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présentée par

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Etude des modèles stochastique et déterministe de Keller-Segel

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"On n'est p't'être pas bien payés, mais qu'est ce qu'on s'marre !"

Les vieux de mon village

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Étude des modèles stochastique et déterministe de Keller-Segel

Résumé

Dans cette thèse, nous étudions le modèle de Keller-Segel : à la fois l'EDP et le système de particules. Ce modèle a attiré beaucoup d'attention à cause du phénomène de compétition étroite entre diffusion et concentration qu'il modélise. Nous nous intéressons plus particulièrement au comportement de ces modèles autour de l'instant de formation d'une masse de Dirac (ou de l'instant d'apparition d'un cluster dont la masse représente une proportion positive du nombre total de particules). Ce travail est divisé en cinq parties.

Dans une première partie, nous étudions finement le comportement des collisions pour le système de particules. Plus précisément ce système de particules consiste en N mouvements browniens dans le plan intéragissant en champ moyen via une intéraction de type Coulomb en $\theta/(Nr)$ où r est la distance entre deux particules et θ un paramètre positif tel que $N > 3\theta$. Selon les valeurs de θ et N il y a deux scénarios possibles, en voici un : le système de particules explose en temps fini en faisant émerger un cluster de k_0 particules, où k_0 est un entier déterministe dépendant de N et θ. Juste avant l'explosion, il y a une infinité de collisions impliquant $k_0 - 1$ particules parmi les k_0 impliquées dans le cluster de l'explosion. Puis avant chaque collision entre $k_0 - 1$ particules, il y a une infinité de collisions entre $k_0 - 2$ particules parmi les $k_0 - 1$ impliquées dans la collision à $k_0 - 1$ particules. De plus, avant chaque collision entre $k_0 - 2$ particules, il y a une infinité de collision de pair de particules impliquées dans la collision entre $k_0 - 2$ particules. Enfin, il n'y a aucune collision entre exactement k particules pour $k \in \{3, \ldots, k_0 - 2\}$.

Dans une deuxième partie, nous nous intéresserons à une preuve simplifiée de non explosion, i.e. d'existence globale, pour l'EDP de Keller-Segel, pour toute donnée initiale measure f_0 tel que $f_0(\mathbb{R}^2) < 8\pi$. La preuve repose sur un calcul de moment *à deux particules*.

Dans une troisième partie, nous prouverons la convergence de la mesure empirique du sytème de particules de Keller-Segel le long d'une sous suite dans les cas sous-critique et critique vers la solution faible de l'EDP de Keller-Segel. On s'inspire dans cette partie du travail de la seconde et on utilise le même argument de moment *à deux particules*.

Dans une quatrième partie, nous présenterons un projet en cours mais bien engagé. Il s'agit d'introduire une extension possible du système de particules au delà du temps d'explosion. Plus précisément, il s'agit du même système de particules pour tout les temps antérieurs au temps de formation du premier cluster, puis après ce temps, ce cluster est remplacé par une particule de masse équivalente à la somme des masses des particules impliquées dans ce cluster et on applique les lois d'intéraction entre les particules en tenant compte de la masse de chacune. On réitère ce procédé à chaque formation de cluster jusqu'à obtenir à la fin une seule particule équivalente de masse égale à la somme des masses de toute les particules, i.e égale à N. Nous montrerons en quoi cette extension peut être considérée comme naturelle.

Enfin, dans une cinquième partie, nous présenterons des simulations qui illustrent les résultats de la première partie.

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Chapitre 1

Introduction

1.1 Introduction aux équations de McKean-Vlasov dans le cas Lipschitz

Soit $d \geq 1$. Nous nous munissons de $b : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ et $\sigma : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathcal{M}_d(\mathbb{R})$ Lipschitz dans le sens où il existe $C > 0$ tel que pour tout x, $y \in \mathbb{R}^d$, tout μ, $\nu \in \mathcal{P}(\mathbb{R}^d)$, on a

$$|b(x,\mu) - b(y,\nu)| + |\sigma(x,\mu) - \sigma(y,\nu)| \leq C(|x-y| + \mathcal{W}_2(\mu,\nu)),$$

où $\mathcal{W}_2(\mu,\nu) = \inf_{\substack{X\sim\mu \\ Y\sim\nu}}(\mathbb{E}[\|X-Y\|^2])^{1/2}$ est la 2-distance de Wasserstein. On pose, pour tout $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}(\mathbb{R}^d)$, $a(x,\mu) = \sigma(x,\mu)\sigma(x,\mu)^T$. On peut écrire $a(x,\mu) = (a_{i,j}(x,\mu))_{1\leq i,j,\leq d}$. Nous considérons l'EDP

$$\partial_t f_t = \frac{1}{2} \sum_{1\leq i,j\leq d} \partial_{x_i}\partial_{x_j}[a_{i,j}(x,f_t)f_t] - \nabla \cdot [b(x,f_t)f_t], \tag{1.1}$$

où $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ est l'inconnue. On écrit abusivement f_t pour la mesure sur $\mathbb{R}^d$ de densité f_t. Nous admettrons que grâce aux hypothèses faites sur b et σ, (1.1) a une unique solution faible.

Il s'agit d'une équation de Fokker-Planck non linéaire, qui peut être pensée comme modélisant le comportement de particules diffusant selon la *diffusivité* σ et intéragissant entre elles et avec l'environnement selon le *drift* b. Le point de vue utilisé est le point de vue *eulérien* : on s'intéresse à l'évolution de la densité de particules $f_t(x)$ au temps t et à la position x. Un autre point de vue est toutefois possible : on s'intéresse plutôt au comportement d'une particule typique plongée dans la masse totale de ses semblables, on parle du point de vue *lagrangien*. L'un des buts de ce paragraphe est de montrer dans le cas simple de coefficients Lipschitz le lien entre ces deux points de vue.

L'équation du point de vue *lagrangien* est la suivante

$$\mathrm{d}X_t = \sigma(X_t,f_t)\mathrm{d}B_t + b(X_t,f_t)\mathrm{d}t, \tag{1.2}$$

où pour $t \geq 0$, f_t est la loi de X_t et $(B_t)_{t\geq 0}$ est un mouvement brownien de dimension d. Montrons que f_t est solution faible de (1.1). Soit $\varphi \in C_c^\infty(\mathbb{R}^d)$. Par la formule d'Itô, on a

$$\begin{aligned}\mathbb{E}[\varphi(X_t)] =&\mathbb{E}[\varphi(X_0)] + \mathbb{E}\Big[\int_0^t \nabla\varphi(X_s)\cdot b(X_s,f_s)\mathrm{d}s\Big]\\ &+\frac{1}{2}\sum_{1\leq i,j\leq d}\mathbb{E}\Big[\int_0^t \partial_{x_i}\partial_{x_j}\varphi(X_s)\sum_{k=1}^d \sigma_{i,k}(X_s,f_s)\sigma_{j,k}(X_s,f_s)\mathrm{d}s\Big]\\ =&\mathbb{E}[\varphi(X_0)] + \mathbb{E}\Big[\int_0^t \nabla\varphi(X_s)\cdot b(X_s,f_s)\mathrm{d}s\Big]\\ &+\frac{1}{2}\sum_{1\leq i,j\leq d}\mathbb{E}\Big[\int_0^t \partial_{x_i}\partial_{x_j}\varphi(X_s)a_{i,j}(X_s,f_s)\mathrm{d}s\Big].\end{aligned}$$

Utilisant le théorême de Fubini, on obtient,

$$\mathbb{E}[\varphi(X_t)] = \mathbb{E}[\varphi(X_0)] + \int_0^t \mathbb{E}[\nabla\varphi(X_s)\cdot b(X_s,f_s)]\mathrm{d}s + \frac{1}{2}\sum_{1\leq i,j\leq d}\int_0^t \mathbb{E}[\partial_{x_i}\partial_{x_j}\varphi(X_s)a_{i,j}(X_s,f_s)]\mathrm{d}s,$$

d'où, en dérivant par rapport au temps et en se rappellant la définition de f_t,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\varphi f_t = \int_{\mathbb{R}^d}\nabla\varphi(x)b(x,f_t)f_t(x)\mathrm{d}x + \frac{1}{2}\sum_{1\leq i,j\leq d}\int_{\mathbb{R}^d}\partial_{x_i}\partial_{x_j}\varphi(x)a_{i,j}(x,f_t)f_t(x)\mathrm{d}x,$$

ce qui est le résultat désiré.

On a donc un point de vue probabiliste pour étudier les phénomènes modélisés par les EDP du type de (1.1). Il faut comprendre que dans le cas de cette équation, nous faisons l'approximation qu'il y a un nombre infini de particules, mais il est plus naturel de considérer qu'il n'y en a qu'un nombre fini. Prenons $N \geq 1$ et étudions

$$\text{pour tout } i \in [\![1,N]\!], \quad \mathrm{d}X_t^{i,N} = \sigma(X_t^{i,N},\mu_t^N)\mathrm{d}B_t^i + b(X_t^{i,N},\mu_t^N)\mathrm{d}t, \tag{1.3}$$

où $(B_t)_{t\geq 0} := (B_t^1,\dots,B_t^N)_{t\geq 0}$ est un mouvement brownien de dimension dN et pour tout $t \geq 0$, $\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$ est la mesure empirique du systême de particule $(X_t^{1,N},\dots X_t^{N,N})_{t\geq 0}$.

Nous nous intéressons au cas particulier où

$$\text{pour } x\in\mathbb{R}^d, \mu\in\mathcal{P}(\mathbb{R}^d) \quad b(x,\mu) = \tilde{b}(x, K_1\star\mu(x)) \text{ et } \sigma(x,\mu) = \tilde{\sigma}(x,K_2\star\mu(x)),$$

où $\tilde{b}$ et $\tilde{\sigma}$ sont Lipschitz et pour tout noyau $K : (\mathbb{R}^d)^2 \to \mathbb{R}^d$ mesurable, $x \in \mathbb{R}^d$ et $\mu \in \mathcal{P}(\mathbb{R}^d)$, on note $K\star\mu(x) = \int_{\mathbb{R}^d} K(x,y)\mu(\mathrm{d}y)$. De plus K_1 et K_2 sont des noyaux Lipschitz, i.e pour tout $i \in \{1,2\}$, il existe $\|K_i\|_{Lip} > 0$ tel que pour tout x, x', y, $y' \in \mathbb{R}^d$, $\|K_i(x,y) - K_i(x',y')\| \leq \|K_i\|_{Lip}(\|x-x'\| + \|y-y'\|)$. On suppose de plus que K_1 et K_2 sont bornés.

L'enjeu ici est de montrer que l'approximation de (1.2) par (1.3) est pertinente. Il y a plusieurs méthodes possibles, ici nous choisissons de présenter la méthode par couplage synchrone. Il s'agit de prendre un couplage particulier du système de N particules et de N copies indépendantes de (1.2). On essaie de contrôler la distance de Wasserstein entre ces systèmes de particules et on montre qu'elle tend vers 0 lorsque N tend vers l'infini.

Théorème 1.1. *Pour tout $T > 0$, en posant pour tout $i \in [\![1, N]\!]$,*

$$X_t^{i,N} = X_0^{i,N} + \int_0^t \sigma(X_s^{i,N}, \mu_s^N)\mathrm{d}B_s^i + \int_0^t b(X_s^{i,N}, \mu_s^N)\mathrm{d}s,$$

et

$$\overline{X}_t^{i,N} = \overline{X}_0^{i,N} + \int_0^t \sigma(\overline{X}_s^i, f_s)\mathrm{d}B_s^i + \int_0^t b(\overline{X}_s^i, f_s)\mathrm{d}s,$$

où $(B_t)_{t\geq 0} := (B_t^1, \ldots, B_t^N)_{t\geq 0}$ est un mouvement brownien de dimension dN, il existe c_1, $c_2 > 0$ tels que pour tout $N \geq 1$,

$$\frac{1}{N}\sum_{i=1}^N \mathbb{E}\Big[\sup_{t\leq T}|X_s^{i,N} - \overline{X}_t^{i,N}|^2\Big] \leq \frac{c_1}{N}e^{c_2 T}.$$

Démonstration. On a pour $T \geq 0$, $t \in [0, T]$, et $i \in [\![1, N]\!]$, par l'inégalité de Cauchy-Schwartz,

$$\begin{aligned}\sup_{t\in[0,T]}\|\overline{X}_t^{i,N} - X_t^{i,N}\|^2 \leq & 3\|\overline{X}_0^{i,N} - X_0^{i,N}\|^2 + 3\sup_{t\in[0,T]}\Big\|\int_0^t(\sigma(\overline{X}_s^{i,N}, f_s) - \sigma(X_s^{i,N}, \mu_s^N)\mathrm{d}B_s^i\Big\|^2 \\ & + 3\sup_{t\in[0,T]}\Big\|\int_0^t(b(\overline{X}_s^{i,N}, f_s) - b(X_s^{i,N}, \mu_s^N)\mathrm{d}s\Big\|^2.\end{aligned} \tag{1.4}$$

En utilisant l'inégalité de Burkholder-Davis-Gundy, on obtient l'existence d'une constante $C > 0$ telle que

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\Big\|\int_0^t(\sigma(\overline{X}_s^{i,N}, f_s) - \sigma(X_s^{i,N}, \mu_s^N)\mathrm{d}B_s^i\Big\|^2\Big] \leq C\mathbb{E}\Big[\int_0^T(\sigma(\overline{X}_s^{i,N}, f_s) - \sigma(X_s^{i,N}, \mu_s^N))^2\mathrm{d}s\Big].$$

On en déduit

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\Big\|\int_0^t(\sigma(\overline{X}_s^{i,N}, f_s) - \sigma(X_s^{i,N}, \mu_s^N)\mathrm{d}B_s^i\Big\|^2\Big] \leq S_1 + S_2 + S_3.$$

où

$$\begin{aligned} S_1 &= 3C\|\tilde{\sigma}\|_{Lip}^2\|K_2\|_{Lip}^2\mathbb{E}\Big[\int_0^T\|\overline{X}_s^{i,N} - X_s^{i,N}\|^2\mathrm{d}s\Big] \\ S_2 &= 3C\|\tilde{\sigma}\|_{Lip}^2\mathbb{E}\Big[\int_0^T(K_2 \star f_s(\overline{X}_s^{i,N}) - K_2 \star \overline{\mu}_s^N(\overline{X}_s^{i,N}))^2\mathrm{d}s\Big], \\ S_3 &= 3C\|\tilde{\sigma}\|_{Lip}^2\mathbb{E}\Big[\int_0^T(K_2 \star \overline{\mu}_s^N(\overline{X}_s^{i,N}) - K_2 \star \mu_s^N(\overline{X}_s^{i,N}))^2\mathrm{d}s\Big], \end{aligned}$$

avec $\overline{\mu}_s^N = N^{-1}\sum_{i=1}^N \delta_{\overline{X}_s^{i,N}}$. En effet, pour $s \in [0, T]$, comme $(\sigma(\overline{X}_s^{i,N}, f_s) - \sigma(X_s^{i,N}, \mu_s^N))^2$ vaut

$$\Big(\sigma(\overline{X}_s^{i,N}, f_s) - \sigma(\overline{X}_s^{i,N}, \overline{\mu}_s^N) + \sigma(\overline{X}_s^{i,N}, \overline{\mu}_s^N) - \sigma(\overline{X}_s^{i,N}, \mu_s^N) + \sigma(\overline{X}_s^{i,N}, \mu_s^N) - \sigma(X_s^{i,N}, \mu_s^N)\Big)^2,$$

par l'inégalité de Cauchy-Schwartz on majore $(\sigma(\overline{X}_s^{i,N}, f_s) - \sigma(X_s^{i,N}, \mu_s^N))^2$ par

$3(\sigma(\overline{X}_s^{i,N}, f_s) - \sigma(\overline{X}_s^{i,N}, \overline{\mu}_s^N))^2 + 3(\sigma(\overline{X}_s^{i,N}, \overline{\mu}_s^N) - \sigma(\overline{X}_s^{i,N}, \mu_s^N))^2 + 3(\sigma(\overline{X}_s^{i,N}, \mu_s^N) - \sigma(X_s^{i,N}, \mu_s^N))^2$.

