left(\frac{\lambda_1}{\lambda_1 + C/\log N}\right)^{N^{\alpha_1^*/2}-1}.$$

In particular, on the event $\mathcal{E}_N$ none of the excursions of $(L_1^N(s))$ above $N^{\alpha_1^*/2}$ will exceed the value $2N^{\alpha_1^*/2}$ with a probability bounded by the quantity in the right hand side of (22). This yields

$$\limsup_{N\to+\infty} \mathbb{P}\left(\sup_{s\in[AN^{\alpha_1^*/2}, N^\gamma]} L_1^N(s) \geq 2N^{\alpha_1^*/2}\right)$$

$$= \limsup_{N\to+\infty} \mathbb{P}\left(\left\{\sup_{s\in[AN^{\alpha_1^*/2}, N^\gamma]} L_1^N(s) \geq 2N^{\alpha_1^*/2}\right\} \cap \mathcal{E}_N\right)$$

$$\leq \limsup_{N\to+\infty} \lambda_1 N^\gamma\left(\frac{\lambda_1}{\lambda_1 + C/\log N}\right)^{N^{\alpha_1^*/2}-1} = 0.$$

It remains to control $(L_1^N(s))$ on the time interval $[N^{\alpha_1^*/2}, AN^{\alpha_1^*/2}]$. This is done with the help of Relation (20), which gives the identity

$$\lim_{N\to+\infty} \mathbb{P}\left(\sup_{s\in[N^{\alpha_1^*/2}, AN^{\alpha_1^*/2}]} L_1^N(s) < 2\lambda_1 AN^{\alpha_1^*/2}\right) = 1.$$

Taking $A_\gamma = 2 + 2\lambda_1 A$, the lemma is proved.

A.2. **Proof of Lemma 2.** We first consider the initial time $N^{\alpha_1^*/2}$. By Lemma 1, there exist some constants C_1, C_2 and $C_2' > 0$ such that

$$\lim_{N\to\infty} \mathbb{P}\left(L_1^N(N^{\alpha_1^*/2}) \le C_1 N^{\alpha_1^*/2} \text{ and } C_2' N^{\alpha_1^*/2} \le L_2^N(N^{\alpha_1^*/2}) \le C_2 N^{\alpha_1^*/2}\right) = 1.$$

Let us first complete this result by showing that for every $0 < \varepsilon < \alpha_1^*/4$, one has

(23) $$\lim_{N\to\infty} \mathbb{P}\left(L_1^N(N^{\alpha_1^*/2}) \ge C_1' N^{\alpha_1^*/2-\varepsilon}\right) = 1$$

for some constant $C_1' = C_1'(\varepsilon) > 0$. By Proposition 2, there exists $C = C(\varepsilon) > 0$ such that

$$\lim_{N\to+\infty} \mathbb{P}\left(L_1^N(N^{\alpha_1^*/2-\varepsilon}) \ge CN^{\alpha_1^*/2-\varepsilon}\right) = 1.$$

Another use of Proposition 2 and Lemma 1 shows that the process L_1^N has transitions such that

$$\begin{cases} x \to x + 1 \text{ at rate } \lambda_1 \\ x \to x - 1 \text{ at a rate } \le \mu_1 \dfrac{\alpha_1^*}{(1+\alpha_1^*)} = \lambda_1 \end{cases}$$

on the time interval $[N^{\alpha_1^*/2-\varepsilon}, N^{\alpha_1^*/2}]$. Consequently, the process $(L_1^N(s+N^{\alpha_1^*/2-\varepsilon}))$ is stochastically bounded from below by $(CN^{\alpha_1^*/2-\varepsilon}+X(s))$ on the time interval $[0, N^{\alpha_1^*/2}-N^{\alpha_1^*/2-\varepsilon}]$, where X denotes a symmetric random walk starting at 0 and jumping up and down by 1 at rate λ_1 in each direction. If $I_N=[0, N^{\alpha_1^*/2} - N^{\alpha_1^*/2-\varepsilon}]$, Chebishev's and then Doob's Maximal Inequality applied to the martingale $(X(s))$ show that, for any $\kappa>0$,

