Yoan Tardy
Curriculum Vitae

PERSONAL DETAILS

Birth	April 2, 1995
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SCIENTIFIC INTERESTS

Interacting particle systems

Partial differential equations

Large deviation principles

Lévy processes

Dirichlet form theory

Branching Brownian motions

CURRENT SITUATION

Hadamard Lecturer — Since 2023
under the supervision of Giovanni Conforti, CMAP, X, Palaiseau, France
We study the large deviation principle of the Branching Brownian motion.

EDUCATION

PhD in Probability — 2020-2023
under the supervision of Nicolas Fournier, Sorbonne Université, Paris, France
Title : Etude des modèles stochastique et déterministe de Keller-Segel.

Master "Probabilité et Modèles Aléatoires" — 2019-2020
Sorbonne Université, Paris, France

Internship in research in Probability — 2018
4 months at UBC, Vancouver, under the supervision of Martin Barlow and Mathav Murugan, about random walks in dynamical environment with percolation.

ENS Ulm — 2016-2020
Paris, France

MPSI-MP* — 2013-2016
Lycée du Parc, Lyon, France

PUBLICATIONS

[1] N. Fournier, Y.T. A simple proof of non-explosion for measure solutions of the Keller-Segel equation. **Kinet. Relat. Models.** Vol. 16, 178-186, 2023.

[2] Y.T. Convergence of the empirical measure for the Keller-Segel model in both subcritical and critical cases. **Submitted**, https://arxiv.org/abs/2205.04968

[3] N. Fournier, Y.T. Collisions of the supercritical Keller-Segel particle system. To appear in **J. Eur. Math. Soc.**, https://arxiv.org/abs/2110.08490

ORAL COMMUNICATIONS

Seminary of the Probability Unit — Oct 2023
CMAP, Palaiseau, France

Summer school "mean field models — Jun 2023
IRMAR, Rennes, France

Séminaire de l'équipe de probabilités et statistiques — Nov 2022
LAGA, Villetaneuse, France

Last Workshop of the ANR EFI — Nov 2022
ICJ, Lyon, France

Les Probabilités de demain — Nov 2022
IHP, Paris, France

Workshop "Particle systems in Mathematical Biology" — Jul 2022
Hausdorff Institute for Mathematics, Bonn, Germany

GTT of the LJLL — Feb 2022
Sorbonne Université, Paris, France

GTT of the LPSM — Feb 2022
Sorbonne Université, Paris, France

Les probas du vendredi — Feb 2022
Sorbonne Université, Paris, France

Seminary of SAMM — Jan 2022
Paris 1, Paris, France

Seminary of PEIPS — Dec 2021
CMAP, Palaiseau, France

Seminary Gaussbuster — Jun 2021
IRMAR, Rennes, France

GTT of the LPSM Jun 2021
Sorbonne Université, Paris, France

TEACHING

Tutorials for "Modélisation de phénomènes aléatoires" 2023-2024
Polytechnique, Ingénieur 2A, 40h.

Tutorials for "Probability: stochastic processes" 2023-2024
Polytechnique, Bachelor 3, 12h.

Tutorials for "Probabilités II" 2020-2022
Sorbonne Université, Paris, L3, 36h.

Tutorials for "Théorie de la mesure et probabilités" 2020-2022
Sorbonne Université, L3, 36h.

Tutorials for "Mathématiques pour les études scientifiques II" 2019-2020
Sorbonne Université, L1, 18h.

RESEARCH STATEMENT

Introduction

My research interests are concerned with the boundary between PDE's and probability theory. I work especially on the Keller-Segel PDE and the Keller-Segel particle system in both the critical and supercritical cases.

Simple proof of non explosion for measure of the Keller-Segel equation

We introduce the following equation modeling chemotaxis in $\mathbb{R}^2$ introduced by Keller and Segel [31] (see also Patlak [40]), with unkown (f, c),

$$\partial_t f_t + \nabla \cdot (f_t \nabla c_t) = \Delta f_t, \tag{1}$$

$$\Delta c_t = -f_t. \tag{2}$$

Chemotaxis is the biologic phenomenon where bacterias diffuse in their environement and emit a chemoattractant which attracts themselves. At time $t > 0$ and position $x \in \mathbb{R}^2$ we denote by $f_t(x)$ the density of cells and by $c_t(x)$ the density of chemoattractant. One can informally observe that since (2) is a Poisson equation, we get $\nabla c_t = K \star f_t$ where for $x \in \mathbb{R}^2 \setminus \{0\}$, $K(x) = -x/(2\pi\|x\|^2)$ and $K(0) = 0$. We can thus write

$$\partial_t f_t + \nabla \cdot (f_t K \star f_t) = \Delta f_t. \tag{3}$$

Moreover since the right hand side of (1) can be expressed as the divergence of a vector, the total mass of the solution must be constant, so we can set $M := \int_{\mathbb{R}^2} f_0(\mathrm{d}x) = \int_{\mathbb{R}^2} f_t(\mathrm{d}x)$ for all $t \geq 0$.