On conclut car

$$\begin{aligned}\Big(\sigma(\overline{X}_s^{i,N}, \mu_s^N) - \sigma(X_s^{i,N}, \mu_s^N)\Big)^2 &\leq \|\tilde{\sigma}\|_{Lip}^2 \Big(\frac{1}{N}\sum_{j=1}^N (K_2(\overline{X}_s^{i,N}, X_s^{j,N}) - K_2(X_s^{i,N}, X_s^{j,N})\Big)^2 \\ &\leq \|\tilde{\sigma}\|_{Lip}^2 \|K_2\|_{Lip}^2 \|\overline{X}_s^{i,N} - X_s^{i,N}\|^2.\end{aligned}$$

Par définition de $\overline{\mu}_s^N$,

$$S_2 = 3C\|\tilde{\sigma}\|_{Lip}^2 \mathbb{E}\Big[\int_0^T \Big(\frac{1}{N}\sum_{j=1}^N (K_2 \star f_s(\overline{X}_s^{i,N}) - K_2(\overline{X}_s^{i,N}, \overline{X}_s^{j,N})\Big)^2 \mathrm{d}s\Big],$$

donc en développant le carré,

$$S_2 = \frac{3C\|\tilde{\sigma}\|_{Lip}^2}{N^2} \sum_{1 \leq j,k \leq N} \int_0^T A_{i,j,k}(s)\mathrm{d}s,$$

où pour tout i, j, $k \in [\![1, N]\!]$, tout $s \geq 0$,

$$A_{i,j,k}(s) = \mathbb{E}[(K_2 \star f_s(\overline{X}_s^{i,N}) - K_2(\overline{X}_s^{i,N}, \overline{X}_s^{j,N}))((K_2 \star f_s(\overline{X}_s^{i,N}) - K_2(\overline{X}_s^{i,N}, \overline{X}_s^{k,N}))].$$

Puis on obtient

$$S_2 \leq \frac{36C\|\tilde{\sigma}\|_{Lip}^2 \|K_2\|_\infty^2 T}{N} + \frac{3C\|\tilde{\sigma}\|_{Lip}^2}{N^2} \sum_{j,k \text{ tels que } i,j,k \text{ distincts}} \int_0^T A_{i,j,k}(s)\mathrm{d}s.$$

En effet $|\{(j,k) \in [\![1, N]\!]^2 : j = k \text{ ou } i = j \text{ ou } k = i\}|$ est majoré par

$$|\{(j,k) \in [\![1, N]\!]^2 : j = k\}| + 2|\{(j,k) \in [\![1, N]\!]^2 : j = i\}| \leq 3N.$$

Or pour tout $i, j, k \in [\![1, N]\!]$ distincts on a $A_{i,j,k} = 0$. En effet, comme $\overline{X}_s^{i,N}$, $\overline{X}_s^{j,N}$, $\overline{X}_s^{k,N}$ sont mutuellement indépendants, on obient en conditionnant par $\overline{X}_s^{i,N}$,

$$\begin{aligned}A_{i,j,k}(s) &= \mathbb{E}\Big[\mathbb{E}[K_2 \star f_s(\overline{X}_s^{i,N}) - K_2(\overline{X}_s^{i,N}, \overline{X}_s^{j,N}) | \overline{X}_s^{i,N}]\mathbb{E}[K_2 \star f_s(\overline{X}_s^{i,N}) - K_2(\overline{X}_s^{i,N}, \overline{X}_s^{k,N}) | \overline{X}_s^{i,N}]\Big] \\ &= \mathbb{E}[(K_2 \star f_s(\overline{X}_s^{i,N}) - K_2 \star f_s(\overline{X}_s^{i,N}))^2] \\ &= 0,\end{aligned}$$

où l'on a utilisé que pour $k \neq i$ la loi conditionnelle de $\overline{X}_s^{k,N}$ sachant $\overline{X}_s^{i,N}$ est f_s. Finalement,

$$S_2 \leq \frac{36C\|\tilde{\sigma}\|_{Lip}^2 \|K_2\|_\infty^2 T}{N}.$$

Enfin, en utilisant une inégalité de convexité on obtient

$$S_3 \leq 3C\|\tilde{\sigma}\|_{Lip}^2 \|K_2\|_{Lip}^2 \mathbb{E}\Big[\int_0^T \frac{1}{N}\sum_{i=1}^N \|\overline{X}_s^{i,N} - X_s^{i,N}\|^2 \mathrm{d}s\Big].$$

On obtient des estimées similaires pour la partie de droite de (1.4) faisant intervenir le drift, puis on conclut avec le Lemme de Gronwall. □

La méthode par couplage, reposant sur des arguments trajectoriels, s'avère inefficace dans des cas où les noyaux d'intéractions sont trop singuliers comme dans le cas de l'équation de Keller-Segel, et il faut alors se rabattre sur d'autres méthodes comme par exemple la méthode par compacité qui sera introduite dans le Chapitre 2.

1.2 Le modèle de Keller-Segel

Durant cette thèse, nous nous intéressons à un modèle précis, le modèle de Keller-Segel. Le modèle de Keller-Segel parabolique-parabolique modélise le mouvement de bactéries diffusant dans le plan, libérant un chemoattractant diffusant lui aussi, et qui attire ces mêmes bactéries :

$$\partial f_t = \Delta f_t - \nabla \cdot [f_t \nabla c_t] \tag{1.5}$$

$$\varepsilon \partial_t c_t = \Delta c_t + f_t, \tag{1.6}$$

où pour $t \geq 0$ et $x \in \mathbb{R}^2$, $f_t(x)$ et $c_t(x)$ représentent respectivement la densité de bactéries et la concentration de chemoattractant au temps t et à la position x, et $\varepsilon > 0$ est un paramètre modélisant l'inverse de la vitesse de diffusion du chemoattractant. Ce qui rend ce modèle difficile à étudier, outre les singularités qui apparaîtront aussi dans le cas parabolique-elliptique, est le fait que les bactéries ne s'attirent pas directement entre elles mais passent par un intermédiaire dont la concentration est c_t bouge aussi à son rythme. Cela dit en pratique, le chemoattractant bouge beaucoup plus vite que les bactéries, ce qui nous amène a considérer l'approximation $\varepsilon = 0$ où l'on suppose que le chemoattractant a le temps à chaque instant $t \geq 0$ d'atteindre son état stationnaire avant que les bactéries n'aient eu le temps de bouger. C'est ce que l'on appelle le modèle parabolique-elliptique :

$$\partial f_t = \Delta f_t - \nabla \cdot [f_t \nabla c_t] \tag{1.7}$$

$$\Delta c_t = -f_t. \tag{1.8}$$

Dans ce modèle, cela revient à ce que les bactéries soient attirées par elles même car la concentration du chemoattractant au temps t ne dépend plus que de la position des bactéries au temps t. De manière plus précise, on peut exprimer c_t en fonction de f_t et donc on obtient une équation fermée sur $(f_t)_{t\geq 0}$. Plus précisément l'équation (1.8) donne :

$$c_t = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y| f_t(y)\mathrm{d}y, \tag{1.9}$$

et donc $\nabla c_t = (2\pi)^{-1} K \star f_t$ où $\star$ désigne la convolution et pour $x \in \mathbb{R}^2 \setminus \{0\}$, $K(x) = -\|x\|^{-2}x$ et $K(0) = 0$. Ainsi (1.7) devient

$$\partial f_t = \Delta f_t - (2\pi)^{-1}\nabla \cdot [f_t K \star f_t]. \tag{1.10}$$

Remarquons d'abord que la partie droite de (1.10) se met sous la forme divergence, ce qui implique la conservation de la masse totale au fil du temps. On peut donc poser $M = f_0(\mathbb{R}^2)$ et on a ainsi pour tout $t \geq 0$, $M = f_t(\mathbb{R}^2)$. De plus, en notant $x = (x^1, x^2)$ pour $x \in \mathbb{R}^2$, on remarque en utilisant plusieurs intégrations par parties que pour $i \in \{1, 2\}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} x^i f_t(x) \mathrm{d}x = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K^i(x - y) f_t(x) f_t(y) \mathrm{d}x \mathrm{d}y,$$

où $K = (K^1, K^2)$. Etant donné que pour $x \in \mathbb{R}^2$, $K(-x) = -K(x)$, on déduit que la quantité $\int_{\mathbb{R}^2} x f_t(x) \mathrm{d}x$ est constante au fil du temps. Le barycentre des bactéries restant immobiles et le problème des collisions étant invariant par translation de toute les bactéries par un même vecteur, on peut se ramener au cas où $\int_{\mathbb{R}^2} x f_t(x) \mathrm{d}x = \int_{\mathbb{R}^2} x f_0(\mathrm{d}x) = 0$ pour tout $t \geq 0$.

Cette équation est tout d'abord intéressante car le noyau de convolution K est très singulier, les questions d'existence et unicité sont encore ouvertes dans certains cas. Ce qui est spécifiquement intéressant dans cette équation est la compétition entre deux phénomènes opposés. La partie laplacien de (1.10) implique une diffusion des bactéries qui les fait tendre à s'étaler dans l'espace, donc à globalement faire s'éloigner les bactéries entre elles, mais la deuxième partie du membre de droite décrit l'attraction des bactéries les unes vers les autres à cause du chemoattractant. Ainsi, le fait de savoir si les bactéries ont tendance à se rapprocher où à s'éloigner n'est pas clair étant donné que ces deux phénomènes contradictoires sont de nature différente. Savoir qui de la diffusion ou de la concentration l'emporte en définitive est une question importante dans la visée de démontrer l'existence d'une solution à cette équation, en effet si la concentration l'emporte, la singularité de l'équation va être massivement visitée et alors l'équation sera "vraiment" singulière. Dans le cas où la diffusion l'emporte, on peut espèrer qu'alors les bactéries ne visitent pas vraiment le lieu de la singularité et alors la singularité serait "fausse" en un sens, ce qui permettrait de définir une solution.

Néanmois, un simple calcul informel nous permet d'anticiper quel phénomène devrait l'emporter sur l'autre. Considérons $d_t = \int_{\mathbb{R}^2} \|x\|^2 f_t(x) \mathrm{d}x$ la dispersion des bactéries au temps t. On a

$$d_t' = \int_{\mathbb{R}^2} \|x\|^2 \partial_t f_t(x) \mathrm{d}x = \int_{\mathbb{R}^2} \|x\|^2 \Delta f_t(x) \mathrm{d}x - \frac{1}{2\pi} \int_{\mathbb{R}^2} \|x\|^2 \nabla \cdot [f_t K \star f_t](x) \mathrm{d}x.$$

Dans la première intégrale du membre de droite, deux intégration par parties successives donnent $\int_{\mathbb{R}^2} \|x\|^2 \Delta f_t(x) \mathrm{d}x = 4M$. Dans la deuxième intégrale, une intégration par parties donne

$$\int_{\mathbb{R}^2} \|x\|^2 \nabla \cdot [f_t K \star f_t](x) \mathrm{d}x = -2 \int_{\mathbb{R}^2} x \cdot K \star f_t(x) f_t(x) \mathrm{d}x = 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{x \cdot (x - y)}{\|x - y\|^2} f_t(x) f_t(y) \mathrm{d}x \mathrm{d}y.$$

Or, en symétrisant, on obtient

$$\int_{\mathbb{R}^2} \|x\|^2 \nabla \cdot [f_t K \star f_t](x) \mathrm{d}x = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\|x - y\|^2}{\|x - y\|^2} f_t(x) f_t(y) \mathrm{d}x \mathrm{d}y = M^2.$$

On conclut que

$$d_t = d_0 + (2\pi)^{-1} M(8\pi - M)t \quad \text{pour tout } t \geq 0, \tag{1.11}$$

cela fait supposer qu'il y a trois cas :

- le cas sous-critique : $M < 8\pi$. Alors, $d_t \to \infty$ quand $t \to \infty$, cela laisse penser que la diffusion l'emporte sur la concentration, l'équation devrait en principe admettre une solution en un sens relativement classique pour tout temps.
- le cas sur-critique : $M > 8\pi$. Dans ce cas la concentration semble l'emporter. En effet, comme $8\pi - M < 0$, on remarque que $d_t \to -\infty$ quand $t \to \infty$. Cela est absurde car $d_t \geq 0$ pour tout $t \geq 0$, cela donc suggère qu'à un moment toute les bactéries se rassemblent en un point, donnant une dispersion totale nulle, et le système n'est plus défini à partir de ce moment.
- le cas critique : $M = 8\pi$. Dans ce cas $(d_t)_{t\geq 0}$ est constant au fil du temps, ce qui suggère un équilibre entre la diffusion et la concentration. On pense qu'il y a ici aussi création d'un amas comme dans le cas sur-critique, mais qui se produit en un temps infini. Des solutions dans un sens relativement classique devraient exister mais ce cas semble plus délicat que le premier.

1.2.1 Le cas sous-critique

Le cas sous-critique est un cas qui a déjà largement été analysé. S'il semble clair grâce aux calculs précèdents que dans le cas sur-critique il n'y a pas de solution globale au sens classique, il n'est pas clair qu'il y a existence de solutions dans le cas où $M < 8\pi$. Nous allons décrire informellement la preuve de Blanchet-Dolbeault-Perthame [9] de ce fait.

On veut montrer qu'à priori, $\int_{\mathbb{R}^2} f_t(x)|\log f_t(x)| < \infty$ pour tout $t \geq 0$, car alors cela voudra dire que f_t sera une "vraie" fonction et pas une masse de Dirac par exemple. Nous ne montrerons que la partie de la preuve qui explique pourquoi $\int_{\mathbb{R}^2} f_t(x)\log f_t(x)$ est majorée par une quantité finie. En effet cela est l'élément crucial de la preuve car la minoration est en principe simple à montrer via l'inégalité de Jensen en utilisant la concavité de $x \mapsto x\log(x)$ et parce que la masse totale de f_t est constante, même si il faut écrire plus de détails pour montrer proprement pourquoi cela permet de conclure.

L'idée est de regarder l'énergie libre du système :

$$E[f_t] := \int_{\mathbb{R}^2} f_t(x)\log f_t(x)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^2} f_t(x)c_t(x)\mathrm{d}x.$$

En la dérivant on obtient

$$\frac{\mathrm{d}}{\mathrm{d}t}E[f_t] = \int_{\mathbb{R}^2} \partial_t f_t(x)(1+\log f_t(x))\mathrm{d}x - \int_{\mathbb{R}^2} \partial_t f_t(x)c_t(x)\mathrm{d}x. \tag{1.12}$$

En effet, d'après (1.9), on a $\int_{\mathbb{R}^2} f_t(x)c_t(x)\mathrm{d}x = -(2\pi)^{-1}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log|x-y|f_t(x)f_t(y)\mathrm{d}x\mathrm{d}y$, et comme cette expression est quadratique en f_t on obtient le résultat voulu.