$$\mathbb{P}\left(\inf_{s\in I_N} L_1^N(s + N^{\alpha_1^*/2-\varepsilon}) \le CN^{\alpha_1^*/2-\varepsilon}-N^\kappa\right) \le \mathbb{P}\left(\inf_{s\in I_N} X(s) \le -N^\kappa\right)$$

$$\le \frac{1}{N^{2\kappa}}\mathbb{E}\left(\left(\sup_{s\in I_N}|X(s)|\right)^2\right) \le \frac{2}{N^{2\kappa}}\mathbb{E}\left(X\left(N^{\alpha_1^*/2}-N^{\alpha_1^*/2\varepsilon}\right)^2\right) \le \frac{2\lambda_1}{N^{2\kappa}}N^{\alpha_1^*/2},$$

where the last inequality is obtained by recalling that the quadratic variation of X at time $s\ge 0$ is $\lambda_1 s$. Since $\varepsilon < \alpha_1^*/4$, κ can be chosen so that $\alpha_1^*/4 < \kappa < \alpha_1^*/2 - \varepsilon$, and then

$$\lim_{N\to+\infty} \mathbb{P}\left(\inf_{s\in I_N} L_1^N(s) \le CN^{\alpha_1^*/2-\varepsilon} - N^\kappa\right) = 0.$$

Relation (23) follows.

The next step is to show that

$$\lim_{N\to\infty} \mathbb{P}\left(\inf_{s\in[N^{\alpha_1^*/2}, N^\gamma]} \left(Y_1^N(s) + Y_2^N(s)\right) < \alpha_1^* - \varepsilon\right) = 0.$$

From Lemma 1, we know that, with a probability converging to 1, the relation $C_1 N^t \le L_2^N(N^t) \le C_2 N^t$ holds for every $t\in[\alpha_1^*/2, \gamma]$, which implies that

$$Y_2^N(N^t) = t + \mathcal{O}(1/\log N), \quad \forall t \in [\alpha_1^*/2, \gamma]$$

holds with a probability tending to 1 as N tends to infinity. Hence, all one has to prove is that

$$(24) \qquad \lim_{N\to+\infty} \mathbb{P}\left(\inf_{t\in[\alpha_1^*/2,\gamma]} \left(Y_1^N(N^t) - \alpha_1^* + \varepsilon + t\right) < 0 \right) = 0.$$

Let

$$\nu^N \stackrel{\text{def.}}{=} \inf\left\{ t \geq \alpha_1^*/2 : Y_1^N(N^t) < \alpha_1^* - t - \varepsilon/2 \right\}.$$

By Relation (23), necessarily $\nu^N > \alpha_1^*/2$ with probability tending to 1 as N becomes large. On the event $\{\nu^N < \gamma\}$ and on the time interval $[N^{\nu^N}, N^{\nu^N+\varepsilon/4}]$, the process $(L_1^N(s))$ is stochastically bounded from below by $(\lfloor N^{\alpha_1^* - \nu^N - \varepsilon/2}\rfloor - X_{2,+}(s - \nu^N))$, where $(X_{2,+}(s))$ is a birth and death process starting at 0 and reflected at 0, for which the transition $x \mapsto x-1$ occurs at rate λ_1 and $x \mapsto x+1$ at rate

$$\mu_1 \frac{\alpha_1^* - \nu^N - \varepsilon/2 + \nu^N + \varepsilon/4}{1 + \alpha_1^* - \varepsilon/4} = \lambda_1 - C\varepsilon$$

for some constant $C > 0$. Consequently, setting $\gamma_0 = (\alpha_1^*/2 + \varepsilon/4) \wedge \gamma$ and using Proposition 1, one obtains

$$\mathbb{P}\left(\inf_{\alpha_1^*/2 \leq t \leq \gamma_0} Y_1^N(N^t) - \alpha_1^* + t + \varepsilon \leq 0 \right)$$

$$= \mathbb{P}\left(\inf_{\alpha_1^*/2 \leq t \leq \gamma_0} \left(Y_1^N(N^t) - \alpha_1^* + t + \varepsilon\right) \leq 0, \nu^N < (\alpha_1^*/2 + \varepsilon/4) \wedge \gamma \right)$$