This model is interesting because of the tight competition between diffusion and attraction. As shown rigourously in Jäger Luckhaus [29], the conclusion of this competition depends on M. Setting $V_t = \int_{\mathbb{R}^2} \|x\|^2 f_t(\mathrm{d}x)$ we formally get by using multiple integration by part that $V_t' = M(4 - M/(2\pi))$. This suggests that there are three cases:

- If $M < 8\pi$ (the subcritical case), then $V_t > 0$ for all $t > 0$, so the diffusion should win over the concentration and the equation should be well posed,
- if $M > 8\pi$ (the supercritical case), then there must have blow up in finite time since V must be nonnegative for all time, more precisely we expect the emergence of a Dirac mass,
- if $M = 8\pi$ (the critical case), then the objects should be well posed since V is constant.

In particular, it is not clear that a solution exists, especially in the supercritical case where a blow up must occurs in finite time, in the sense of the emergence of a Dirac mass.

Biler-Karch-Laurençot-Nadzieja [4]-[5] build a global weak solution in the radially symmetric case for all M such that $M \leq 8\pi$ and all measure inital condition.

Blanchet-Dolbeault-Perthame [6] proved the existence of a global weak *free energy* solution to (1)-(2) for initial conditions $f_0 \in L^1(\mathbb{R}^2)$ for all M such that $M < 8\pi$.

Bedrossian-Masmoudi [3] have shown that there exists a local mild solution (which is stronger than a weak solution) for all measure initial condition f_0 satisfying the property that $\max_{x\in\mathbb{R}^2} f_0(\{x\}) < 8\pi$. Observe that whenever $M < 8\pi$, this condition always holds. Moreover, Wei [46] proved the existence of a global mild solution for all $f_0 \in L^1(\mathbb{R}^2)$ in the subcritical and critical cases, and local solution in the supercritical case. Combining these two results, one can find that we can build global mild solution if f_0 is a measure initial data such that $f_0(\mathbb{R}^2) \leq 8\pi$ and $\max_{x\in\mathbb{R}^2} f_0(\{x\}) < 8\pi$.

The following result proved in Fournier-Tardy [24] is weaker, but the arguments are much simpler, and more robust in that the same strategy can be applied to approximate the solution by particle systems.

Theorem 1. *Consider a nonnegative measure f_0 on $\mathbb{R}^2$ with mass M strictly smaller than 8π. There exists a global weak solution f to* (3) *with initial condition f_0. Moreover, for all $\gamma \in (M/(4\pi), 2)$, there is a constant $A_{M,\gamma} > 0$ depending only on M and γ such that for all $T > 0$,*

$$\int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \|x - y\|^{\gamma-2} f_s(\mathrm{d}x) f_s(\mathrm{d}y)\mathrm{d}s \leq A_{M,\gamma}(1 + T). \tag{4}$$

The idea of the proof is to use a *two-particle* moment argument, implying an *a priori* bound with the same flavor as (4).

Precise study of the collisions

Since we will use a probabilistic approach, we slightly reformulate the problem. Until now we consider the equivalent formulation of (3):

$$\partial_t f_t + 2\pi\theta\nabla \cdot (f_t K \star f_t) = \frac{1}{2}\Delta f_t, \tag{5}$$

with initial mass $\int_{\mathbb{R}^2} f_0(\mathrm{d}x) = 1$. Indeed, by a change of variable, one can show that the threshold 8π we had on the mass becomes the threshold 2 on the attractivity parameter θ.

An interesting way to study the equation (5) is to use the Lagrangian point of view by following the motion of one typical cell immersed in an infinite amount of cells. We consider the following SDE

$$\mathrm{d}X_t = \mathrm{d}B_t + 2\pi\theta K \star f_t(X_t)\mathrm{d}t, \tag{6}$$

where $f_t = \mathcal{L}(X_t)$ for all $t \geq 0$ and $(B_t)_{t\geq 0}$ is a 2-dimensionnal Brownian motion. Informally, f_t must solve (5). We consider the classical mean-field approximation of (6) in $\mathbb{R}^2$,