Ainsi, en injectant (1.7) dans (1.12), on obtient

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E[f_t] &= -\int_{\mathbb{R}^2}[\nabla f_t(x) - f_t(x)\nabla c_t(x)]\cdot[\nabla\log(f_t(x)) - \nabla c_t(x)]\mathrm{d}x \\ &= -\int_{\mathbb{R}^2} f_t(x)\Big[\frac{\nabla f_t(x)}{f_t(x)} - \nabla c_t(x)\Big]\cdot[\nabla\log(f_t(x)) - \nabla c_t(x)]\mathrm{d}x \\ &= -\int_{\mathbb{R}^2} f_t(x)\|\nabla\log(f_t(x)) - \nabla c_t(x)\|^2\mathrm{d}x \\ &\leq 0.\end{aligned}$$

Ainsi, pour tout $t \geq 0$, $E[f_t] \leq E[f_0]$. D'autre part, nous allons utiliser l'inégalité logarithmique de Hardy-Littlewood-Sobolev :

Lemme 1.2. *Pour tout $g \in L^1(\mathbb{R}^2)$ positive tel que $g \log g$ et $g \log(1 + \|x\|^2) \in L^1(\mathbb{R}^2)$, si $\int_{\mathbb{R}^2} g(x)\mathrm{d}x = M$ on a*

$$\int_{\mathbb{R}^2} g(x) \log g(x)\mathrm{d}x + \frac{2}{M} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} g(x)g(y) \log|x-y|\mathrm{d}x\mathrm{d}y \geq -C_M,$$

où $C_M := M(1 + \log \pi - \log M)$.

Notons que les constantes dans cette inégalité sont optimales. Rappellons (1.9). Soit $\theta \in (0,1)$. Pour tout $t \geq 0$, $E[f_0]$ majore

$$(1-\theta) \int_{\mathbb{R}^2} f_t(x) \log f_t(x)\mathrm{d}x + \theta\Big(\int_{\mathbb{R}^2} f_t(x) \log f_t(x)\mathrm{d}x + \frac{1}{4\pi\theta} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f_t(x)f_t(y) \log|x-y|\mathrm{d}x\mathrm{d}y\Big).$$

On choisit θ de tel sorte que $(4\pi\theta)^{-1} = 2M^{-1}$, ce qui est possible car $M/(8\pi) \in (0,1)$. Ainsi, en utilisant le Lemme 1.2, on obtient $(1-\theta)\int_{\mathbb{R}^2} f_t(x) \log f_t(x)\mathrm{d}x - \theta C_M \leq E[f_0]$ et ainsi,

$$\int_{\mathbb{R}^2} f_t(x) \log f_t(x)\mathrm{d}x \leq \frac{E[f_0] + \theta C_M}{1-\theta} < \infty.$$

1.2.2 Le cas critique

Comme suggéré précédemment, on peut montrer l'existence dans un sens relativement classique de l'équation (1.7)-(1.8) dans le cas $M = 8\pi$. Cela a été aussi largement analysé, notamment par Bedrossian-Masmoudi [2], Biler-Karch-Laurençot-Nadzieja [4]-[5], Wei [51]. En s'intéressant au cas critique on se rend compte que contrairement au cas sous-critique, si l'on part d'une condition initiale qui est une masse de Dirac alors on n'arrive pas à définir le système. En effet si l'on considère moralement qu'on a formation d'un amas en temps infini dans le cas général, si l'on part directement de l'amas alors les bactéries sont déjà dans leur état stationnaire et donc y restent. Il semble que la condition à respecter pour une condition initiale f_0 pour avoir une solution globale est $\max_{x\in\mathbb{R}^2} f_0(\{x\}) < 8\pi$ où f_0 est une mesure de masse totale égale à 8π. On verra que c'est encore cette condition qu'il faudra respecter dans le cas sur-critique pour avoir une solution locale en temps. Le Chapitre 3 est assez proche des travaux de [4]-[5], à la différence qu'ils considèrent des solutions radiales ce qui leur permet de considérer une équation plus simple en changeant de variable.

1.2.3 Le cas sur-critique

Le cas où $M > 8\pi$ est le cas le plus singulier et le plus intéressant de ce problème, en effet (1.11) suggère qu'un amas se forme en temps fini, ne laissant aucun espoir à l'existence d'une solution dans un sens classique car la solution serait une solution à valeurs dans l'espace des mesures et des masses de Dirac se formeraient en temps fini. Néanmoins des résultats sur l'existence locale en temps de solutions *tempérées* ont été démontrés, notamment par Bedrossian Masmoudi [2] sous la condition $\max_{x\in\mathbb{R}^2} f_0(\{x\}) < 8\pi$. Cela dit, il n'a pas été démontré que le temps de vie de la solution ainsi construite coïncide avec le temps de formation d'une masse de Dirac. Suzuki [48] a démontré l'apparition d'une masse de Dirac de masse 8π au temps d'explosion, dans le cas où $\mathbb{R}^2$ est remplacé par un domaine borné à bords lisses.

Plus généralement, une question qui semble intéressante et qui est encore ouverte est la descripton du comportement d'une solution dans le cas sur-critique après formation d'une masse de Dirac. Velazquez [49]-[50] étudia ce cas de manière informelle en considèrant que la solution se décompose en une partie absolument continue par rapport à la mesure de Lebesgue et en plusieurs masses de Dirac qui intéragissent entre elles. Dolbeault-Schmeiser [16] ont apporté une justification partielle aux propositions de Velazquez en arrivant à définir un modèle entre deux instant de formation de masse de Dirac, mais ils n'arrivent pas à construire une solution qui décrit tout le processus et en particulier la formation d'une ou plusieurs masses de Dirac qui seront ensuites amenées à intéragir entre elles et avec la partie absolument continue.

1.2.4 Un pas vers les probabilités

Comme étudié dans la sous-section 1.1, il y a une version probabiliste de l'équation (1.10). Quitte à faire certains changements de variables, il est équivalent de considèrer l'équation

$$\partial_t f_t = \frac{1}{2}\Delta f_t - \theta \nabla \cdot [f_t K \star f_t],$$

partant d'une condition initiale f_0 telle que $\int_{\mathbb{R}^2} f_0(x)\mathrm{d}x = 1$. On vient de normaliser la masse totale de bactéries à 1 et la discussion sur la masse devient alors une discussion sur la *sensitivité* au chemoattractant $\theta > 0$. Le phénomène de transition à $M = 8\pi$ du problème précèdent devient un phénomène de transition à $\theta = 2$ ici : on observe des amas si les bactéries se sentent assez attirées par le chemoattractant. En suivant les idées présentées dans la sous-section 1.1, on considère l'équivalent *lagrangien* de ce problème *eulérien* :

$$\mathrm{d}X_t = \mathrm{d}B_t + \theta K \star f_t(X_t)\mathrm{d}t,$$

où $f_t = \mathcal{L}(X_t)$. Ici on suit le mouvement d'une particule typique immergée parmi une infinité de ses semblables. Toujours en suivant les idées de la sous-section 1.1 on considère le système de particules correspondant au même point de vue centré sur les individus où l'on considère que l'on a uniquement $N \geq 2$ bactéries :

$$\text{pour } i \in [\![1, N]\!], \quad \mathrm{d}X_t^{i,N} = \mathrm{d}B_t^i - \frac{\theta}{N} \sum_{j \neq i} \frac{X_t^{i,N} - X_t^{j,N}}{\|X_t^{i,N} - X_t^{j,N}\|^2} \mathrm{d}t. \tag{1.13}$$

C'est cette équation que nous appellons le système de particules de Keller-Segel.

1.3 Familiarisation informelle avec les formes de Dirichlet

Un outil indispensable à la rédaction du chapitre 2 est la théorie des formes de Dirichlet. Cette théorie est particulièrement délicate à manipuler c'est pourquoi une partie entière de l'introduction lui est consacrée. Une présentation plus rigoureuse et agrémentée de références précises sera faite en Appendix 2.13, ici nous en resterons à une présentation plus informelle et intuitive qui semble nécessaire étant donné le haut niveau technique de cette théorie.

L'idée générale de cette théorie est de fournir une description plus analytique de processus stochastiques comme les processus de diffusions. Dans un souci de clarté nous nous restreindrons au cas de ces processus à valeurs dans un ouvert de $\mathbb{R}^d$, la généralisation à d'autres espaces d'état étant bien sûr disponible dans le livre de Fukushima [24].

Soit $d \geq 1$ et O un ouvert de $\mathbb{R}^d$. On définit la topologie de la compactification à un point sur $O_\triangle = O \cup \{\triangle\}$. Plus précisément c'est la topologie engendrée par les ouverts de la topologie trace sur O et par les ensembles de la forme $K^c \cup \{\triangle\}$ où K est un compact de O pour la topologie trace. On dit qu'une suite $(x_n)_{n\geq 0}$ d'éléments de $O_\triangle$ converge dans cette topologie vers $x \in O_\triangle$ si et seulement si

- $x \in O$ et $x_n \to x$ quand $n \to \infty$ dans le sens usuel,
- ou $x = \triangle$ et pour tout K compact de O il exists $n_K \geq 0$ tel que pour tout $n \geq n_K$, $x_n \notin K$.

Une application de $\mathbb{R}_+$ vers $O_\triangle$ est continue au sens où elle l'est pour la topologie de la compactification à un point sur $O_\triangle$.

Soit $\mathbb{Y} = (\Omega^Y, \mathcal{M}, (Y_t)_{t\geq 0}, (\mathbb{P}_y)_{y\in E_\Delta})$ un processus de Markov à valeurs dans $O_\triangle$, continu, tel que $\mathbb{P}_y(Y_0 = y) = 1$ pour tout $y \in O_\triangle$ et avec $\triangle$ un état absorbant, i.e $Y_t = \triangle$ pour tout $t \geq 0$ sous $\mathbb{P}_\triangle$. Le temps de vie de $\mathbb{Y}$ est défini par $\zeta^Y = \inf\{t \geq 0 : Y_t = \triangle\}$.

On définit les noyaux de transition de $\mathbb{Y}$ pour tout $t \geq 0$, $y \in E_\Delta$ par

$$P_t^Y(y, \mathrm{d}z) = \mathbb{P}_y^Y(t < \zeta^Y, Y_t \in \mathrm{d}z).$$

Soit α une mesure Radon sur O. On dit que $\mathbb{Y}$ est α-symétrique si pour tout $t \geq 0$, toutes fonctions mesurables φ, $\psi : O \to \mathbb{R}_+$, $\int_O (P_t^Y \varphi)\psi \mathrm{d}\alpha = \int_O \varphi(P_t^Y \psi)\mathrm{d}\alpha$. Remarquons que dans le cas où $\mathbb{Y}$ est un processus réversible et récurrent positif de probabilité invariante α, alors $\mathbb{Y}$ est α-symétrique. Si on regarde ce même processus mais tué quand il touche un sous ensemble O' de O alors il n'est en général plus récurrent positif mais on peut facilement montrer qu'il est $\alpha|_{O'}$-symétrique.

Fixons α une mesure Radon sur O et supposons que $\mathbb{Y}$ soit α-symétrique. On peut alors définir l'espace de Dirichlet $(\mathcal{E}^Y, \mathcal{F}^Y)$ associé à $\mathbb{Y}$ dans $L^2(O, \alpha)$ de la manière suivante :

$$\mathcal{F}^Y = \left\{ \varphi \in L^2(O, \alpha) : \lim_{t\to 0} \int_O \frac{\varphi - P_t^Y \varphi}{t} \varphi \mathrm{d}\alpha \text{ existe } \right\}$$

$$\mathcal{E}^Y(\varphi, \psi) = \lim_{t\to 0} \int_O \frac{\varphi - P_t^Y \varphi}{t} \psi \mathrm{d}\alpha \quad \text{pour tout } \varphi, \psi \in \mathcal{F}^Y.$$

On définit le générateur $(\mathcal{A}^Y, \mathcal{D}_{\mathcal{A}^Y})$ de $\mathbb{Y}$ de la manière suivante

$$\mathcal{D}_{\mathcal{A}^Y} = \left\{ \varphi \in \mathcal{F}^Y : \exists h \in L^2(O, \alpha) \text{ tel que } \forall \psi \in \mathcal{F}^Y, \text{ on a } \mathcal{E}^Y(\varphi, \psi) = -\int_O h\psi \mathrm{d}\alpha \right\}$$

et dans ce cas $\mathcal{A}^Y \varphi = h$. On remarque que l'expression de la forme de Dirichlet $\mathcal{E}^Y$ devient pour $\varphi \in \mathcal{D}_{\mathcal{A}^Y}$, $\psi \in \mathcal{F}^Y$, $\mathcal{E}^Y(\varphi, \psi) = -\int_O \psi \mathcal{A}^Y \varphi \mathrm{d}\alpha$.

On dit que la forme de Dirichlet $(\mathcal{E}^Y, \mathcal{F}^Y)$ est régulière si il existe un noyau $\mathcal{C} \subset C_c^\infty(O) \cap \mathcal{F}^Y$ tel que $\mathcal{C}$ est dense dans $\mathcal{F}^Y$ pour la norme $\|\cdot\| = \mathcal{E}^Y(\cdot, \cdot) + \|\cdot\|_{L^2(O,\alpha)}$ et dans $C_c^\infty(O)$ pour la norme uniforme.

Deux formes de Dirichlet régulières $(\mathcal{E}, \mathcal{F})$ et $(\mathcal{E}', \mathcal{F}')$ qui ont un noyau commun $\mathcal{C}$ tel que pour tout $\varphi \in \mathcal{C}$, $\mathcal{E}(\varphi, \varphi) = \mathcal{E}'(\varphi, \varphi)$ sont nécessairement égales, i.e $\mathcal{E} = \mathcal{E}'$ et $\mathcal{F} = \mathcal{F}'$.

Prenons le cas du mouvement brownien sur $\mathbb{R}$. On peut montrer que ce processus a une forme de Dirichlet associée $(\mathcal{E}, \mathcal{F})$ sur $L^2(\mathbb{R}, \ell)$ avec ℓ la mesure de Lebesgue et que son générateur est la moitié de la dérivée seconde. Alors informellement, pour $\varphi \in \mathcal{F}$, il faut penser

$$\mathcal{E}^Y(\varphi, \varphi) = -\frac{1}{2} \int_{\mathbb{R}} \varphi \varphi'' \mathrm{d}\ell = \frac{1}{2} \int_{\mathbb{R}} |\varphi'|^2 \mathrm{d}\ell,$$

où l'on a utilisé une intégration par parties à la dernière égalité. On interprète la forme de Dirichlet comme une énergie.

La forme de Dirichlet d'un processus le caractérise en un sens que nous ne préciserons pas. Pour le comprendre informellement, remarquons que si l'on prend $\varphi \in C_c^\infty(O)$ alors par la formule d'Itô,

$$\mathbb{E}_f[\varphi(Y_t)] = \mathbb{E}_f[\varphi(Y_0)] + \int_0^t \mathbb{E}_f[\mathcal{A}^Y \varphi(Y_s)]\mathrm{d}s,$$

où $f \in L^2(O, \alpha)$ est la densité de la loi de Y_0. Si l'on note $T_t f$ la loi de X_t, on a

$$\begin{aligned}\int_O \varphi T_t f &= \int_O \varphi f + \int_0^t \int_O (\mathcal{A}^Y \varphi) \frac{T_s f}{\alpha} \mathrm{d}\alpha \mathrm{d}s \\ &= \int_O \varphi f \mathrm{d}\alpha - \int_0^t \mathcal{E}^Y(\varphi, \alpha^{-1} T_s f)\mathrm{d}s.\end{aligned}$$

Ainsi en un sens, la donnée de $\mathcal{E}^Y$ caractérise les lois finies dimensionnelles.