$$\leq \mathbb{P}\left(\sup_{\nu^N \leq t \leq \gamma_0} \left(X_{2,+}(N^t - N^{\nu^N}) - \lfloor N^{\alpha_1^* - \tau^N - \varepsilon/2}\rfloor + \lceil N^{\alpha_1^* - t - \varepsilon}\rceil \right) \geq 0 \right)$$

$$\leq \mathbb{P}\left(\sup_{s \leq N^{\gamma_0}} X_{2,+}(s) \geq N^{\alpha_1^* - \gamma - \varepsilon/2} \left(1 - N^{-\varepsilon/2}\right) \right)$$

$$\leq \left(\lambda_1 N^{\gamma_0} + 1\right) \left(\frac{\lambda_1 - C\varepsilon}{\lambda_1}\right)^{N^{\alpha_1^* - \gamma - \varepsilon/2}/2}.$$

The quantity in the right hand side of the last relation provides an upper bound on the probability that an excursion of $(Y_1(N^t))$ exceeds $\alpha_1^* - t - \varepsilon$ for t in the time interval $(\alpha_1^*/2, \alpha_1^*/2 + \varepsilon/4)$. By repeating the procedure a finite number of times to cover the time interval $(N^{\alpha_1^*/2}, N^\gamma)$, one finally obtains Relation (24).

Similar arguments show that

$$\lim_{N\to\infty} \mathbb{P}\left(\sup_{s\in[N^{\alpha_1^*/2},N^\gamma]} Y_1^N(s) + Y_2^N(s) > \alpha_1^* + \varepsilon \right) = 0,$$

and the lemma is proved.

A.3. **Proofs of Lemmas 3 and 5.** Both proofs use the same idea. Let us start by the proof of Lemma 3.

First, recall from the paragraph before the statement of Lemma 3 that one has fixed some $\eta, \varepsilon > 0$ and one considers a time interval of the form $[T_N, T_N + N^{\alpha_1^*/2}]$ with $T_N \in [N^{\alpha_1^*/2+\eta}, N^{(\alpha_1^* \wedge 1)-\eta}]$. Recall also the definition of the stopping time $\tau^N(T_N)$ given in (14). One wants to show that if $L_1^N L_2^N(T_N)/N^{\alpha_1^*} \in (1+\varepsilon, 1+3\varepsilon/2)$, then the probability that $L_1^N L_2^N$ exceeds $(1+2\varepsilon)N^{\alpha_1^*}$ before $\tau^N(T_N)$ is exponentially small. From Proposition 4, one knows the behavior of L_2^N over this interval of time and, in particular, we can bound the values it takes over $[T_N, \tau^N(T_N)]$ with a

probability arbitrarily close to 1. Hence, the proof essentially consists in showing that L_1^N does not depart too much from $N^{\alpha_1^*}/L_2^N$.

Let $\delta > 0$ and define

$$A_\delta \stackrel{\text{def.}}{=} \left\{ \sup_{s \in [N^{\alpha_1^*/2+\eta}, N^{(\alpha_1^* \wedge 1) - \eta}]} \left| \frac{L_2^N(s)}{s} - \kappa \right| \leq \delta \right\},$$

where $\kappa \stackrel{\text{def.}}{=} \mu_2(\rho_2 - \rho_1)$. By Proposition 4, this event has a probability tending to 1 as N goes to infinity.

Let us work conditionally on the event that $L_1^N L_2^N(T_N) = \lfloor (1 + 3\varepsilon/2)N^{\alpha_1^*} \rfloor$. A simple coupling argument shows that it is enough to consider this case. To ease the notation, one does not mention this conditioning in the notation. On the event A_δ, one thus has

$$L_1^N(T_N) \leq \ell_N \stackrel{\text{def.}}{=} \frac{(1 + 3\,\varepsilon/2)}{\kappa - \delta} \frac{N^{\alpha_1^*}}{T_N}.$$