$$\mathrm{d}X_t^{i,N} = \mathrm{d}B_t^i - \frac{\theta}{N}\sum_{j=1}^{N} \frac{X_t^{i,N} - X_t^{j,N}}{\|X_t^{i,N} - X_t^{j,N}\|^2}\mathrm{d}t, \tag{7}$$

where $(B_t)_{t\geq 0} := (B_t^1, \dots, B_t^N)_{t\geq 0}$ is a $2N$-dimensionnal Brownian motion. Since $x \mapsto -x/\|x\|^2$ is a very singular kernel, the existence of the process (7) is not guaranted. In fact, in the case where $\theta \geq 2$, one can prove the existence of such a process as long as there is no collision, but it can't exist in the classical sense for all times, because of the emergence of a cluster of particle which stay glued together once they collide. Thus we need to use the Dirichlet form theory to give a sense to (6), at least as long as there is no sticky collision. For the sake of conciseness and clarity we will act as if there was a classical solution to (7) and ignore subtilities implied by the use of the Dirichlet form theory.

We set for all $L \subset [\![1, N]\!]$, all $x = (x^1, \dots, x^N) \in (\mathbb{R}^2)^N$,

$$S_L(x) = \frac{1}{|L|}\sum_{i\in L} x^i \quad \text{and} \quad R_L(x) = \sum_{i\in L} \|x^i - S_L(x)\|^2. \tag{8}$$

Moreover we say for all $L \subset [\![1, N]\!]$ that a L-collision occurs at time t if $R_L(X_t^N) = 0$ and for all $i \notin L$, $R_{L\cup\{i\}}(X_t^N) > 0$.

The question of the behaviour of the collisions of the process $(X_t^N)_{t\geq 0}$ is a difficult one because of the tight competition between the diffusion of the Brownian motion and the attraction due to the attractive and singular kernel. We present our result about this problem proved in Fournier-Tardy [25].

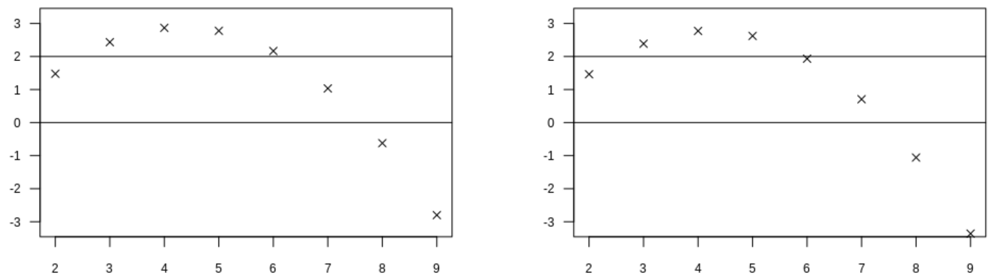
Informally, using the Itô formula, one gets that for all $L \subset [\![1, N]\!]$, $(R_L(X_t))_{t\geq 0}$ is almost a squared Bessel process with dimension $d_{\theta,N}(|L|)$, where for all $k \in [\![1, N]\!]$,

$$d_{\theta,N}(k) = (k-1)\Big(2 - \frac{\theta k}{N}\Big).$$

One has to compare the dimensions with 0 and 2. Indeed, according to Revuz-Yor [42], a squared Bessel process of dimension δ will

- never hit 0 if $\delta \geq 2$,
- hit 0 infinitely many and then immediately be reflected if $\delta \in (0, 2)$, we speak of reflective collision
- hit 0 and stay at 0 if $\delta \leq 0$, we speak of sticky collision.

Figure 1: Plot of $d_{\theta,N}(k)$ as a function of $k \in [\![2, N]\!]$ with $N = 9$ and with $\theta = 2.35$ (left) and $\theta = 2.42$ (right).



$k_0 = 8$, $k_1 = 7$, $k_2 = 7$ $\qquad$ $k_0 = 8$, $k_1 = 7$, $k_2 = 6$

Fixing N and θ, we plot $d_{N,\theta}(k)$ as a function of k, see Figure 1.

We will place ourselves in the case where N and θ are chosen such that $2 > d_{\theta,N}(k_2) \geq d_{\theta,N}(k_1) \geq d_{\theta,N}(k_0)$ where $k_0 = \lceil 2N/\theta \rceil$, $k_1 = k_0 - 1$ and $k_2 = k_0 - 2$ (this correspond to the second picture.)

