Conditional propagation of chaos for mean field system of neurons

Xavier Erny [1]

Joint work with: Eva Löcherbach [2] and Dasha Loukianova [1]

[1]Université d'Evry (LaMME)
[2]Université Paris 1 Panthéon-Sorbonne (SAMM)

ICMNS 2021, 28th of June

1. Model
   - Neural network model
   - Limit system
2. Propagation of chaos
   - Martingale problem
   - Convergence of $(\mu^N)_N$

Modeling in neuroscience

Neural activity $=$ Set of spike times

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Network of N neurons :

- $\begin{aligned} Z^{N,i} &= \text{set of spike times of neuron } i \\ &= \text{point process with intensity } f(X^{N,i}_{t-})\end{aligned}$
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$X^{N,i}$ solves an SDE directed by $(Z^{N,j})_{1 \leq j \leq N}$

Mean field limit

$N-$neurons network :

$$dX_t^{N,i} = b(X_t^{N,i})dt + \sum_{j=1}^{N} u^{ji}(t)dZ_t^{N,j}$$

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- diffusive scaling $N^{-1/2}$ (CLT) :
  [E. et al. (2019)] random and centered $u^{ji}(s)$

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- drift : $-\alpha x$ models an exponantial loss of the potential
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[De Masi et al. (2015)] and [Fournier & Löcherbach (2016)]
Generalization to McKean-Vlasov frame [Andreis et al. (2018)]

Diffusive scaling

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j\neq i}}^{N} U^j(t) dZ_t^{N,j} - X_{t-}^{N,i} dZ_t^{N,i}$$

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Dynamic of $X^{N,i}$:

- $X_t^{N,i} = X_s^{N,i} e^{-\alpha(t-s)}$ if the system does not jump in $[s,t[$
- $X_t^{N,i} = X_{t-}^{N,i} + \frac{U^j(t)}{\sqrt{N}}$ if a neuron $j \neq i$ emits a spike at t
- $X_t^{N,i} = 0$ if neuron i emits a spike at t ($\rightarrow$ repolarization)

Limit system : heuristic (1)

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^{N} U^j(t) dZ_t^{N,j} - X_{t-}^{N,i} dZ_t^{N,i}$$

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$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + d\bar{M}_t - \bar{X}_{t-}^i d\bar{Z}_t^i$$

with :

- $M_t^N \underset{N \to \infty}{\longrightarrow} \bar{M}_t$
- $\bar{Z}^i$ point process with intensity $f(\bar{X}_t^i)$

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$$\langle \bar{M} \rangle_t = \lim_N \langle M^N \rangle_t = \lim_N \sigma^2 \int_0^t \frac{1}{N} \sum_{j=1}^{N} f(X_s^{N,j}) ds$$

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Then $\bar{M}$ should satisfy

$$\bar{M}_t = \sigma \int_0^t \sqrt{\lim_N \frac{1}{N} \sum_{j=1}^{N} f(\bar{X}_s^j)} dW_s = \sigma \int_0^t \sqrt{\lim_N \bar{\mu}_s^N(f)} dW_s$$

with $\bar{\mu}^N := \frac{1}{N} \sum_{j=1}^{N} \delta_{\bar{X}^j}$

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μ is the limit of empirical measures of $(\bar{X}^i)_{i\geq 1}$ exchangeable by Proposition (7.20) of [Aldous (1983)] μ is the directing measure of $(\bar{X}^i)_{i\geq 1}$ (conditionally on μ, $\bar{X}^i$ i.i.d.$\sim \mu$)

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$f'(x) \leq C(1 + |x|)^{-(1+\varepsilon)}$ $(\varepsilon > 0)$