Again by a coupling argument, one can assume that $L_1^N(T_N)$ is equal to this upper bound, which is the worst-case scenario. Because on A_δ one has $L_2^N(s) \leq (\kappa + \delta)s$ for every s, the relation $L_1^N L_2^N(s) < (1 + 2\varepsilon)N^{\alpha_1^*}$ will holds for any $s \leq \tau^N(T_N)$ if

$$(25) \qquad \sup_{s \in [T_N, \tau^N(T_N)]} L_1^N(s) < \frac{(1 + 2\varepsilon)}{(\kappa + \delta)} \frac{N^{\alpha_1^*}}{T_N + N^{\alpha_1^*/2}},$$

where the expression for the denominator uses the fact that by construction, one has the relation $\tau^N(T_N) \leq T_N + N^{\alpha_1^*/2}$. Hence, Relation (25) is what is proved below. Observe that $T_N \geq N^{\alpha_1^*/2+\eta} \gg N^{\alpha_1^*/2}$, and so, such an inequality is possible for N large enough whenever δ is chosen small enough so that

$$(26) \qquad \frac{1 + 3\varepsilon/2}{\kappa - \delta} < \frac{1 + 2\varepsilon}{\kappa + \delta}$$

holds. Now, the definition of $\tau^N(T_N)$ and the bounds on L_2^N imply that for every $s \in [T_N, \tau^N(T_N)]$,

$$L_1^N(s) \geq \bar{\ell}_N \stackrel{\text{def.}}{=} \frac{(1 + \varepsilon)N^{\alpha_1^*}}{(\kappa + \delta)(T_N + N^{\alpha_1^*/2})} \quad \text{and} \quad L_2^N(s) \geq (\kappa - \delta)T_N,$$

$(L_1^N(s))$ is therefore stochastically bounded by $(\ell_N + X_+(s - T_N))$, where $(X_+(s))$ is a birth and death process reflected at $\bar{\ell}_N - \ell_N$, with birth rate λ_1 and a death rate given by

$$\mu_1 \frac{\log\left(\bar{\ell}_N\right) + \log((\kappa - \delta)T_N)}{\log N + \log\left(\bar{\ell}_N\right) + \log((\kappa - \delta)T_N)}$$

$$= \mu_1 \frac{\alpha_1^*}{1 + \alpha_1^*} + \frac{C_0 \log\left((\kappa - \delta)(1 + \varepsilon)/(\kappa + \delta)\right)}{\log N} = \mu_1 \frac{\alpha_1^*}{1 + \alpha_1^*} + \frac{C\varepsilon}{\log N},$$

for some positive constants C_0 and C. by definition of α_1^*, one has the relation $\alpha_1^*/(1+\alpha_1^*) = \rho_1 = \lambda_1/\mu_1$, the infinitesimal drift of X_+ is thus equal to $-C\varepsilon/(\log N)$.

By Proposition 1 (by writing $(\lambda/\mu)^x = \exp(-\log(\mu/\lambda)x)$), one thus has

$$\mathbb{P}\left(\sup_{s\in[0,\tau^N(T_N)-T_N]} X_+(s) \geq \frac{(1+2\varepsilon)N^{\alpha_1^*}}{(\kappa+\delta)(T_N+N^{\alpha_1^*/2})} - \ell_N\right)$$
$$\leq \left(\lambda_1\left(\tau^N(T_N)-T_N\right)+1\right)$$
$$\times \exp\left\{-\log\left(1+\frac{C\varepsilon}{\lambda_1\log N}\right)\left(\frac{(1+2\varepsilon)N^{\alpha_1^*}}{(\kappa+\delta)(T_N+N^{\alpha_1^*/2})}-\ell_N-2\right)\right\}.$$