L'intérêt de cette théorie est quelle permet de donner un sens à des processus qui n'en ont pas au sens classique. On considère informellement le processus $(Y_t)_{t\geq 0}$ tué en sortant de O et vérifiant l'EDS :

$$\mathrm{d}Y_t = \sqrt{2}\mathrm{d}B_t - \nabla U(Y_t)\mathrm{d}t,$$

où $(B_t)_{t\geq 0}$ est un mouvement brownien de dimension d et $U : \mathbb{R}^d \to \mathbb{R}$ une application mesurable sur O telle que $e^{-U}\mathrm{d}\ell$ soit une mesure Radon sur O où ℓ est la mesure de Lebesue sur $\mathbb{R}^d$. Dans le cas général on ne sait bien sûr pas construire un tel processus, U n'étant pas nécessairement dérivable et pouvant être singulier au bord de O (comme ce sera le cas dans le cas du système de particules de Keller-Segel, voir Chapitre 2). Néanmoins, formellement $(Y_t)_{t\geq 0}$ a pour mesure invariante $\alpha := e^{-U}\mathrm{d}\ell$ Radon par hypothèse. Si l'on note $T_t f$ la loi de Y_t où $(Y_t)_{t\geq 0}$ part de Y_0 où Y_0 a une loi de densité f, alors $(T_t f)_{t\geq 0}$ vérifie l'EDP suivante :

$$\partial_t T_t f = \nabla \cdot \Big(\alpha \nabla\Big(\frac{T_t f}{\alpha}\Big)\Big).$$

En effet, comme $\mathrm{d}Y_t = \sqrt{2}\mathrm{d}B_t + \nabla \log \alpha(Y_t)\mathrm{d}t$, par la formule d'Itô on obtient pour tout $\varphi \in C_c^\infty(O)$,

$$\mathbb{E}_f[\varphi(Y_t)] = \mathbb{E}_f[\varphi(Y_0)] + \int_0^t \mathbb{E}_f[\mathcal{A}^Y \varphi(Y_s)]\mathrm{d}s,$$

où $\mathcal{A}^Y \varphi = \alpha^{-1}\nabla \cdot \Big(\alpha \nabla \varphi\Big)$. Ainsi, on obtient,

$$\int_O \varphi T_t f = \int_O \varphi f + \int_0^t \int_O (\mathcal{A}^Y \varphi) T_t f \mathrm{d}s,$$

ce qui nous donne le résultat grâce à deux intégration par parties.

Ainsi, on remarque que

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_O \Big(\frac{T_t f}{\alpha}\Big)^2 \mathrm{d}\alpha &= 2 \int_0 \partial_t T_t f (T_t f \alpha^{-1}) \\ &= -2 \int_O \Big\| \nabla \Big(\frac{T_t f}{\alpha}\Big)\Big\|^2 \mathrm{d}\alpha.\end{aligned}$$

En intégrant, on obtient

$$\int_0^t 2\int_O \left\|\nabla\Big(\frac{T_t f}{\alpha}\Big)\right\|^2 \alpha \leq \int_O \Big(\frac{f}{\alpha}\Big)^2 \alpha.$$

Si l'on suppose au départ que $f/\alpha \in L^2(O,\alpha)$, alors on a informellement $T_t f/\alpha \in H^1(O,\alpha)$ pour presque tout $t \geq 0$, on a donc une majoration à priori de la solution ce qui laisse entrevoir la possibilité d'une preuve de l'existence de $T_t f$, et donc avec plus de travail de l'existence de $(Y_t)_{t\geq 0}$.

Pour creuser l'analogie avec l'énergie, nous définissons la capacité relative à $(\mathcal{E}^Y, \mathcal{F}^Y)$ d'un ensemble. On note $\mathcal{U}$ l'ensemble des ouverts de O et, pour $A \in \mathcal{U}$, on définit

$$\mathcal{L}_A = \{u \in \mathcal{F}^Y : u \geq 1, \alpha - p.p \text{ sur } A\},$$
$$\mathrm{Cap}(A) = \inf_{u\in\mathcal{L}_A} \mathcal{E}_1^Y(u,u) \text{ si } \mathcal{L}_A \neq \emptyset \quad \text{ et } \quad \mathrm{Cap}(A) = \infty \text{ si } \mathcal{L}_A = \emptyset,$$

où pour $u \in \mathcal{F}^Y$, $\mathcal{E}_1^Y(u,u) = \mathcal{E}^Y(u,u) + \int_O u^2 \mathrm{d}\alpha$. Finalement on définit pour tout $A \subset O$,

$$\mathrm{Cap}(A) = \inf_{B\in\mathcal{U}, A\subset B} \mathrm{Cap}(B).$$

De manière totalement heuristique, la capacité doit être pensée comme une capacité électrique et elle est obtenue en minimisant une fonctionnelle d'énergie à la manière du principe de moindre action. En un sens, la capacité doit être pensée comme l'inverse de la résistance et les probabilités de présence du processus comme le courant électrique. Si la capacité est nulle, alors la résistance est infinie et le courant de probabilité ne passe pas : un ensemble de capacité nulle n'est jamais visité par le processus.

De là émerge la notion de propriété vérifiée quasi-partout. Il s'agit d'une propriété vérifiée pour tout point excepté un ensemble de capacité nulle. Encore une fois, intuitivement une propriété vérifiée presque partout est une propriété vérifiée de partout du point de vue du processus. Par exemple pour un mouvement brownien sur $\mathbb{R}$, il n'existe pas d'ensemble de capacité nulle étant donné que ce processus visite tout $\mathbb{R}$ presque sûrement, par contre en dimension 2, les singletons sont de capacité nulle car jamais visités par le brownien, mais les droites ne sont pas de capacité nulle.

La notion de capacité est cruciale pour faire le lien entre le processus $\mathbb{Y}$ et son espace de Dirichlet $(\mathcal{E}^Y, \mathcal{F}^Y)$, nous allons tenter d'illustrer cette importance via un exemple.

Prenons l'exemple de $\mathbb{M} = (\Omega^M, \mathcal{M}^M, (M_t)_{t\geq 0}, (\mathbb{P}_m)_{m\in\mathbb{R}})$ un mouvement brownien de dimension 1. Si l'on suppose que l'on connaît $\mathcal{A}^M\varphi$ comme élément de $L^2(\mathbb{R},\ell)$ pour φ dans une classe suffisamment large de fonctions, alors cela ne suffit pas à caractériser le mouvement brownien. En effet si l'on prend l'application $\mathcal{A} : \varphi \to \Delta\varphi \mathbb{1}_{x\neq 0}$ alors pour tout φ on aura $\mathcal{A}^M\varphi = \mathcal{A}\varphi$ dans $L^2(\mathbb{R},\ell)$. Or la dynamique induite par $\mathcal{A}$ est celle d'un mouvement brownien qui s'arrête en 0 dès lors qu'il atteint 0. Ce qui bloque dans cet exemple est le fait que $\mathrm{Cap}^M(\{0\}) \neq 0$, ainsi 0 est un point potentiellement visité par le processus et ainsi il nous faut connaître l'action du générateur en ce point. De manière plus générale, la notion d'ensemble négligable semble inadaptée pour décrire un processus via sa forme de Dirichlet, et la notion d'ensembles de capacité nulle semble être la bonne notion.

Le reste de cette section est consacré à la présentation de résultats expliquant comment certaines opérations effectuées sur une diffusion $\mathbb{Y}$ sont traduites dans le monde des formes Dirichlet.

1.3.1 Transience et Récurrence

On dit qu'un borélien A de O est $(P_t^Y)_{t\geq 0}$-invariant si pour tout $\varphi \in L^2(O,\alpha)$, tout $t>0$, on a $P_t^Y(\mathbb{1}_A\varphi) = \mathbb{1}_A P_t^Y \varphi$ α-p.p. On dit que $(\mathcal{E}^Y, \mathcal{F}^Y)$ est irréductible si pour tout A ensemble $(P_t^Y)_{t\geq 0}$-invariant, $\alpha(A)=0$ ou $\alpha(O/A)=0$.

On dit que $(\mathcal{E}^Y, \mathcal{F}^Y)$ est récurrent si pour tout $\varphi \in L^1(O,\alpha)$ positif, pour α-presque tout $y\in O$, on a $\mathbb{E}_y[\int_0^\infty \varphi(Y_s)\mathrm{d}s] \in \{0,\infty\}$.

On dit que $(\mathcal{E}^Y, \mathcal{F}^Y)$ est transient si pour tout $\varphi \in L^1(O,\alpha)$ positif, pour α-presque tout $y\in O$, on a $\mathbb{E}_y[\int_0^\infty \varphi(Y_s)\mathrm{d}s] < \infty$ avec la convention $\varphi(\triangle)=0$.

Enfin, si $(\mathcal{E},\mathcal{F})$ est un espace de Dirichlet régulier alors il est soit récurrent, soit transient.

1.3.2 Unicité

Deux processus de diffusion sur un ouvert de $\mathbb{R}^d$ qui ont le même espace de Dirichlet partagent aussi la même notion de quasi-partout, à condition que la capacité de chaque compact soit finie (ce qui sera toujours le cas dans ce manuscrit). De plus, les fonctions de transition des deux processus coincident en dehors d'un ensemble de capacité nulle.

1.3.3 Martingales

Si l'on suppose que $\mathrm{Supp}(\alpha)=O$, alors si $\varphi : O\to\mathbb{R}$ appartient à $\mathcal{D}_{\mathcal{A}^Y}$ tel que φ et $\mathcal{A}^Y\varphi$ soient bornés, alors si l'on définit

$$M_t^\varphi = \varphi(Y_t) - \varphi(Y_0) - \int_0^t \mathcal{A}^Y\varphi(Y_s)\mathrm{d}s,$$

avec la convention $\varphi(\triangle) = \mathcal{A}^Y\varphi(\triangle) = 0$, on a $(M_t^\varphi)_{t\geq 0}$ est une $\mathbb{P}_y^Y$-martingale dans la filtration canonique de $(Y_t)_{t\geq 0}$ quasi-partout. C'est par cette formule que l'on retrouve la formule d'Itô à condition de l'appliquer à des φ adaptés, les conditions à vérifier étant restrictives. Par exemple comme $\mathcal{D}_{\mathcal{A}^Y}\subset L^2(O,\alpha)$, une contrainte d'intégrabilité apparaît alors qu'elle ne semble pas nécessaire pour appliquer Itô au processus de départ. Il faut dont approcher les fonctions φ d'intérêt par des fonctions adaptées.

1.3.4 Changement de temps

On suppose $\mathrm{Supp}(\alpha) = O$ et $g : O \to (0,\infty)$ continue tel que $g(\triangle)=0$ et on pose $A_t = \int_0^t g(Y_s)\mathrm{d}s$ et son inverse généralisé $\rho_t = \inf\{s>0 : A_s > t\}$. On pose finalement $X_t = Y_{\rho_t}\mathbb{1}_{\rho_t<\infty} + \triangle\mathbb{1}_{\{\rho_t=\infty\}}$. Alors il est bien connu que $(X_t)_{t\geq 0}$ est aussi une diffusion sur O. Si $(Y_t)_{t\geq 0}$ était une diffusion sur O alors on peut écrire pour $\varphi\in C^2(O)$,

$$\varphi(Y_t) = \varphi(Y_0) + M_t^\varphi + \int_0^t \mathcal{A}^Y\varphi(Y_s)\mathrm{d}s,$$

et ainsi en changeant le temps on obtient

$$\varphi(Y_{\rho_t}) = \varphi(Y_0) + N_t^\varphi + \int_0^t g^{-1}(Y_{\rho_s})\mathcal{A}^Y\varphi(Y_{\rho_s})\mathrm{d}s,$$

où $(N_t^\varphi)_{t\geq 0} = (M_{\rho_t}^\varphi)_{t\geq 0}$ est une martingale pour la filtration $(\mathcal{M}_{\rho_t}^Y)_{t\geq 0}$ puisque la suite $(\rho_t)_{t\geq 0}$ est une suite de temps d'arrêts. Cela est une preuve informelle du fait que le générateur de $(Y_{\rho_t})_{t\geq 0}$ est $g^{-1}\mathcal{A}^Y$. Enfin pour trouver la probabilité invariante de $(Y_{\rho_t})_{t\geq 0}$ il suffit de trouver la mesure μ telle que pour tout $\varphi \in C_c^\infty(O)$, $\int_0 g^{-1}\mathcal{A}^Y\varphi \mathrm{d}\mu = 0$. Mais on voit qu'il suffit de prendre $\mu = g\alpha$. Ainsi cela suggère que $(X_t)_{t\geq 0}$ a un espace de Dirichlet $(\mathcal{E}^X, \mathcal{F}^X)$ associé sur $L^2(O, g\alpha)$ tel que pour tout $\varphi \in \mathcal{F}^X$,

$$\mathcal{E}^X(\varphi,\varphi) = -\int_O \varphi g^{-1}\mathcal{A}^Y\varphi g \mathrm{d}\alpha = -\int_O \varphi\mathcal{A}_Y\varphi \mathrm{d}\alpha.$$

1.3.5 Concaténation

Prenons deux processus de diffusion à valeurs dans deux ouverts O_1, O_2 de respectivement $\mathbb{R}^{d_1}$, $\mathbb{R}^{d_2}$ indépendants $(Y_t^1)_{t\geq 0}$ et $(Y_t^2)_{t\geq 0}$ qui sont respectivement α_1-symétrique et α_2-symétrique. Alors le processus $(X_t)_{t\geq 0} = (Y_t^1, Y_t^2)_{t\geq 0}$ est aussi un processus de Markov de générateur $\mathcal{A}^X$ tel que pour tout φ assez régulière, pour tout $y_1 \in O_1$, tout $y_2 \in O_2$,

$$\mathcal{A}^X\varphi(y_1,y_2) = \mathcal{A}^{Y^1}(\varphi(\cdot,y_2))(y_1) + \mathcal{A}^{Y^2}(\varphi(y_1,\cdot)).(y_2).$$

Comme précèdemment, il suffit pour obtenir la mesure invariante de trouver une mesure μ telle que pour tout φ assez régulière, $\int_{O_1\times O_2}\mathcal{A}^X\varphi \mathrm{d}\mu = 0$. Or, on a pour tout $y_1 \in O_1$, tout $y_2 \in O_2$,

$$\int_{O_1}\mathcal{A}^{Y^1}(\varphi(\cdot,y_2))\mathrm{d}\alpha_1 = \int_{O_2}\mathcal{A}^{Y^2}(\varphi(y_1,\cdot))\mathrm{d}\alpha_2 = 0,$$

et donc $\int_{O_1\times O_2}\mathcal{A}^X\varphi \mathrm{d}\alpha_1\otimes\alpha_2 = 0$, ce qui impose que $(X_t)_{t\geq 0}$ soit $\alpha_1\otimes\alpha_2$-symétrique. Ainsi, on s'attend à ce que le processus $(X_t)_{t\geq 0}$ ait un espace de Dirichlet sur $L^2(O_1\times O_2, \alpha_1\otimes\alpha_2)$ et que pour tout $\varphi \in \mathcal{F}^X$,

$$\begin{aligned}\mathcal{E}^X(\varphi,\varphi) = &-\int_{O_1}\int_{O_2}\varphi(y_1,y_2)\mathcal{A}^{Y^1}[\varphi(\cdot,y_2)](y_1)\mathrm{d}\alpha_1(y_1)\mathrm{d}\alpha_2(y_2)\\ &-\int_{O_2}\int_{O_1}\varphi(y_1,y_2)\mathcal{A}^{Y^2}[\varphi(y_1,\cdot)](y_2)\mathrm{d}\alpha_2(y_2)\mathrm{d}\alpha_1(y_1).\end{aligned}$$

Prenons l'exemple de $(B_t)_{t\geq 0} = (B_t^1, B_t^2)_{t\geq 0}$, mouvement brownien de dimension 2. Pour chaque $i \in [\![1,2]\!]$ la forme de Dirichlet $\mathcal{E}^i$ de $(B_t^i)_{t\geq 0}$ est défini par $\mathcal{E}^i(\varphi,\varphi) = \int_\mathbb{R}|\varphi'|^2\mathrm{d}\ell$ avec ℓ la mesure de Lebesque sur $\mathbb{R}$. Ainsi par le résultat précédent, si l'on note $\mathcal{E}$ la forme de Dirichlet de $(B_t)_{t\geq 0}$, pour tout $\varphi \in \mathcal{F}$ on a

$$\mathcal{E}(\varphi,\varphi) = \frac{1}{2}\int_\mathbb{R}\int_\mathbb{R}|\partial_1\varphi|^2\mathrm{d}\ell\otimes\ell + \frac{1}{2}\int_\mathbb{R}\int_\mathbb{R}|\partial_2\varphi|^2\mathrm{d}\ell\otimes\ell = \frac{1}{2}\int_{\mathbb{R}^2}\|\nabla\varphi\|^2\mathrm{d}\ell_2,$$

avec ℓ_2 la mesure de Lebesgue sur $\mathbb{R}^2$.

