Particle system with random interaction

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Journée francilienne d'accueil des postdoctorant·e·s
Institut Henri Poincaré, 20 October 2021

Particle systems

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- each particle has a state
- each state follows a dynamic
  (ex : differential equation, ODE, PDE, SDE, SPDE...)

Discrete time interaction

Frame modelization :

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The activity of each particle is modeled by a point process

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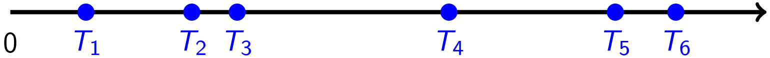

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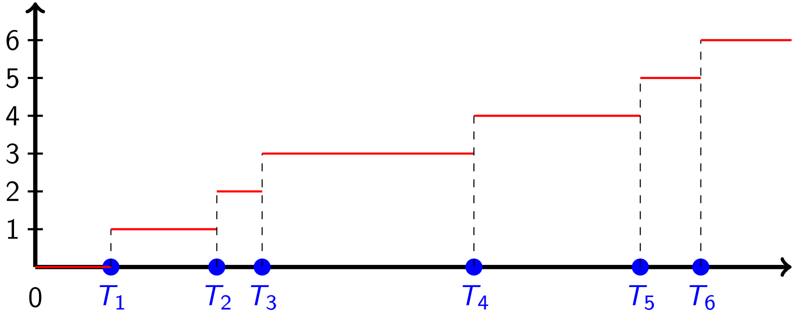
- a stair function on $\mathbb{R}_+$: $Z_t = Z([0,t])$



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Context : consider Z point process with rate λ
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Why : Let Z point process with constant rate $\lambda > 0$, and $t \geq 0, n \in \mathbb{N}^*$
Conditionally to $\{Z_t = n\}$, the T_k $(1 \leq k \leq n)$ are iid uniform on $[0, t]$

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- if particle j creates event at time t, $X_t^{N,i} = X_{t-}^{N,i} + u(j,i,t)$

Neural network model

- particle = neuron

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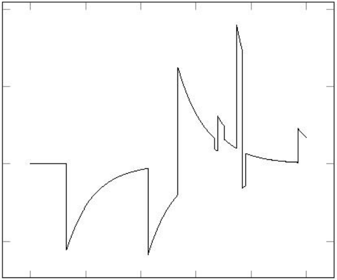

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- linear scaling N^{-1} (LLN) :
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Why making $N \to \infty$:

- it is natural $N \approx 86.10^9$
- the limit system can be easier to simulate and study

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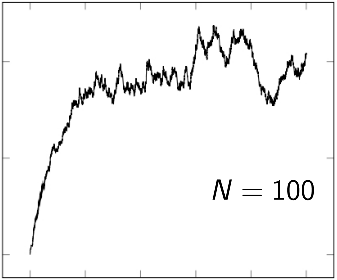

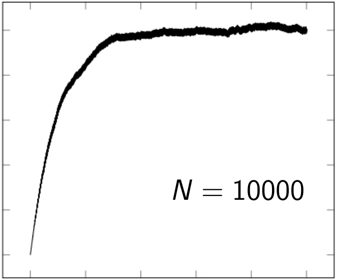

Limit system (1)

$N-$neuron system

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Result

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\left|X_s^{N,i} - \bar{x}_s^i\right|\right] \leq C_t \cdot N^{-1/2}$$

Sketch of proof

$$\boxed{dX_t^N = -\alpha X_t^N dt + \frac{1}{N}\sum_{j=1}^N dZ_t^{N,j}}$$

where $Z^{N,j}$ has rate $f(X_t^N)$

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Infinitesimal generator of X^N **:** for g smooth enough,

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$$N \longrightarrow +\infty \quad : \quad \bar{A}g(x) = -\alpha x g'(x) + f(x)g'(x)$$

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Infinitesimal generator of X^N : for g smooth enough,

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McKean-Vlasov equation :

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Result [Andreis et al. (2018)] (and [E. (2021)])

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\left|X_s^{N,i} - \bar{X}_s^i\right|\right] \leq C_t \cdot N^{-1/2}$$

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Previously : all the particles within a system are similar

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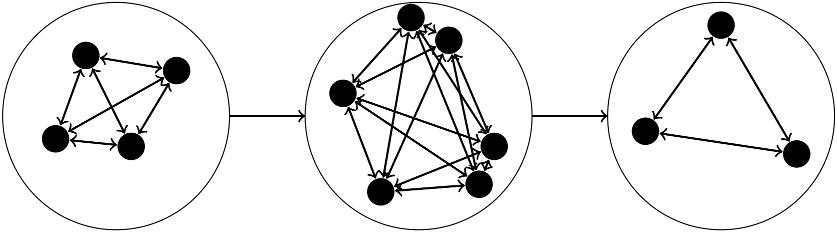
Example : neural retina



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Thank you for your attention !

Questions ?