But by definition,

$$\tau^N(T_N)-T_N \leq N^{\alpha_1^*/2},$$

and

$$\frac{(1+2\varepsilon)N^{\alpha_1^*}}{(\kappa+\delta)(T_N+N^{\alpha_1^*/2})} - \ell_N = \frac{N^{\alpha_1^*}}{T_N}\left[\frac{1+2\varepsilon}{(\kappa+\delta)(1+N^{\alpha_1^*/2}/T_N)} - \frac{1+3\varepsilon/2}{\kappa-\delta}\right],$$

in which the term within the square brackets tends to a positive constant as N tends to infinity by (26). Thus, up to taking a smaller constant to accommodate the power of N multiplying the exponential, and performing a Taylor expansion of $\log(1+C\varepsilon/(\lambda_1\log N))$, one can deduce that there exists a constant $C_1(\varepsilon) > 0$ such that

$$\mathbb{P}\left(\sup_{s\in[0,\tau^N(T_N)-T_N]} X_+(s) \geq \frac{(1+2\varepsilon)N^{\alpha_1^*}}{(\kappa+\delta)(T_N+N^{\alpha_1^*/2})} - \ell_N\right)$$
$$\leq \exp\left(-C_1\frac{N^{\alpha_1^*}}{T_N\log N}\right).$$

(Formally, C_1 also depends on δ which itself depends on ε.) This implies (25) and proves Lemma 3.

The proof of Lemma 5 is similar. Indeed, to obtain the desired upper bound, this time one starts from $L_1^N(T_N)=(1+\varepsilon)N^{\alpha_1^*}/[(\kappa-\delta)T_N]$ and shows that on the time interval $[T_N, T_N+N^{\alpha_1^*/2}]$, the process $(L_1^N(t))$ never exceeds the quantity $(1+3\varepsilon/2)N^{\alpha_1^*}/[(\kappa+\delta)T_N]$ with a probability that has the required form. The only difference here is that one has to control the number of excursions of $(L_1^N(t))$ above $(1+\varepsilon)N^{\alpha_1^*}/[(\kappa-\delta)T_N]$ on the time interval $[T_N, T_N+N^{\alpha_1^*/2}]$. This number is obviously bounded by the number of jumps of size $+1$ performed by $(L_1^N(t))$ during this lapse of time, which itself is stochastically bounded by a Poisson random variable with parameter $\lambda_1 N^{\alpha_1^*/2}$. Thus, for any $C_2 > \lambda_1$, there exists $C_3 > 0$ such that

$$\mathbb{P}\left(\text{at least } C_2 N^{\alpha_1^*/2} \text{ excursions on } [T_N, T_N+N^{\alpha_1^*/2}]\right) \leq e^{-C_3 N^{\alpha_1^*/2}}.$$

Consequently,

$$\mathbb{P}\left(\sup_{s\in[T_N, T_N+N^{\alpha_1^*/2}]} L_1^N L_2^N(s) \geq \left(1+3\varepsilon/2\right)N^{\alpha_1^*}\right)$$
$$\leq e^{-C_3 N^{\alpha_1^*/2}} + C_2 N^{\alpha_1^*/2}\exp\left(-C_4\frac{N^{\alpha_1^*}}{T_N\log N}\right)$$
$$\leq \exp\left(-C_5\frac{N^{\alpha_1^*}}{T_N\log N}\right)$$

for some positive constants $C_4(\varepsilon), C_5(\varepsilon) > 0$ depending on ε, where the last inequality uses the fact that $N^{\alpha_1^*}/T_N \leq N^{\alpha_1^*/2-\eta}$. The proof of Lemma 5 is thus complete.

A.4. **Proof of Lemma 4.** The worst case to consider here is when

$$L_1^N(T_N) = \tilde{\ell}_N \stackrel{\text{def.}}{=} \frac{(1 + 3\varepsilon/2)N^{\alpha_1^*}}{(\kappa - \delta)T_N}.$$

Since $L_2^N(s) \leq (\kappa+\delta)(T_N + N^{\alpha_1^*/2})$ on the time interval considered, the probability to estimate is bounded from above by the probability that $(L_1^N(t))$ does not go below

$$m_N \stackrel{\text{def.}}{=} \frac{(1 + \varepsilon)N^{\alpha_1^*}}{[(\kappa + \delta)(T_N + N^{\alpha_1^*/2})]}$$

on the time interval $[T_N, T_N + N^{\alpha_1^*/2}/(\log N)]$. Using the same type of coupling as before, on $[T_N, \tau^N(T_N)]$, $(L_1^N(t))$ is stochastically bounded by $(\tilde{\ell}_N + X_+(t - T_N))$, where $(X_+(t))$ is a birth and death process reflected at $m_N - \tilde{\ell}_N < 0$ with birth rate λ_1 and death rate