1.3.6 Tuage

Prenons encore un processus de diffusion $(X_t)_{t\geq 0}$ à valeurs dans O et tuons le lorsqu'il sort d'un ouvert U de O. On pose $\tau_U = \inf\{t \geq 0 : X_t \notin U\}$ et pour $t \geq 0$, $Y_t = X_t$ si $t < \tau_U$ et $Y_t = \triangle$ un état cimetière dans l'autre cas. Alors $(Y_t)_{t\geq 0}$ est un processus de diffusion et a un espace de Dirichlet associé sur $L^2(U, \alpha|_U)$ tel que pour tout $\varphi \in \mathcal{F}^Y$,

$$\mathcal{E}^Y(\varphi,\varphi) = -\int_U \varphi\mathcal{A}^X\varphi \mathrm{d}\alpha.$$

1.3.7 Girsanov

Nous illustrons le théorème de Girsanov dans le monde des formes de Dirichlet par un exemple. Considérons $(X_t)_{t\geq 0}$ le processus à valeurs dans $\mathbb{R}^d$ vérifiant

$$\mathrm{d}X_t = \sqrt{2}\mathrm{d}B_t + \frac{\nabla\mu(X_t)}{\mu(X_t)}\mathrm{d}t,$$

tué lorsqu'il sort de O un ouvert de $\mathbb{R}^d$, avec $\mu : \mathbb{R}^d \to \mathbb{R}$. Soit $\phi : \mathbb{R}^d \to \mathbb{R}$. On s'intéresse à trouver, à partir de l'espace de Dirichlet $(\mathcal{E}^X, \mathcal{F}^X)$ de $(X_t)_{t\geq 0}$, l'espace de Dirichlet $(\mathcal{E}^Y, \mathcal{F}^Y)$ du processus $(Y_t)_{t\geq 0}$ vérifiant

$$\mathrm{d}Y_t = \sqrt{2}\mathrm{d}B_t + \frac{\nabla\mu(Y_t)}{\mu(Y_t)}\mathrm{d}t + \frac{\nabla\phi(Y_t)}{\phi(Y_t)}\mathrm{d}t,$$

tué lorsqu'il sort de O, ce qui revient en un sens à comprendre quelle est l'action de l'application du théorème de Girsanov dans le monde des formes de Dirichlet.

On remarque que le générateur de $(X_t)_{t\geq 0}$ est

$$\mathcal{A}^X = \Delta + \frac{\nabla\mu}{\mu}\cdot\nabla = \frac{1}{\mu}\nabla\cdot[\mu\nabla],$$

et ainsi on cherche une mesure ν tel que $\int \mathcal{A}^X\varphi\mathrm{d}\nu = 0$ pour tout φ. Si ν a une densité que l'on note abusivement ν, il suffit que

$$\nabla\cdot\left[\mu\nabla\left(\frac{\nu}{\mu}\right)\right] = 0.$$

On remarque que $\nu = \mu$ convient, ainsi $(X_t)_{t\geq 0}$ est μ-symétrique. On obtient alors pour $\varphi \in \mathcal{F}^X$,

$$\mathcal{E}^X(\varphi,\varphi) = -\int_O \varphi\frac{1}{\mu}\nabla\cdot[\mu\nabla\varphi]\mathrm{d}\mu = \int_O \|\nabla\varphi\|^2\mathrm{d}\mu,$$

où l'on a effectué une intégration par parties à la dernière égalité.

De la même manière, comme

$$\frac{\nabla\mu}{\mu} + \frac{\nabla\phi}{\phi} = \frac{\nabla(\mu\phi)}{\mu\phi},$$

en appliquant les calculs précèdents, on obtient que le processus $(Y_t)_{t\geq 0}$ est un processus $\mu\phi$-symétrique vérifiant pour $\varphi \in \mathcal{F}^Y$,

$$\mathcal{E}^Y(\varphi,\varphi) = \int_O \|\nabla\varphi\|^2\phi\mathrm{d}\mu.$$

On fixe $T > 0$ et on pose $\mathrm{d}\mathbb{Q} = \mathcal{E}(L)_T\mathrm{d}\mathbb{P}$ où pour $t \geq 0$,

$$L_t = -2^{-1/2}\int_0^t \frac{\nabla\phi(X_s)}{\phi(X_s)}\cdot\mathrm{d}B_s.$$

On admet que les hypothèses nécessaires à l'application du théorème de Girsanov sont réunis, alors la loi du processus $(X_t)_{t\geq 0}$ sous $\mathbb{Q}$ est égal à la loi du processus $(Y_t)_{t\geq 0}$. Ainsi, l'application du théorème de Girsanov pour faire apparaître le terme $\phi^{-1}\nabla\phi$ dans l'EDS de départ semble correspondre à la multiplication par ϕ de la mesure invariante et à l'introduction d'un terme multiplicatif ϕ dans l'intégrale de Dirichlet. Cela est généralisé plus bas, voir Chapitre 2 et encore plus dans Fukushima [24].

1.4 Vue d'ensemble des chapitres

1.4.1 Chapitre 2

Suzuki [48] a prouvé que pour l'EDP de Keller-Segel dans le cas sur-critique dans un domaine borné, les solutions doivent exploser en temps fini et la solution mesure converge quand le temps converge vers le temps d'explosion vers une mesure qui se décompose en la somme d'une mesure absolument continue par rapport à la mesure de Lebesgue et une somme de mesure de Dirac de masse égale à 8π. De plus, les points où l'aggrégation a lieu n'appartiennent pas au bord du domaine. Ce travail est en lien avec celui de Herrero et Velázquez [28] où ils ont prouvé l'existence d'une solution radiale vérifiant les mêmes propriétés dans le cas où le domaine est une boule. A notre connaissance, il n'existe pas de résultats analogues pour le système de particules. En outre, le résultat que nous prouvons ici présente une description très précise du comportement des particules juste avant l'explosion.

Nous rappellons l'équation (1.13) :

$$\mathrm{d}X_t^{i,N} = \mathrm{d}B_t^i - \frac{\theta}{N}\sum_{j\neq i}\frac{X_t^{i,N} - X_t^{j,N}}{\|X_t^{i,N} - X_t^{j,N}\|^2}\mathrm{d}t,$$

où $\theta > 0$ est la sensitivité chimique des bactéries au chemoattractant. On éssaie ici de décrire précisément le comportement du processus proche du temps d'explosion, en particulier on se demande comment intéragissent les deux phénomènes opposés qui sont la diffusion et la concentration à cet instant. Par exemple, prenons $N = 3$ et supposons qu'on ait choisi les paramètres de telle sorte qu'une collision entre 3 bactéries arrive. Le drift est-il plus fort que la diffusion dans la mesure où la collision à trois particules a lieu sans qu'il ne puisse y avoir de collisions entre 2 particules juste avant ou au contraire est-ce que la diffusion l'emporte sur le drift dans le sens où juste avant la collision à trois particules, les bactéries vibrent tellement qu'elles entrent en collision deux à deux ?

Une première difficulté que l'on rencontre en étudiant cette équation est qu'elle n'admet pas de solution au sens classique, étant donné que le drift est trop singulier. Il est alors nécessaire d'utiliser une autre théorie comme la la théorie des formes de Dirichlet pour donner un sens à cette équation. Nous ne rentrons pas dans le détail de l'utilisation de cette théorie dans cette introduction et nous ferons comme si le processus était bien défini au sens classique, tout en se rappelant que l'utilisation de la formule d'Itô ou du théorème de Girsanov, par exemple, n'est licite qu'au sens de la théorie des formes de Dirichlet, ce qui cache beaucoup de détails techniques.

On pose pour $x \in (\mathbb{R}^2)^N$, pour $K \subset [\![1, N]\!]$,

$$S_K(x) = \frac{1}{|K|}\sum_{k\in K} x^k \quad \text{et} \quad R_K(x) = \sum_{k\in K}\|x^k - S_K(x)\|^2.$$

On appelle une solution de (1.13) (au sens de Dirichlet) un processus $KS(\theta, N)$ et on pose $\zeta = \inf\{t \geq 0 : X_t \notin E_{k_0}\}$ où $k_0 = \lceil 2N/\theta \rceil$ et pour $k \geq 0$,

$$E_k = \Big\{x \in (\mathbb{R}^2)^N : \text{ pour tout } K \subset [\![1, N]\!] \text{ tel que } |K| = k, \text{ on a } R_K(x) > 0\Big\}.$$

Il s'agit de l'ensemble des configurations où il n'y a jamais k particules (ou plus) au même endroit.

Ainsi, en appliquant informellement la formule d'Itô on obtient :

$$\mathrm{d}R_K(X_t) = 2\sqrt{R_K(X_t)}\mathrm{d}W_t^K + d_{\theta,N}(|K|)\mathrm{d}t + \text{ Int}(K, K^c)_t\mathrm{d}t,$$

où $(W_t^K)_{t\geq 0}$ est un mouvement brownien de dimension 1, $d_{\theta,N}(k) = (k-1)(2 - k\theta/N)$ pour tout $k \geq 0$, et $\text{Int}(K, K^c)_t$ est un terme d'intéraction en les particules indexées par K et celles indexées par K^c.

Si l'on néglige le terme d'intéraction, alors $(R_K(X_t))_{t\geq 0}$ vérifie l'équation d'un carré de Bessel de dimension $d_{\theta,N}(|K|)$, où l'on rappelle qu'un carré de Bessel de dimension δ :

- ne touche jamais 0 si $\delta \geq 2$,
- touche 0 et est réfléchi une infinité de fois si $\delta \in (0, 2)$,
- touche 0 puis reste collé à 0 pour toujours si $\delta \leq 0$.

Ainsi, en comparant la valeur de $d_{\theta,N}(|K|)$ avec 0 et 2, cela nous donne intuitivement quelles collisions sont susceptibles d'arriver. C'est ce qui est fait dans le lemme suivant :

Lemma 1.3. *Fixons $\theta > 0$ et $N \geq 2$ tel que $N > \theta$. Pour $k_0 = \lceil \frac{2N}{\theta} \rceil \geq 3$, on a*

$$d_{\theta,N}(k) > 0 \quad \textit{if} \quad k \in [\![2, k_0 - 1]\!] \quad \textit{and} \quad d_{\theta,N}(k) \leq 0 \quad \textit{if} \quad k \geq k_0. \tag{1.14}$$

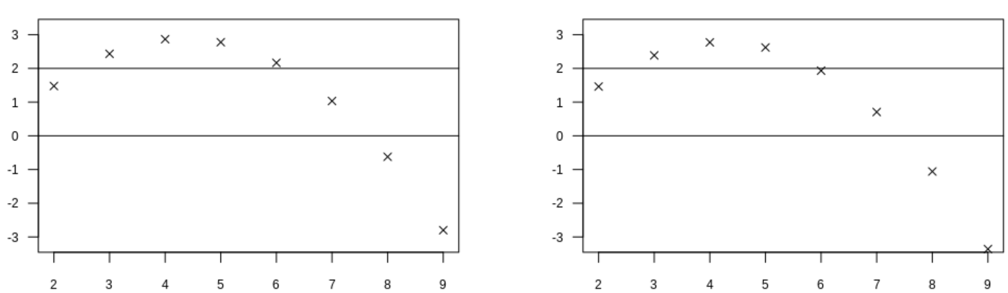
On définit aussi $k_1 = k_0 - 1$, et

$$k_2 = \begin{cases} k_0 - 2 & si \quad d_{\theta,N}(k_0 - 2) < 2, \\ k_0 - 1 & si \quad d_{\theta,N}(k_0 - 2) \geq 2. \end{cases}$$

Si $\theta \geq 2$ et $N > 3\theta$, alors $k_0 \in [\![7, N]\!]$ et on a

- *$d_{\theta,N}(2) \in (0, 2)$;*
- *$d_{\theta,N}(k) \geq 2$ si $k \in [\![3, k_2 - 1]\!]$;*
- *$d_{\theta,N}(k) \in (0, 2)$ si $k \in \{k_2, k_1\}$;*
- *$d_{\theta,N}(k) \leq 0$ si $k \geq k_0$.*

FIGURE 1.1 – Plot of $d_{\theta,N}(k)$ as a function of $k \in [\![2, N]\!]$ with $N = 9$ and with $\theta = 2.35$ (left) and $\theta = 2.42$ (right).



$k_0 = 8$, $k_1 = 7$, $k_2 = 7$ $\qquad$ $k_0 = 8$, $k_1 = 7$, $k_2 = 6$

Pour $K \subset [\![1, N]\!]$, et $x \in (\mathbb{R}^2)^N$, on dit qu'il y a une K-collision dans la configuration x si $R_K(x) = 0$ et pour tout $i \notin K$, $R_{K\cup\{i\}}(x) > 0$. On déduit du lemme précédent les résultats suivant :

Proposition 1.4. *Fixons $\theta \in (0, 2)$ et $N \geq 2$. Considérons $(X_t)_{t\geq 0}$ un processus $KS(\theta, N)$ introduit ci-haut. Pour tout $x \in E_2$, on a $\mathbb{P}_x(\zeta = \infty) = 1$.*

Ce résultat vient du fait que si $\theta \in (0,2)$, alors $d_{\theta,N}(k) \geq 2$ pour tout $k \geq 3$ et il n'y a alors aucune collision trop "singulière" pour empècher la bonne définition du processus. Le résultat principal est le suivant.

Théorème 1.5. *Supposons que $\theta \geq 2$, que $N > 3\theta$ et rappellons (voir Lemme 1.3) que $k_0 \in [\![7, N]\!]$, $k_1 = k_0 - 1$ et $k_2 \in \{k_0 - 1, k_0 - 2\}$. Considérons $(X_t)_{t\geq 0}$ un processus $KS(\theta, N)$ introduit ci-haut. Pour tout $x \in E_2$, on a $\mathbb{P}_x$-p.s. les propriétés suivantes :*

(i) ζ est fini et $X_{\zeta-} = \lim_{t\to\zeta-} X_t$ existe pour la topologie usuelle de $(\mathbb{R}^2)^N$;

(ii) il existe $K_0 \subset [\![1, N]\!]$ de cardinal $|K_0| = k_0$ tel qu'il y a une K_0-collision dans la configuration $X_{\zeta-}$, et pour tout $K \subset [\![1, N]\!]$ tel que $|K| > k_0$, il n'y a pas de K-collision dans la configuration $X_{\zeta-}$;

(iii) pour tout $t \in [0, \zeta)$ et tout $K \subset K_0$ de cardinal $|K| = k_1$, il y a une infinité de K-collisions durant (t, ζ) et aucun de ces instants de K-collision n'est isolé ;

(iv) si $k_2 = k_0 - 2$, alors pour tout $L \subset K \subset K_0$ tel que $|L| = k_2$ et $|K| = k_1$, pour tout instant $t \in (0, \zeta)$ de K-collision et tout $s \in [0, t)$, il y a une infinité de L-collisions durant (s, t) et aucun de ces instants de L-collision n'est isolé ;

(v) pour tout $K \subset [\![1, N]\!]$ de cardinal $|K| \in [\![3, k_2 - 1]\!]$, il n'y a pas de K-collision durant $[0, \zeta)$;

(vi) pour tout $L \subset K \subset K_0$ tel que $|L| = 2$ et $|K| = k_2$, pour tout instant $t \in (0, \zeta)$ de K-collision et tout $s \in [0, t)$, il y a une infinité de L-collisions durant (s, t) et aucun de ces instants de L-collision n'est isolé.