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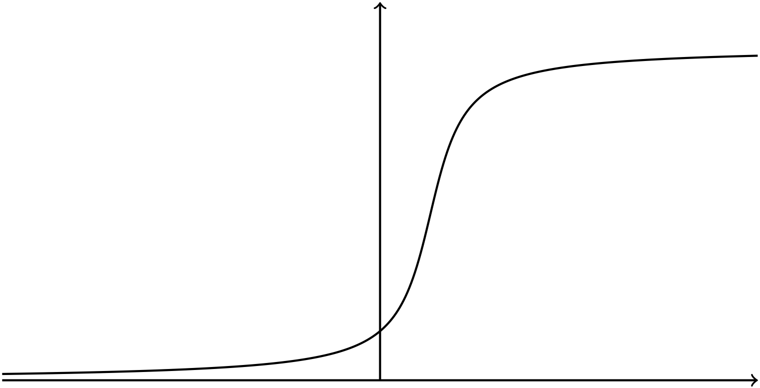

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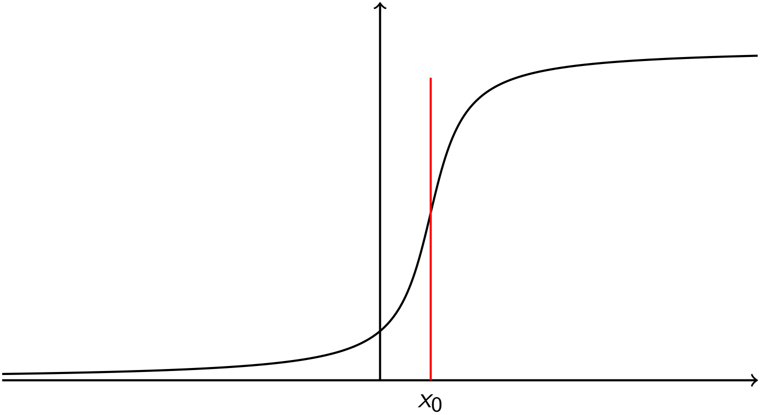

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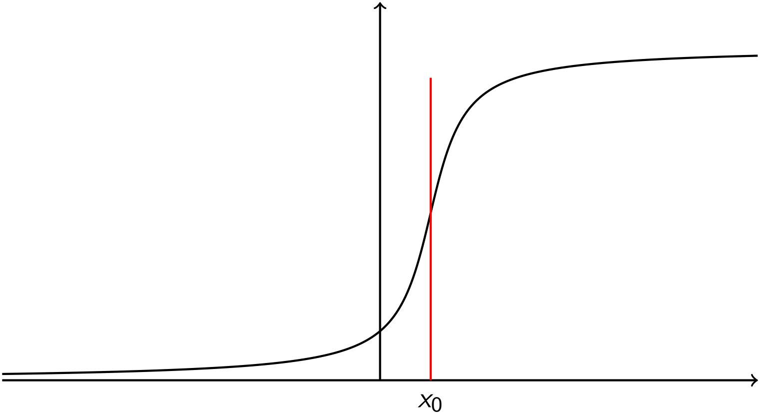
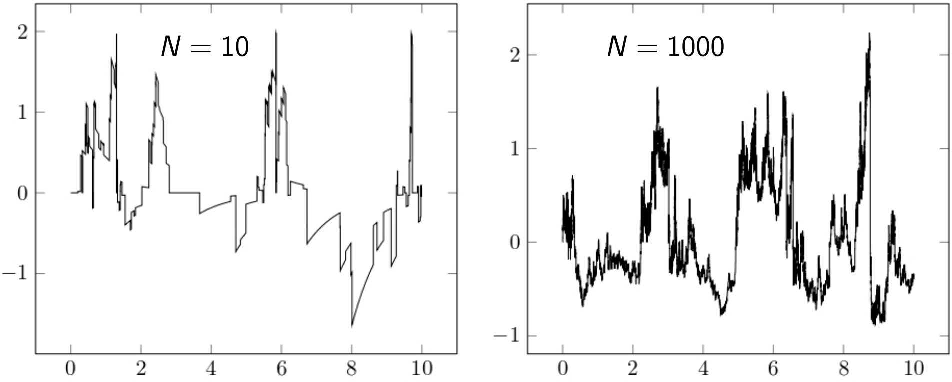


"Neuron i active / inactive" $\approx$ "$X^{N,i} > x_0$ / $X^{N,i} < x_0$"