$$\mu_1 \frac{\log m_N + \log((\kappa - \delta)T_N)}{\log N + \log m_N + \log((\kappa - \delta)T_N)} = \mu_1 \frac{\alpha_1^*}{1 + \alpha_1^*} + \frac{C\varepsilon}{\log N} = \lambda_1 + \frac{C\varepsilon}{\log N},$$

where $C > 0$. One denotes by $(X(s))$ the non-reflected birth and death process with the same initial point. In particular, $(X(s))$ is a random walk whose drift is equal to $-C\varepsilon/(\log N)$. Consequently,

$$\mathbb{P}\left(\tau^N(T_N) - T_N > \frac{N^{\alpha_1^*/2}}{\log N}\right) \leq \mathbb{P}\left(\inf_{s \in [0, N^{\alpha_1^*/2}/\log N]} X_+(s) > m_N - \tilde{\ell}_N\right)$$

$$\leq \mathbb{P}\left(X\left(\frac{N^{\alpha_1^*/2}}{\log N}\right) > m_N - \tilde{\ell}_N\right)$$

$$= \mathbb{P}\left(X\left(\frac{N^{\alpha_1^*/2}}{\log N}\right) + C\varepsilon\frac{N^{\alpha_1^*/2}}{(\log N)^2} > m_N - \tilde{\ell}_N + C\varepsilon\frac{N^{\alpha_1^*/2}}{(\log N)^2}\right) \leq \exp\left(-C_2 \frac{N^{\alpha_1^*/2}}{(\log N)^2}\right)$$

for some $C_2(\varepsilon) > 0$, where the last line uses standard large deviations principles applied to the centered random walk

$$\left(X(t) + \frac{C\varepsilon}{(\log N)}t\right)$$

and the fact that

$$\frac{N^{\alpha_1^*}}{T_N} = o\left(\frac{N^{\alpha_1^*/2}}{(\log N)^2}\right) \quad \text{implies} \quad \left|m_N - \tilde{\ell}_N\right| = o\left(\frac{N^{\alpha_1^*/2}}{(\log N)^2}\right).$$

A.5. **Proof of Lemma 6.** The proof is a combination of the arguments used in the proofs of Lemmas 3, 4 and 5. Indeed, let s_η be defined by

$$N^{s_\eta} = N^{\alpha_1^*/2+\eta} - N^{\alpha_1^*/2+\eta/2}, \quad \text{i.e.} \quad s_\eta = \frac{\alpha_1^*}{2} + \eta + \frac{\log(1 - N^{-\eta/2})}{\log N}.$$

Since $s_\eta > \alpha_1^*/2$, by Theorem 1, for a given small $\delta > 0$, the event

$$\left\{ L_1^N(N^{s_\eta}) \in \left[N^{\alpha_1^* - s_\eta - \delta}, N^{\alpha_1^* - s_\eta + \delta} \right] = \left[\frac{N^{\alpha_1^*/2 - \eta - \delta}}{1 - N^{-\eta/2}}, \frac{N^{\alpha_1^*/2 - \eta + \delta}}{1 - N^{-\eta/2}} \right] \right\}$$

$$\bigcup \left\{ L_2^N(s) \in \left[(\kappa - \delta) N^{s_\eta}, (\kappa + \delta) N^{\alpha_1^*/2 + \eta} \right], \forall s \in [N^{s_\eta}, N^{\alpha_1^*/2 + \eta}] \right\}$$

has a probability converging to 1 as N becomes large. Recall that $\kappa = \mu_2(\rho_2 - \rho_1)$. As before, via a coupling, one can assume that $L_1^N(N^{s_\eta})$ is equal to the maximal value $N^{\alpha_1^* - s_\eta + \delta}$. For $\varepsilon > 0$, define

$$\ell_N \overset{\text{def.}}{=} \frac{1 + \varepsilon}{\kappa + \delta} N^{\alpha_1^*/2 - \eta} \text{ and } \sigma_N \overset{\text{def.}}{=} \inf \left\{ s \geq N^{s_\eta} : L_1^N(s) \leq \ell_N \right\}.$$