Ce résultat explique par exemple qu'un des scénarios possible est le suivant : à un instant fini $zeta$, un amas de k_0 particules se forme, ces particules restent collées les unes aux autres, et on ne peut plus définir le système au delà de ce temps. Juste avant cette collision, il y a une infinité de collisions à k_1 particules (pour tout sous-ensemble de cardinal k_1 des k_0 particules impliquées dans la collision collante). De plus, avant chaque collision à k_1 particules, il y a une infinité de collisions à k_2 particules (pour tout sous-ensemble de cardinal k_2 des k_1 particules qui collisionnent). Avant chaque collision à k_2 particules, il y a une infinité de collisions binaires. Mais durant tout ce temps, il n'y a jamais de collisions à k particules, pour $k \in [\![3, k_2 - 1]\!]$.

Une approximation forte qui a été faite dans l'analyse informelle ci-dessus est la négligeance du terme d'intéraction $\mathrm{Int}(K, K_t^c)$ qui n'est absolument pas simple. En effet, dès qu'une particule indexée par K se rapproche d'une particule indexée par K^c, ce qui est censé arriver souvent d'après notre analyse, ce terme devient très singulier. Nous allons éssayer d'expliquer brièvement comment oublier ce terme. Nous allons nous mettre dans le cas simplifié où les paramètres sont tels que $k_1 = N$ et l'on veut montrer que l'on a des collisions à $k_2 = N - 1$ particules avant toute collision à $k_1 = N$ particules.

Une remarque fondamentale que l'on fait sur ce processus est qu'il se décompose en trois parties : un "milieu" $M_t = S_{[\![1,N]\!]}(X_t^N)$, un "rayon" $D_t = R_{[\![1,N]\!]}(X_t^N)$ et un "angle"

$$(U_t^i)_{i\in[\![1,N]\!]} = \Big(\frac{X_{\rho_t}^i - M_{\rho_t}}{\sqrt{D_{\rho_t}}}\Big)_{i\in[\![1,N]\!]},$$

où ρ est l'inverse généralisé de $A_t = \int_0^t (D_s)^{-1} \mathrm{d}s$. Ce qui est remarquable est que le triplet (M, D, U) est composé de trois processus indépendants. De plus, si l'on note $b = (b^1, \dots, b^N)$ avec pour

$i \in [\![1, N]\!]$, $x \in (\mathbb{R}^2)^N$,

$$b^i(x) = -\frac{\theta}{N} \sum_{j \neq i} \frac{x^i - x^j}{\|x^i - x^j\|^2},$$

alors U vérifie une EDS autonome :

$$\mathrm{d}U_t = \pi_H \pi_{U_t^\perp}(\mathrm{d}B_t + b(U_t))\mathrm{d}t + \frac{2N-3}{2} U_t \mathrm{d}t,$$

où $(B_t)_{t \geq 0}$ est un mouvement brownien de dimension $2N$, π_H est la projection orthonormale sur $H = \{x \in (\mathbb{R}^2)^N : S_{[\![1,N]\!]}(x) = 0\}$, et pour $x \in (\mathbb{R}^2)^N$, $y \in (\mathbb{R}^2)^N$, $\pi_{x^\perp}(y) = y - \|x\|^{-2}(x \cdot y)x$. On remarque que U vérifie l'équation du système de particules de Keller-Segel mais sur la sphère de H. Etant donné qu'essentiellement la même équation est satisfaite, on retrouve environ les mêmes équations pour les processus de variance empirique, en effet on a

$$\mathrm{d}R_K(U_t) = 2\sqrt{R_K(U_t)(1 - R_K(U_t))}\mathrm{d}W_t^K + d_{\theta,N}(|K|)\mathrm{d}t + \mathrm{Int}_t(K, K^c)\mathrm{d}t + \mathrm{Norm}_t \mathrm{d}t,$$

où l'on a en plus des termes de normalisation Norm_t étant donné qu'on contraint le processus à rester sur la sphère unité de H que nous notons $\mathbb{S}$. Tout d'abord, les termes de normalisations sont facilement négligeables. On se retrouve alors dans le même cas que précèdemment avec une équation qui ressemble à celle de l'équation d'un carré de Bessel mais avec un terme d'intéraction singulier. La différence ici est qu'il y a des contraintes géomètriques qui permettent de contrôler le terme d'intéraction. En effet comme pour tout $t \geq 0$, U_t est dans la sphère de H, alors $S_{[\![1,N]\!]}(U_t) = 0$ et $\sum_{i=1}^N \|U_t^i\|^2 = 1$. Ainsi, si l'on prend $K = [\![1, N-1]\!]$, alors si $R_{[\![1,N-1]\!]}(U_t)$ est très petit, alors $U_t^1, \ldots, U_t^{N-1}$ sont tous environ égaux à un certain $u \in \mathbb{S}$. Or, comme $S_{[\![1,N]\!]}(U_t) = 0$, si l'on imagine N grand, on a

$$0 = S_{[\![1,N]\!]}(U_t) \approx \frac{N-1}{N} u + \frac{U_t^N}{N} \approx u.$$

Donc $U_t^1, \ldots U_t^{N-1}$ sont très proches de 0. Or, $\sum_{i=1}^N \|U_t^i\|^2 = 1$, donc

$$1 = \sum_{i=1}^N \|U_t^i\|^2 \approx \|U_t^N\|^2.$$

Dans ce cas les particules indexées par $[\![1, N-1]\!]$ et la particule indexée par N sont éloignées d'une distance environ 1, donc le terme d'intéraction est négligeable et donc $R_{[\![1,N-1]\!]}(U)$ se comporte localement comme un carré de Bessel de dimension $d_{\theta,N}(N-1)$. On conclut qu'à chaque fois que $R_{[\![1,N]\!]}(U)$ est assez petit, il y a une probabilité (uniformémement) positive que ce processus touche 0. Par Borel-Cantelli, il suffit de montrer que cela se produit infiniment souvent pour conclure. Il suffit alors de montrer que le processus $R_K(U)$ est très proche de 0 régulièrement. Mais c'est le cas car U satisfait une équation autonome. Il est donc Markov, de plus à valeurs dans un compact, donc est récurrent positif. Cet argument est en fait faux en toute généralité car nous traiterons des jeux de paramètre où U explose et est donc transient, mais il se trouve que la transience implique aussi la visite régulière de cette partie de l'espace.

1.4.2 Chapitre 3

L'existence d'une solution de type *énergie libre* globale dans le cas sous-critique a été démontré par Blanchet-Dolbeault-Perthame [9] dans le cas de conditions initiales $f_0 \in L^1_+(\mathbb{R}^2)$, d'entropie et de second moment finis. Au même moment, Biler-Karch-Laurençot-Nadzieja [4, 5] ont démontré l'existence d'une solution de type *énergie libre* faible globale dans le cas radial pour toute mesure initiale telle que $f_0(\mathbb{R}^2) \leq 8\pi$ et $f_0(\{0\}) = 0$.

Bedrossian-Masmoudi [2] ont prouvé sous la condition $\max_{x\in\mathbb{R}^2} f_0(\{x\}) < 8\pi$ l'existence d'une solution *tempérée* même dans le cas sur-critique, mais locale en temps. De plus Wei [51] a construit une solution *tempérée* globale dans les cas sous-critique et critique pour toute condition initiale $f_0 \in L^1(\mathbb{R}^2)$. En combinant ces résultats, on obtient l'existence de solutions globales pour toute condition initiale mesure f_0 tel que $f_0(\mathbb{R}^2) \leq 8\pi$ et $\max_{x\in\mathbb{R}^2} f_0(\{x\}) < 8\pi$.

Le résultat que nous prouvons dans ce chapitre n'est pas nouveau et en combinant les résultats de [2] et [51] on obtient un résultat encore plus fort. Néanmoins, la preuve de non-explosion que nous présentons est robuste, et sensiblement plus courte et simple. De plus, les arguments proposés ici, poussés un peu plus loin, permettraient de traîter aussi le cas critique, mais cela sera fait sous une autre forme dans le Chapitre 4.

L'idée est de trouver une estimation à priori de la solution à l'EDP qui justifie qu'il ne peut y avoir de formation de masse de Dirac. Plus précisemment, on montre le théorème suivant

Théorème 1.6. *Fixons $M \in (0, 8\pi)$ et supposons que f_0 soit une mesure sur $\mathbb{R}^2$ de masse totale M. Il existe une solution globale faible f à (1.10) avec condition initiale f_0. De plus, pour tout $\gamma \in (M/(4\pi), 2)$, il existe une constante $A_{M,\gamma} > 0$ dépendant uniquement de M et γ telle que pour tout $T > 0$,*

$$\int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \|x-y\|^{\gamma-2} f_s(\mathrm{d}x) f_s(\mathrm{d}y) \mathrm{d}s \leq A_{M,\gamma}(1+T). \tag{1.15}$$

L'idée ici est de dériver la quantité

$$S_t = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (\|x-y\| \wedge 1)^\gamma f_t(\mathrm{d}x) f_t(\mathrm{d}y).$$

Les parties du calcul relatives à la diffusion et à l'effet de concentration donnent chacune un terme proportionnel à $\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \|x-y\|^{\gamma-2} f_t(\mathrm{d}x) f_t(\mathrm{d}y)$. La partie difficile consiste à s'assurer que ces termes ne se compensent pas tout à fait de sorte qu'il existe une constante $c > 0$ tel que

$$S_t \geq S_0 + c \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \|x-y\|^{\gamma-2} f_t(\mathrm{d}x) f_t(\mathrm{d}y).$$

L'argument consiste en des calculs élémentaires qui symétrisent correctement les quantités d'intérêt, mais nous n'utilisons aucune inégalité fonctionnelle. Ainsi, comme S_t est borné pour tout $t \geq 0$, on a le résultat.

1.4.3 Chapitre 4

Il y a déjà quelques résultats connus sur la propagation du chaos dans le contexte du système de particules de Keller-Segel, i.e. sur la justification rigoureuse de l'approximation de la solution de (1.10) par la mesure empirique de la solution de (1.13) quand N tend vers l'infini.

Godinho-Quininao [25] ont prouvé la propagation du chaos sans utiliser de sous-suite en suivant les idées de Fournier-Hauray-Mischler [19] dans le cas où le noyau d'intéraction K est remplacé par noyau moins singulier $-x/(2\pi\|x\|^{1+\alpha})$ avec $\alpha \in (0, 2)$.

Olivera-Richard-Tomasevic [41], en utilisant une méthode s'appuyant sur les semigroupes développé par Flandoli [24], ont montré la convergence d'un système de particules où informellement K est remplacé par $-x/(\varepsilon_N + \|x\|^2)$ avec ε_N très grand devant $N^{-1/d}$.

La convergence du système de particules avec le noyau d'intéraction K a été démontré par Fournier-Jourdain [20] dans le cas (très) sous-critique $\theta < 1/2$. On souligne le fait que ce résultat est le même que celui que nous démontrons mais dans le cas où $\theta < 1/2$ et où une hypothèse supplémentaire sur le moment est faite.

Enfin, des estimées sur la vitesse de convergence de la mesure empirique dans le cas $\theta < 2$ ont été démontrés par Bresch-Jabin-Wang [11] en utilisant une *méthode par énergie libre*. La convergence obtenue est dans un sense fort quand $\mathbb{R}^2$ est remplacé par le tore pour des conditions initiales $f_0 \in W^{2,\infty}$ sur le tore.

Bien que la convergence que nous démontrons soit dans un sens plus faible que dans [11] et uniquement le long d'une sous-suite, la preuve semble bien plus simple et nous traitons le cas de l'espace entier et pas simplement du tore. De plus nous ne supposons pas de régularité sur la condition initiale f_0 qui peut être une masse de Dirac. Enfin, nous traitons le cas critique $\theta = 2$ qui semble n'avoir jamais été traité auparavant.

Dans ce chapitre, on montre la propagation du chaos dans le cas où $\theta \leq 2$. Le cas sous-critique $\theta < 2$ est plus simple que le cas $\theta = 2$, car dans ce cas il n'y a que des collisions doubles qui interviennent, et ces collisions sont plus faciles à gérer car à ce moment le processus visite un peu moins la singularité en un sens.

On note $\mathcal{P}^*_{sym,1}((\mathbb{R}^2)^N)$ l'ensemble des mesures de probabilité échangeables sur $(\mathbb{R}^2)^N$ avec un moment d'ordre 1 fini tel que pour tout $F_0^N \in \mathcal{P}^*_{sym,1}((\mathbb{R}^2)^N)$,

$$F_0^N(\{\text{There exists } i \neq j \in [\![1, N]\!] \text{ such that } x^i = x^j\}) = 0.$$

Théorème 1.7. *Soit $\theta \in (0, 2)$ et $f_0 \in \mathcal{P}(\mathbb{R}^2)$. Pour chaque $N \geq N_0 := (1 + \lceil 2/(2-\theta) \rceil) \vee 5$, considérons $F_0^N \in \mathcal{P}^*_{sym,1}((\mathbb{R}^2)^N)$ et $(X_t^{i,N})_{t \geq 0, i \in [\![1,N]\!]}$ un processus $KS(\theta, N)$ (i.e. une solution de* (1.13)*) de loi initiale F_0^N, ainsi que le processus empirique pour tout $t \geq 0$, $\mu_t^N := N^{-1} \sum_{i=1}^N \delta_{X_t^{i,N}}$, qui p.s. appartient à $\mathcal{P}(\mathbb{R}^2)$. On suppose que μ_0^N converge faiblement vers f_0 en probabilité lorsque $N \to \infty$.*

(i) La suite $((\mu_t^N)_{t \geq 0})_{N \geq N_0}$ est tendue dans $C([0, \infty), \mathcal{P}(\mathbb{R}^2))$.