Simulations of $X^{N,1}$



Convergence of $(X^{N,i})_{1\leq i\leq N}$

$$dX_t^{N,i} = -\alpha X_t^{N,i}dt + \frac{1}{\sqrt{N}}\sum_{\substack{j=1\\ j\neq i}}^{N} U^j(t)dZ_t^{N,j} - X_{t-}^{N,i}dZ_t^{N,i}$$

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with :

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NS condition (Proposition (7.20) of [Aldous (1983)]) :
$\mu^N := \sum_{j=1}^N \delta_{X^{N,j}}$ converges to $\mu := \mathcal{L}(\bar{X}^1|W)$ in $\mathcal{P}(D)$

Outline of the proof

Step 1. $(\mu^N)_N$ is tight on $\mathcal{P}(D)$
Equivalent condition : $(X^{N,1})_N$ is tight on D
Proof : Aldous' criterion

Step 2. Identifying the limit distribution of $(\mu^N)_N$
Proof : any limit of μ^N is solution of a martingale problem

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$$Lg(m, x^1, x^2) = -\alpha x^1 \partial_1 g(x) - \alpha x^2 \partial_2 g(x) + \frac{\sigma^2}{2} m(f) \sum_{i,j=1}^{2} \partial_{i,j}^2 g(x)$$

$$+ f(x^1)(g(0, x^2) - g(x)) + f(x^2)(g(x^1, 0) - g(x))$$

Convergence of μ^N to the solution of $(\mathcal{M})$

Let μ be the limit of (a subsequence of) μ^N
$\mathcal{L}(\mu)$ is solution of $(\mathcal{M})$ if

$$\mathbb{E}\left[F(\mu)\right] = 0$$

for any F of the form

$$F(m) := \int_{D^2} m \otimes m(d\gamma)\phi_1(\gamma_{s_1})...\phi_k(\gamma_{s_k})\Big[\phi(\gamma_t) - \phi(\gamma_s) - \int_s^t L\phi(m_r, \gamma_r)dr\Big]$$

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$$+\alpha \int_s^t \gamma_r^1 \partial_1 \phi(\gamma_r)dr + \alpha \int_s^t \gamma_r^2 \partial_2 \phi(\gamma_r)dr$$
$$-\int_s^t f(\gamma_r^1)(\phi(0,\gamma_r^2) - \phi(\gamma_r))dr - \int_s^t f(\gamma_r^2)(\phi(\gamma_r^1, 0) - \phi(\gamma_r))dr$$
$$-\frac{\sigma^2}{2}\int_s^t m_r(f) \sum_{i_1,i_2=1}^{2} \partial^2_{i_1,i_2}\phi(\gamma_r)dr\Big]$$

The expression of $F(\mu^N)$

$$F(\mu^N) :=$$

$$\int_{D^2} \mu^N \otimes \mu^N(d\gamma)\phi_1(\gamma_{s_1})...\phi_k(\gamma_{s_k})\Big[\phi(\gamma_t) - \phi(\gamma_s)$$

$$+ \alpha \int_s^t \gamma_r^1 \partial_1 \phi(\gamma_r)dr + \alpha \int_s^t \gamma_r^2 \partial_2 \phi(\gamma_r)dr$$

$$- \int_s^t f(\gamma_r^1)(\phi(0, \gamma_r^2) - \phi(\gamma_r))dr$$

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$$-\int_s^t f(X_r^{N,i})(\phi(0,X_r^{N,j})-\phi(X_r^{N,i},X_r^{N,j}))dr$$

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Convergence of $(\mu^N)_N$

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^{N} U^j(t) dZ_t^{N,j} - X_{t-}^{N,i} dZ_t^{N,i}$$

$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t - \bar{X}_{t-}^i d\bar{Z}_t^i$$

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- let μ be the limit of a converging subsequence
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- $\mu = \mathcal{L}(\bar{X}^1 | W)$ is the only limit of $(\mu^N)_N$

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Thank you for your attention !

Questions ?