One first shows that $\sigma_N < N^{\alpha_1^*/2 + \eta}$ holds with probability tending to 1 as N becomes large. On the time interval $[N^{s_\eta}, \sigma_N]$, the process $(L_1^N(s))$ is stochastically bounded by

$$\left(\frac{N^{\alpha_1^*/2 - \eta + \delta}}{1 - N^{-\eta/2}} + X(s - N^{s_\eta}) \right),$$

where $(X(s))$ is a birth and death process on $\mathbb{Z}$ starting at 0 with birth rate λ_1 and a death rate given by

$$\mu_1 \frac{\log \left(\ell_N \right) + \log((\kappa - \delta) N^{\alpha_1^*/2 + \eta}(1 - N^{-\eta/2}))}{\log N + \log \left(\ell_N \right) + \log((\kappa - \delta) N^{\alpha_1^*/2 + \eta}(1 - N^{-\eta/2}))}$$

$$= \mu_1 \frac{\alpha_1^*}{1 + \alpha_1^*} + \frac{C\varepsilon}{\log N} = \lambda_1 + \frac{C\varepsilon}{\log N},$$

for some constant $C > 0$. Hence, as in the proof of Lemma 4, one has

$$\mathbb{P}\left(\sigma_N > N^{\alpha_1^*/2 + \eta} \right) \leq \mathbb{P}\left(\inf_{s \in [0, N^{\alpha_1^*/2 + \eta/2}]} X(s) > \frac{1 + \varepsilon}{\kappa + \delta} N^{\alpha_1^*/2 - \eta} - \frac{N^{\frac{\alpha_1^*}{2} - \eta + \delta}}{1 - N^{-\eta/2}} \right)$$

$$\leq \mathbb{P}\left(X(N^{\alpha_1^*/2 + \eta/2}) + \frac{C\varepsilon N^{\alpha_1^*/2 + \eta/2}}{\log N} > -C_1 N^{\alpha_1^*/2 - \eta + \delta} + \frac{C\varepsilon N^{\alpha_1^*/2 + \eta/2}}{\log N} \right).$$

This last term converges to 0 as N tends to infinity whenever $\delta < 3\eta/2$, since then $N^{\alpha_1^*/2 - \eta + \delta}$ is negligible compared to $N^{\alpha_1^*/2 + \eta/2}/\log N$. As before, one uses standard large deviation estimates on centered random walks.

Secondly, one can see that conditionally on the event $\{\sigma_N < N^{\alpha_1^*/2 + \eta}\}$, the process $(L_1^N(s))$ stays below the value $(1 + 3\varepsilon/2) N^{\alpha_1^*/2 - \eta}/(\kappa + \delta)$ on the time interval $[\sigma_N, N^{\alpha_1^*/2 + \eta}]$ with a probability tending to 1. It is proved using exactly the same method as in the proof of Lemma 5.

The quantity $\delta > 0$ is chosen sufficiently small that $(1 + \varepsilon)/(\kappa + \delta) > 1/\kappa$. One just has to prove that

$$\lim_{N \to +\infty} \mathbb{P}\left(\sigma_N < N^{\alpha_1^*/2 + \eta}, \sup_{T_N \leq s \leq N^{\alpha_1^*/2 + \eta}} L_1^N(s) \leq \frac{1 + 3\varepsilon/2}{\kappa + \delta} N^{\alpha_1^*/2 - \eta} \right) = 1,$$

hence, with probability tending to 1,

$$L_1^N L_2^N(N^{\alpha_1^*/2 + \eta}) \leq \frac{1 + 3\varepsilon/2}{\kappa + \delta} N^{\alpha_1^*/2 - \eta} \times (\kappa + \delta) N^{\alpha_1^*/2 + \eta} = \left(1 + \frac{3}{2} \varepsilon \right) N^{\alpha_1^*}.$$

Lemma 6 is proved.