(ii) Pour toute suite $(N_k)_{k \geq 0}$ telle que $(\mu_t^{N_k})_{t \geq 0}$ converge en loi dans $C([0, \infty), \mathcal{P}(\mathbb{R}^2))$ lorsque $k \to \infty$ vers un certain $(\mu_t)_{t \geq 0}$, cette limite $(\mu_t)_{t \geq 0}$ est p.s une solution faible de (1.10) *partant de $\mu_0 = f_0$. De plus, pour tout $T > 0$, tout $\gamma \in (\theta, 2)$.*

$$\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \|x - y\|^{\gamma - 2} \mu_t(\mathrm{d}x)\mu_t(\mathrm{d}y)\mathrm{d}t\Big] < \infty.$$

On note que les idées de la preuve sont assez similaires aux calculs du Chapitre 3. Le cas critique est plus difficile. On pose pour tout $N \geq 2$,

$$\text{pour tout } \ell > 0, \quad \tau_3^{N,\ell} = \inf\{t \geq 0 : \text{ il existe } K \subset [\![1, N]\!] \text{ avec } |K| = 3 : R_K(X_t^N) = \ell^{-1}\},$$
$$\tau_3^N = \lim_{\ell \to \infty} \tau_3^{N,\ell}.$$

Théorème 1.8. *Supposons $\theta = 2$. Soit $f_0 \in \mathcal{P}(\mathbb{R}^2)$ vérifiant $\max_{x\in\mathbb{R}^2} f_0(\{x\}) < 1$. Pour chaque $N \geq N_0 := 5$, considérons $F_0^N \in \mathcal{P}^*_{sym,1}((\mathbb{R}^2)^N)$ et $(X_t^{i,N})_{t\in[0,\tau_3^N), i\in[\![1,N]\!]}$ un processus $KS(2,N)$ sur $[0,\tau_3^N)$ de loi initiale F_0^N. On pose $\mu_t^N := N^{-1}\sum_{i=1}^N \delta_{X_t^{i,N}}$ et on suppose que μ_0^N converge faiblement vers f_0 en probabilité lorsque $N \to \infty$.*

Il existe une suite d'entiers strictement croissante $(\ell_N)_{N\geq N_0}$ telle qu'en posant $\beta_N = \tau_3^{N,\ell_N}$, on ait $\lim_{N\to\infty} \beta_N = \infty$ en probabilité et

(i) la suite $((\mu^N_{t\wedge\beta_N})_{t\geq 0})_{N\geq 5}$ est tendue dans $C([0,\infty),\mathcal{P}(\mathbb{R}^2))$;

(ii) pour toute suite $(N_k)_{k\geq 0}$ tel que $(\mu^{N_k}_{t\wedge\beta_{N_k}})_{t\geq 0}$ converge en loi, lorsque $k \to \infty$, dans $C([0,\infty),\mathcal{P}(\mathbb{R}^2))$, vers un certain $(\mu_t)_{t\geq 0}$, alors μ_t est p.s une solution faible de (1.10) partant de $\mu_0 = f_0$. De plus,

$$\mathbb{E}\Big[\int_0^\infty \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathbb{1}_{\{x=y\}}\mu_t(\mathrm{d}x)\mu_t(\mathrm{d}y)\mathrm{d}t\Big] = 0. \tag{1.16}$$

Il s'agit d'un cas où il y a explosion en temps fini du système de particules. Cela peut paraître étrange étant donné qu'on a déjà vu qu'il n'y a pas d'émergence de cluster dans le cas critique pour l'EDP, du moins pas en temps fini. C'est une des clés pour comprendre le cas critique. Si l'on regarde $\tau_N = \inf\{t \geq 0 : R_{[\![1,N]\!]}(X_t^N) = 0\}$, i.e. l'instant où les N particules sont au même endroit, alors τ_N est p.s. fini pour chaque N, mais $\tau_N \to \infty$ quand $N \to \infty$ en probabilité. Cela se comprend heuristiquement car le temps de la première collision entre au moins 3 particules est le temps de la première collision entre $k_2 = N - 2$ particules d'après le Chapitre 2. Or il suffit que quelques particules aient un comportement atypique et s'éloignent un peu trop des autres pour que cette collision n'arrive pas, et comme on fait tendre N vers l'infini il y aura toujours au moins 3 particules parmis les N qui auront une trajectoire assez originale pour ne pas participer à une telle collision. Ceci rejoint l'idée que dans le cas critique, il y a formation d'un amas en temps infini.

Plus précisément, on va montrer informellement que la mesure empirique arrêtée à l'instant de la première collision entre 3 particules converge vers une solution faible de l'EDP. Comme ce temps diverge vers l'infini quand N tend vers l'infini, alors on peut utiliser des estimées dans le même esprit que celles du Chapitre 3 étant donné que les collisions impliquées ne sont que des collisions entre 2 particules comme dans le cas sous-critique.

1.4.4 Chapitre 5

Velázquez [50] décrit de manière informelle le comportement des solutions de l'EDP de Keller-Segel après l'explosion, dans le cas sur-critique. Il y est décrit une solution mesure qui se décompose en une somme d'une partie absolument continue et d'une somme de mesure Dirac dont chacune aurait une masse total suppérieure à 8π. Les masses de Dirac correspondraient aux différentes formations de clusters, et il y est décrit les intéractions entre ces différents composants. Dolbeault-Schmeiser [16] ont démontré la convergence de l'EDP régularisée avec le noyau $-x/(\varepsilon^2 + \|x\|^2)$ quand ε tend vers 0 vers le modéle prescrit par [50] strictement entre les temps de formation de cluster. Prouver la convergence en incluant ces temps est une question encore ouverte à laquelle nous éssayons de donner une réponse au moins dans le cadre stochastique du système de particules. Néanmoins dans ce chapitre nous ne ferons qu'une présentation informelle d'un projet bien engagé dans cette direction mais inachevé pour le moment. Plus précisément, il s'agit de prolonger le système de particules au delà du temps d'explosion et de décrire les interactions entre les particules et les amas formés par les collisions collantes.

1.4.5 Chapitre 6

Dans ce chapitre nous présenterons quelques simulations illustrant les résultats du Chapitre 2. Notamment, dans le cas surcritique, nous mettrons en évidence la formation d'un cluster et le fait que seulement un certain type de collisions se produisent.

Chapitre 2

Collisions of the supercritical Keller-Segel particle system

Abstract. We study a particle system naturally associated to the 2-dimensional Keller-Segel equation. It consists of N Brownian particles in the plane, interacting through a binary attraction in $\theta/(Nr)$, where r stands for the distance between two particles. When the intensity θ of this attraction is greater than 2, this particle system explodes in finite time. We assume that $N > 3\theta$ and study in details what happens near explosion. There are two slightly different scenarios, depending on the values of N and θ, here is one : at explosion, a cluster consisting of precisely k_0 particles emerges, for some deterministic $k_0 \geq 7$ depending on N and θ. Just before explosion, there are infinitely many $(k_0 - 1)$-ary collisions. There are also infinitely many $(k_0 - 2)$-ary collisions before each $(k_0 - 1)$-ary collision. And there are infinitely many binary collisions before each $(k_0 - 2)$-ary collision. Finally, collisions of subsets of $3, \dots, k_0 - 3$ particles never occur. The other scenario is similar except that there are no $(k_0 - 2)$-ary collisions.

2.1 Introduction and main results

2.1.1 Informal definition of the model

We consider some scalar parameter $\theta > 0$ and a number $N \geq 2$ of particles with positions $X_t = (X_t^1, \dots, X_t^N) \in (\mathbb{R}^2)^N$ at time $t \geq 0$. Informally, we assume that the dynamics of these particles are given by the system of S.D.E.s

$$\mathrm{d}X_t^i = \mathrm{d}B_t^i - \frac{\theta}{N} \sum_{j \neq i} \frac{X_t^i - X_t^j}{\|X_t^i - X_t^j\|^2} \mathrm{d}t, \qquad i \in [\![1, N]\!], \tag{2.1}$$

where the 2-dimensional Brownian motions $((B_t^i)_{t \geq 0})_{i \in [\![1,N]\!]}$ are independent. In other words, we have N Brownian particles in the plane interacting through an attraction in $1/r$, which is Coulombian in dimension 2. Actually, this S.D.E. does not clearly make sense, due to the singularity of the drift, and we will use, as suggested by Cattiaux-Pédèches [13], the theory of Dirichlet spaces, see Fukushima-Oshima-Takeda [24].

2.1.2 Brief motivation and informal presentation of the main results

This particle system is very natural from a physical point of view, because, as we will see, there is a tight competition between the Brownian excitation and the Coulombian attraction. It can also be seen as an approximation of the famous Keller-Segel equation [35], see also Patlak [43]. This nonlinear P.D.E. has been introduced to model the collective motion of cells, which are attracted by a chemical substance that they emit. It is well-known that a phase transition occurs : if the intensity of the attraction is small, then there exist global solutions, while if the attraction is large, the solution explodes in finite time.

We will show that this phase transition already occurs at the level of the particle system (2.1) : there exist global (very weak) solutions if $\theta \in (0,2)$ (subcritical case, see Proposition 2.3 below), but solutions must explode in finite time if $\theta \geq 2$ (supercritical case).

To our knowledge, the supercritical case has not been studied in details, and we aim to describe precisely the explosion phenomenon. Informally, we will show the following (see Theorem 2.5 below). We assume that $\theta \geq 2$ and $N > 3\theta$, we set $k_0 = \lceil 2N/\theta \rceil \in [\![7, N]\!]$. There exists a (very weak) solution $(X_t)_{t\in[0,\zeta)}$ to (2.1), with $\zeta < \infty$ a.s. and such that $X_{\zeta-} = \lim_{t\to\zeta-} X_t$ exists. Moreover, there is a cluster containing precisely k_0 particles in the configuration $X_{\zeta-}$, and no cluster containing strictly more than k_0 particles. Such a cluster containing k_0 particles is inseparable, so that (2.1) is meaningless (even in a very weak sense) after ζ. Just before explosion, there are infinitely many k_1-ary collisions, where $k_1 = k_0 - 1$. If $(k_0-3)(2-(k_0-2)\theta/N) < 2$, we set $k_2 = k_1 - 2$ and just before each k_1-ary collision, there are infinitely many k_2-collisions. Else, we set $k_2 = k_1$. In any case, there are infinitely many binary collisions just before each k_2-ary collision. During the whole time interval $[0,\zeta)$, there are no k-ary collisions, for any $k \in [\![3, k_2-1]\!]$.

This phenomenon seems surprising and original, in particular because of the gap between binary and k_2-ary collisions.

2.1.3 Sets of configurations

We introduce, for all $K \subset [\![1,N]\!]$ and all $x = (x^1,\dots,x^N) \in (\mathbb{R}^2)^N$,

$$S_K(x) = \frac{1}{|K|}\sum_{i\in K} x^i \in \mathbb{R}^2 \quad \text{and} \quad R_K(x) = \sum_{i\in K}\|x^i - S_K(x)\|^2 = \frac{1}{2|K|}\sum_{i,j\in K}\|x^i - x^j\|^2 \geq 0.$$

Here $|K|$ is the cardinal of K and $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^2$. Observe that $R_K(x) = 0$ if and only if all the particles indexed in K are at the same place. We also set, for $k \geq 2$,

$$E_k = \Big\{x \in (\mathbb{R}^2)^N : \forall K \subset [\![1,N]\!] \text{ with cardinal } |K| = k,\ R_K(x) > 0\Big\},$$

which represents the set of configurations with no cluster of k (or more) particles. Observe that $E_k = (\mathbb{R}^2)^N$ for all $k > N$.

2.1.4 Bessel processes

We recall that a squared Bessel process $(Z_t)_{t\geq 0}$ of dimension $\delta \in \mathbb{R}$ is a nonnegative solution, killed when it reaches 0 if $\delta \leq 0$, of the equation

$$Z_t = Z_0 + 2\int_0^t \sqrt{Z_s}\mathrm{d}W_s + \delta t,$$

where $(W_t)_{t\geq 0}$ is a 1-dimensional Brownian motion. We then say that $(\sqrt{Z_t})_{t\geq 0}$ is a Bessel process of dimension δ. This process has the following property, see Revuz-Yor [44, Chapter XI] :

- if $\delta \geq 2$, then a.s., for all $t > 0$, $Z_t > 0$;
- if $\delta \in (0, 2)$, then a.s., Z is reflected infinitely often at 0 ;
- if $\delta \leq 0$, then Z a.s. hits 0 and is then killed.

Applying informally the Itô formula, one finds that $Y_t = \sqrt{Z_t}$ should solve

$$Y_t = Y_0 + W_t + \frac{\delta - 1}{2}\int_0^t \frac{\mathrm{d}s}{Y_s},$$

which resembles (2.1) in that we have a Brownian excitation in competition with an attraction by 0, or a repulsion by 0, depending on the value of δ, proportional to $1/r$. This formula rigorously holds true only when $\delta > 1$, see [44, Chapter XI].

2.1.5 Some important quantities

Consider a (possibly very weak) solution $(X_t)_{t\geq 0}$ to (2.1). As we will see, when fixing a subset $K \subset [\![1, N]\!]$ and when neglecting the interactions between the particles indexed in K and the other ones, one finds that the process $(R_K(X_t))_{t\geq 0}$ behaves like a squared Bessel process with dimension $d_{\theta,N}(|K|)$, where

$$d_{\theta,N}(k) = (k-1)\Big(2 - \frac{k\theta}{N}\Big). \tag{2.2}$$

Similar computations already appear in Haškovec-Schmeiser [26], see also [20]. A little study, see Appendix 2.12, see also Figure 2.1.5 and Subsection 2.1.8 for numerical examples, shows the following facts. For $r \in \mathbb{R}_+$, we set $\lceil r \rceil = \min\{n \in \mathbb{N} : n \geq r\}$.

Lemma 2.1. *Fix $\theta > 0$ and $N \geq 2$ such that $N > \theta$. For $k_0 = \lceil \frac{2N}{\theta} \rceil \geq 3$, we have*

$$d_{\theta,N}(k) > 0 \quad \text{if} \quad k \in [\![2, k_0 - 1]\!] \quad \text{and} \quad d_{\theta,N}(k) \leq 0 \quad \text{if} \quad k \geq k_0. \tag{2.3}$$

We also define $k_1 = k_0 - 1$, and

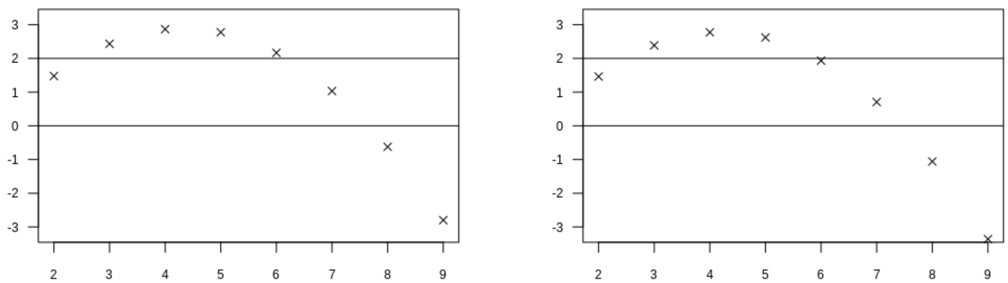
$$k_2 = \begin{cases} k_0 - 2 & \text{if} \quad d_{\theta,N}(k_0 - 2) < 2, \\ k_0 - 1 & \text{if} \quad d_{\theta,N}(k_0 - 2) \geq 2. \end{cases}$$

If $\theta \geq 2$ and $N > 3\theta$, then $k_0 \in [\![7, N]\!]$ and it holds that

- *$d_{\theta,N}(2) \in (0, 2)$;*
- *$d_{\theta,N}(k) \geq 2$ if $k \in [\![3, k_2 - 1]\!]$;*
- *$d_{\theta,N}(k) \in (0, 2)$ if $k \in \{k_2, k_1\}$;*
- *$d_{\theta,N}(k) \leq 0$ if $k \geq k_0$.*

We thus expect that there may be some non sticky k-ary collisions for $k \in \{2, k_2, k_1\}$, some sticky k-ary collisions when $k \geq k_0$, but no k-ary collision for $k \in [\![3, k_2 - 1]\!]$.

FIGURE 2.1 – Plot of $d_{\theta,N}(k)$ as a function of $k \in [\![2, N]\!]$ with $N = 9$ and with $\theta = 2.35$ (left) and $\theta = 2.42$ (right).



$k_0 = 8,\ k_1 = 7,\ k_2 = 7$ $\qquad$ $k_0 = 8,\ k_1 = 7,\ k_2 = 6$

2.1.6 Generator and invariant measure

As we will see in Subsection 2.3.13, the S.D.E. (2.1) cannot have a solution in the classical sense, at least when $d_{\theta,N}(k_1) \in (0, 1)$, because the drift term cannot be integrable in time. We will thus define a solution through the theory of the Dirichlet spaces.