This gives us the intuition that only collisions of k particles occur with $k \in \{2, k_2, k_1, k_0\}$, which seems original. Moreover, we succeeded to show more precisely the following behaviour: a cluster of precisely k_0 particle will emerge and the Dirichlet form theory can't be applied beyond this time, which seems to coincide with our intuition that once k_0 particle collide together, they stay glued together forever. This instant is called the explosion of the process $(X_t^N)_{t\geq 0}$. Before the explosion, there are infinitely many k_1-collision of every subset of k_1 particles from the k_0 particles involved in the cluster of the explosion. Moreover, before each of these k_1-collisions, there are infinitely many k_2-collisions of every subset of k_2 particles from the k_1 particle involved in the k_1-collision. Finally, the same behaviour occurs for 2-collisions before each k_2-collisions. Another remarkable fact is that there is no k-collision for $k \in [\![3, k_2 - 1]\!]$ which seems rather counterintuitive.

Mean-field limit in both subcritical and critical cases

Another interesting question is to understand if the approximation of (6) by (7) is relevant. More precisely, we want to show that $\mu_t^N = \sum_{i=1}^N \delta_{X_t^{i,N}}$ converges in some sense to $(f_t)_{t\geq 0}$ as $N \to \infty$.

This kind of question was first raised by Kac [30] in the view of justifying rigorously the Boltzman equation. McKean [35], Méléard [36] and Mischler-Mouhot [38] brought significant contributions to the theory.

The first result dealing with the Keller-Segel particle system in particular is the one of Godinho-Quininao [27], replacing K by $-x/\|x\|^{1+\alpha}$ with $\alpha \in (0, 1)$, which is a less singular kernel (there is no tight competition anymore between diffusion and attraction). Then Olivera-Richard-Tomasevic [39] where roughly, K is replaced by $-x/(\|x\|^2 + \varepsilon_N)$ with ε_N very large in front of $N^{-1/d}$. The technics used are inspired by a method developped by Flandoli [22] based on semigroups. In the (very) subcritical case $\theta < 1/2$, Fournier-Jourdain [23] proved it up to the extraction of a subsequence. Finally, Bresch-Jabin-Wang [7] proved a convergence with quantitative estimates using a *modulated free entropy* method in the case where $\theta < 2$. The convergence is not up to the extraction of a subsequence, but it holds only for regular initial data, i.e $f_0 \in W^{2,\infty}$, and they simplify the problem by replacing $\mathbb{R}^2$ by a torus.

We now introduce our result in the subcritical case proved in [45].

Theorem 2. *Let $\theta \in (0,2)$ and $f_0 \in \mathcal{P}(\mathbb{R}^2)$. For each $N \geq N_0 := (1 + \lceil 2/(2-\theta) \rceil) \vee 5$, consider $F_0^N \in \mathcal{P}^*_{sym,1}((\mathbb{R}^2)^N)$ and a $KS(\theta, N)$-process $(X_t^{i,N})_{t\geq 0, i \in [\![1,N]\!]}$ with initial law F_0^N, as well as the empirical measure for all $t \geq 0$, $\mu_t^N := N^{-1}\sum_{i=1}^N \delta_{X_t^{i,N}}$, which a.s. belongs to $\mathcal{P}(\mathbb{R}^2)$. We assume that μ_0^N goes weakly to f_0 in probability as $N \to \infty$.*

(i) The sequence $((\mu_t^N)_{t\geq 0})_{N \geq N_0}$ is tight in $C([0,\infty), \mathcal{P}(\mathbb{R}^2))$.

(ii) For any sequence $(N_k)_{k\geq 0}$ such that $(\mu_t^{N_k})_{t\geq 0}$ goes in law in $C([0,\infty), \mathcal{P}(\mathbb{R}^2))$ as $k \to \infty$ to some $(\mu_t)_{t\geq 0}$, this limit $(\mu_t)_{t\geq 0}$ is a.s a weak solution to (1)-(2) *starting from $\mu_0 = f_0$. Moreover, for all $T > 0$, all $\gamma \in (\theta, 2)$.*

$$\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \|x-y\|^{\gamma-2}\mu_t(\mathrm{d}x)\mu_t(\mathrm{d}y)\mathrm{d}t\Big] < \infty.$$

The proof is very similar to the proof of Theorem 1.

We derive the same result in the critical case by using the same proof but with the additionnal difficulty that a cluster must emerge in the particle system (with N being fixed) but not in the PDE (when $N \to \infty$). We use our understanding of the collisions, in particular we use that if one wants to see a collision in the critical case which is not a collision between 2 particles, one needs to have at least $N-2$ particles at the same place (because as explained befor, and since $k_0 = N$, there are only collisions of k particles with $k \in \{2, N-2, N-1, N\}$). However if the number of particle tends to infinity, there will always be some particles with an original trajectory which will deviate too much from the other particles and this means that at the limit, we will never see any $(N-2)$-collisions on a reasonnable time interval.

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