Proof of Lemma 7. Let us define
$$\sigma_N := \inf \left\{ s \geq N^{\alpha_1^* - \varepsilon/2} : L_1^N(s) \geq N^\varepsilon \right\}.$$

One knows from Theorem 1 that $\sigma_N > N^{\alpha_1^* - \eta}$ for any $\eta > 0$. Besides, a simple coupling argument shows that with probability tending to 1, $L_2^N(s) \leq 2\lambda_2 s$ for every $s \in [0, N]$. Hence, on the time interval $[N^{\alpha_1^* - \varepsilon/2}, \sigma_N]$, $(L_2^N(t))$ is stochastically bounded from below by the birth and death process $(\tilde{L}_2^N(t))$ such that
$$\tilde{L}_2^N \left(N^{\alpha_1^* - \varepsilon/2} \right) = L_2^N \left(N^{\alpha_1^* - \varepsilon/2} \right) \sim \kappa N^{\alpha_1^* - \varepsilon/2},$$

and for which transitions $x \mapsto x + 1$ occur at rate λ_2 and $x \mapsto x - 1$ at rate
$$\mu_2 \frac{\varepsilon + [\log(s) + \log(2\lambda_2)]/\log N}{1 + \varepsilon + [\log(s) + \log(2\lambda_2)]/\log N} = \mu_2 \frac{\varepsilon + \log s/\log N}{1 + \varepsilon + \log s/\log N} + \frac{C}{\log N}.$$

But since
$$\lambda_2 > \mu_2 \frac{\varepsilon + \log s/\log N}{1 + \varepsilon + \log s/\log N}$$

is equivalent to
$$\frac{\log s}{\log N} < \frac{\rho_2}{1 - \rho_2} - \varepsilon = \alpha_2^* - \varepsilon,$$

the infinitesimal drift of $(\tilde{L}_2^N(t))$ is bounded from below by some $c_\varepsilon > 0$ on the interval $[N^{\alpha_1^* - \varepsilon/2}, \sigma_N \wedge N^{(\alpha_2^* \wedge 1) - 2\varepsilon}]$. The Ergodic Theorem for Poisson processes thus guarantees that $\tilde{L}_2^N(s)/s$ remains greater than $C_\varepsilon = \kappa + c_\varepsilon/2$ with probability tending to 1 as $N \to \infty$, and so

$$(27) \qquad \lim_{N \to \infty} \mathbb{P} \left(\inf_{s \in [N^{\alpha_1^* - \varepsilon/2}, \sigma_N \wedge N^{(\alpha_2^* \wedge 1) - 2\varepsilon}]} \frac{L_2^N(s)}{s} \geq C_\varepsilon \right) = 1.$$

Now, using this first result together with Theorem 1, for N large enough one can write that on the smaller time interval $[N^{\alpha_1^* - \varepsilon/4}, \sigma_N \wedge N^{(\alpha_2^* \wedge 1) - 2\varepsilon}]$, the process $(L_1^N(t))$ is stochastically bounded from above by $N^{\varepsilon/2} + X_+(\cdot - N^{\alpha_1^* - \varepsilon/4})$, where $(X_+(t))$ is a birth and death process reflected at 0, with birth rate λ_1 and a death rate equal to

$$\mu_1 \frac{\varepsilon/2 + \alpha_1^* - \varepsilon/4 + \log C_\varepsilon/\log N}{1 + \varepsilon/2 + \alpha_1^* - \varepsilon/4 + \log C_\varepsilon/\log N}$$
$$= \mu_1 \frac{\alpha_1^*}{1 + \alpha_1^*} + C'\varepsilon + \frac{C''}{\log N} = \lambda_1 + C'\varepsilon + \frac{C''}{\log N},$$

for some constants C' and $C'' > 0$. Hence, the second part of Proposition 1 enables us to conclude that $\sigma_N > N^{(\alpha_2^* \wedge 1) - 2\varepsilon}$ holds with probability tending to 1. Recalling Relation (27) and the fact that $\sigma_N > N^{(\alpha_2^* \wedge 1) - 2\varepsilon}$ is equivalent to
$$\sup_{s \in [N^{\alpha_1^* - \varepsilon/2}, N^{(\alpha_2^* \wedge 1) - 2\varepsilon}]} L_1^N(s) < N^\varepsilon,$$

Lemma 7 is proved.

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