For $x = (x^1, \dots, x^N) \in (\mathbb{R}^2)^N$ and for $\mathrm{d}x$ the Lebesgue measure on $(\mathbb{R}^2)^N$, we set

$$\mathbf{m}(x) = \prod_{1 \le i \ne j \le N} \|x^i - x^j\|^{-\theta/N} \quad \text{and} \quad \mu(\mathrm{d}x) = \mathbf{m}(x)\mathrm{d}x, \tag{2.4}$$

where $\{1 \le i \ne j \le N\}$ stands for the set $\{(i, j) \in [\![1, N]\!]^2 : i \ne j\}$. Informally, the generator of the solution to (2.1) is given by $\mathcal{L}^X$, where for $\varphi \in C^2((\mathbb{R}^2)^N)$,

$$\mathcal{L}^X \varphi(x) = \frac{1}{2}\Delta\varphi(x) - \frac{\theta}{N} \sum_{1 \le i \ne j \le N} \frac{x^i - x^j}{\|x^i - x^j\|^2} \cdot \nabla_{x^i}\varphi(x) = \frac{1}{2\mathbf{m}(x)}\mathrm{div}[\mathbf{m}(x)\nabla\varphi(x)], \tag{2.5}$$

see (2.11) for the last equality. It is well-defined for all $x \in E_2$ and μ-symmetric. Indeed, an integration by parts shows that

$$\forall\, \varphi, \psi \in C^2_c(E_2), \quad \int_{(\mathbb{R}^2)^N} \varphi \mathcal{L}^X \psi \,\mathrm{d}\mu = -\frac{1}{2}\int_{(\mathbb{R}^2)^N} \nabla\varphi \cdot \nabla\psi \,\mathrm{d}\mu = \int_{(\mathbb{R}^2)^N} \psi \mathcal{L}^X \varphi \,\mathrm{d}\mu. \tag{2.6}$$

As we will see in Proposition 2.30, the measure μ is Radon on $(\mathbb{R}^2)^N$ in the subcritical case $\theta \in (0, 2)$, while it is Radon on E_{k_0} (and not on E_{k_0+1}) in the supercritical case $\theta \ge 2$. This will allow us to use some results found in Fukushima-Oshima-Takeda [24] and to obtain the following existence result.

Proposition 2.2. *We fix $N \ge 2$ and $\theta > 0$ such that $N > \theta$ and recall that $k_0 = \lceil 2N/\theta \rceil$. We set $\mathcal{X} = E_{k_0}$ and $\mathcal{X}_\triangle = \mathcal{X} \cup \{\triangle\}$, where $\triangle$ is a cemetery point. There exists a diffusion $\mathbb{X} = (\Omega^X, \mathcal{M}^X, (X_t)_{t \ge 0}, (\mathbb{P}^X_x)_{x \in \mathcal{X}_\triangle})$ with values in $\mathcal{X}_\triangle$, which is μ-symmetric, with regular Dirichlet space $(\mathcal{E}^X, \mathcal{F}^X)$ on $L^2((\mathbb{R}^2)^N, \mu)$ with core $C^\infty_c(\mathcal{X})$ defined by*

$$\textit{for all} \quad \varphi \in C^\infty_c(\mathcal{X}), \quad \mathcal{E}^X(\varphi, \varphi) = \frac{1}{2}\int_{(\mathbb{R}^2)^N} \|\nabla\varphi\|^2 \mathrm{d}\mu = -\int_{(\mathbb{R}^2)^N} \varphi\mathcal{L}^X\varphi \,\mathrm{d}\mu$$

and such that for all $x \in E_2$, all $t > 0$, the law of X_t under $\mathbb{P}_x$ has a density with respect to the Lebesgue measure on $(\mathbb{R}^2)^N$. We call such a process a $KS(\theta, N)$-process and denote by $\zeta = \inf\{t \geq 0 : X_t = \triangle\}$ its life-time.

We refer to Subsection 2.13.1 for a quick summary about the notions used in this proposition : diffusion (i.e. continuous Hunt process), link between its generator, semi-group and Dirichlet space, definition of the one-point compactification topology endowing $\mathcal{X}_\triangle$, etc. Let us mention that by definition, $\triangle$ is absorbing, i.e. $X_t = \triangle$ for all $t \geq \zeta$. Also, $t \mapsto X_t$ is *a priori* continuous on $[0, \infty)$ only for the one-point compactification topology on $\mathcal{X}_\triangle$, which precisely means that it is continuous for the usual topology of $(\mathbb{R}^2)^N$ during $[0, \zeta)$, and it holds that $\zeta = \lim_{n\to\infty} \inf\{t \geq 0 : X_t \notin \mathcal{K}_n\}$ for any increasing sequence of compact subsets $(\mathcal{K}_n)_{n\geq 1}$ of E_{k_0} such that $\cup_{n\geq 1}\mathcal{K}_n = E_{k_0}$.

As we will see in Remark 2.29, for all $x \in E_2$, under $\mathbb{P}_x^X$, X_t solves (2.1) during $[0, \sigma)$, where $\sigma = \inf\{t \geq 0 : X_t \notin E_2\}$. By the Markov property, this implies X_t solves (2.1) during any open time-interval on which it does not visit $\mathcal{X} \setminus E_2$.

When $\theta < 2$, we have $k_0 > N$ and thus $E_{k_0} = (\mathbb{R}^2)^N$. We will easily prove the following non-explosion result, which is almost contained in Cattiaux-Pédèches [13], who treat the case where $\theta \in (0, 2(N-2)/(N-1))$.

Proposition 2.3. *Fix $\theta \in (0, 2)$ and $N \geq 2$. Consider the $KS(\theta, N)$-process $\mathbb{X}$ introduced in Proposition 2.2. For all $x \in E_2$, we have $\mathbb{P}_x(\zeta = \infty) = 1$.*

When $\theta \geq 2$, we will see that there is explosion. Note that any collision of a set of $k \geq k_0$ particles makes the process leave E_{k_0} and thus explode. However, it is not clear at all at this point that explosion is due to a precise collision : the process could explode because it tends to infinity (which is not hard to exclude) or to the boundary of E_{k_0} with possibly many oscillations.

2.1.7 Main result

To avoid any confusion, let us define precisely what we call a collision.

Definition 2.4. *(i) For $K \subset [\![1, N]\!]$, we say that there is a K-collision in the configuration $x \in (\mathbb{R}^2)^N$ if $R_K(x) = 0$ and if $R_{K\cup\{i\}}(x) > 0$ for all $i \in [\![1, N]\!] \setminus K$.*

(ii) For a $(\mathbb{R}^2)^N$-valued process $(X_t)_{t\in[0,\zeta)}$, we say that there is a K-collision at time $s \in [0, \zeta)$ if there is a K-collision in the configuration X_s.

The main result of this paper is the following description of the explosion phenomenon.

Theorem 2.5. *Assume that $\theta \geq 2$, that $N > 3\theta$ and recall that $k_0 \in [\![7, N]\!]$, $k_1 = k_0 - 1$ and $k_2 \in \{k_0 - 1, k_0 - 2\}$ were defined in Lemma 2.1. Consider the $KS(\theta, N)$-process $\mathbb{X}$ introduced in Proposition 2.2. For all $x \in E_2$, we $\mathbb{P}_x$-a.s. have the following properties :*

(i) ζ is finite and $X_{\zeta-} = \lim_{t\to\zeta-} X_t$ exists for the usual topology of $(\mathbb{R}^2)^N$;

(ii) there is $K_0 \subset [\![1, N]\!]$ with cardinal $|K_0| = k_0$ such that there is a K_0-collision in the configuration $X_{\zeta-}$, and for all $K \subset [\![1, N]\!]$ such that $|K| > k_0$, there is no K-collision in the configuration $X_{\zeta-}$;

(iii) for all $t \in [0, \zeta)$ and all $K \subset K_0$ with cardinal $|K| = k_1$, there is an infinite number of K-collisions during (t, ζ) and none of these instants of K-collision is isolated;

(iv) if $k_2 = k_0 - 2$, then for all $L \subset K \subset K_0$ such that $|L| = k_2$ and $|K| = k_1$, for all instant $t \in (0,\zeta)$ of K-collision and all $s \in [0,t)$, there is an infinite number of L-collisions during (s,t) and none of these instants of L-collision is isolated ;

(v) for all $K \subset [\![1,N]\!]$ with cardinal $|K| \in [\![3, k_2 - 1]\!]$, there is no K-collision during $[0,\zeta)$;

(vi) for all $L \subset K \subset K_0$ such that $|L| = 2$ and $|K| = k_2$, for all instant $t \in (0,\zeta)$ of K-collision and all $s \in [0,t)$, there is an infinite number of L-collisions during (s,t) and none of these instants of L-collision is isolated.

The condition $\theta \geq 2$ is crucial to guarantee that $k_0 \leq N$. On the contrary, we impose $N > 3\theta$ for simplicity, because Lemma 2.1 does not hold true without this assumption. The other cases may also be studied, but we believe this is not very restrictive : N is thought as very large when compared to θ, at least as far as the approximation of the Keller-Segel equation is concerned.

2.1.8 Comments

Let us mention that the very precise values of N and θ influence the value k_2.

(a) If $N = 200$ and $\theta = 4.04$, we have $k_0 = 100$, $k_1 = 99$ and $k_2 = 98$.

(b) If $N = 200$ and $\theta = 4.015$, we have $k_0 = 100$ and $k_1 = k_2 = 99$.

Let us describe informally, in the chronological order, what happens e.g. in case (b) above. We start with 200 particles at 200 different places. During the whole story, there is no k-ary collision for $k = 3, \dots, 98$. Here and there, two particles meet, they collide an infinite number of times, but manage to separate. Then at some times, we have 98 particles close to each other and there are many binary collisions. Then, if a 99-th particle arrives in the same zone (and this eventually occurs), there are infinitely many 99-ary collisions, with infinitely many binary collisions of all possible pairs before each. These 99 particles may manage to separate forever, or for a large time, but if a 100-th particle arrives in the zone (and this situation eventually occurs), then there are infinitely many 99-ary collisions of all the possible subsets and, finally, a 100-ary collision producing explosion, and the story is finished. Informally, the resulting cluster is not able to separate, because the attraction dominates the Brownian excitation, since a Bessel process of dimension $d_{\theta,N}(100) \leq 0$ is absorbed when it reaches 0. We hope to be able, in a future work, to propose and justify a model describing what happens after explosion.

2.1.9 References

In many papers about the Keller-Segel equation, the parameter $\chi = 4\pi\theta$ is used, so that the transition at $\theta = 2$ corresponds to the transition at $\chi = 8\pi$. As already mentioned, this nonlinear P.D.E. has been introduced to model the collective motion of cells, which are attracted by a chemical substance that they emit. It describes the density $f_t(x)$ of particles (cells) with position $x \in \mathbb{R}^2$ at time $t \geq 0$ and writes, in the so-called parabolic-elliptic case,

$$\partial_t f_t(x) + \theta \mathrm{div}_x((K \star f_t)(x) f_t(x)) = \frac{1}{2}\Delta_x f_t(x), \quad \text{where} \quad K(x) = -\frac{x}{|x|^2}. \tag{2.7}$$

Informally, this solution should be the mean-field limit of the particle system (2.1) as $N \to \infty$.

We refer to the recent review paper on (2.7) by Arumugam-Tyagi [1]. The best existence of a global solution to (2.7), including all the subcritical parameters $\theta \in (0,2)$, is due to Blanchet-Dolbeault-Perthame [9]. The blow-up of solutions to (2.7), in the supercritical case $\theta > 2$, has been studied e.g. by Fatkullin [17] and Velasquez [49, 50]. More close to our study, Suzuki [48] has shown, still in the supercritical case, the appearance of a Dirac mass with a precise (critical) weight, at explosion. This is the equivalent, in the limit $N \to \infty$, to the fact that $\lim_{t\to\zeta-} X_t$ exists and corresponds to a K-collision, for some $K \subset [\![1, N]\!]$ with precise cardinal k_0. Let us finally mention Dolbeault-Schmeiser [16], who propose a post-explosion model in the supercritical case.

Concerning particle systems associated with (2.7), let us mention Stevens [47], who studies a physically more complete particle system with two types of particles, for cells and chemo-attractant particles, with a regularized attraction kernel. Haškovec and Schmeiser [26, 27] study a particle system closer to (2.1), but with, again, a regularized attraction kernel.

Cattiaux-Pédèches [13], as well as [20], study the system (2.1) without regularization in the subcritical case : existence of a global solution to (2.1) has been shown in [20] when $\theta \in (0, 2(N-2)/(N-1))$, and uniqueness of this solution has been established in [13]. Also, the theory of Dirichlet spaces has been used in [13] to build a solution to (2.1). Finally, the limit as $N \to \infty$ to a solution of (2.7) is proved in [20] in the very subcritical case where $\theta \in (0, 1/2)$, up to extraction of a subsequence. This last result has been improved by Bresch-Jabin-Wang [11, 12], who remove the necessity of extracting a subsequence and consider the (still very subcritical) case where $\theta \in (0, 1)$. Olivera-Richard-Tomasevic [41] have recently established the $N \to \infty$ convergence of a smoothed version of (2.1), for all the subcritical cases $\theta \in (0, 2)$. Informally, in view of the mean distance between particles, the regularization used in [41] is not far from being physically reasonable. There is also a related paper of Jabir-Talay-Tomasevic [30] about a one-dimensional but more complicated parabolic-parabolic model.

Let us finally mention the seminal paper of Osada [42], see also [19] for a more recent study, which concerns the vortex model : this is very close to (2.1), but the attraction $-x/|x|^2$ is replaced by a rotating interaction $x^\perp/|x|^2$, so that particles never encounter.

2.1.10 Originality and difficulties

To our knowledge, this is the first study of the supercritical Keller-Segel particle system near explosion. We hope that this model, which makes compete diffusion and Coulomb interactions, is very natural from a physical point of view, beyond the Keller-Segel community. The phenomenon we discovered seems surprising and original, in particular because of the gap between binary and k_2-ary collisions. We are not aware of other works, possibly dealing with other models, showing such a behavior.

In Section 2.3, we give the main arguments of the proofs, with quite a high level of precision, but ignoring the technical issues. While it is rather clear, intuitively, that the process explodes in finite time when $\theta \geq 2$ and that no K-collisions may occur for $|K| \in [\![3, k_2 - 1]\!]$, the continuity at explosion is delicate, and some rather deep arguments are required to show that that each k_2-ary collision is preceded by many binary collisions, that each k_1-ary collision is preceded by many k_2-ary collisions, that explosion is preceded by many k_1-ary collisions, and that explosion is due to the emergence of a cluster with precise size k_0 (which more or less says that a possible (k_0+1)-ary collision would necessarily be preceded by a k_0-collision).

Actually, the rigorous proofs are made technically much more involved than those presented in Section 2.3, because we have to use the theory of Dirichlet spaces. Due to the singularity of the interactions and to the occurrence of many collisions near explosion, we can unfortunately not, as already mentioned, deal at the rigorous level directly with the S.D.E. (2.1). We thus have to use some suitable heavy versions of some usual tools such as Itô's formula, Girsanov's theorem, time-change, etc.

2.1.11 Plan of the paper

In Section 2.2, we introduce some notation of constant use. In Section 2.3, we explain the main ideas of the proofs, with quite a high level of precision, but without speaking of the heavy technical issues related to the use of the theory of Dirichlet spaces. Section 2.4 is devoted to the existence of a first version of the Keller-Segel process, namely without the property that $\mathbb{P}_x^X \circ X_t^{-1}$ has a density, and we introduce a spherical Keller-Segel process. In Section 2.5, we show that the Keller-Segel process enjoys a crucial and noticeable decomposition in terms of a 2-dimensional Brownian motion, a squared Bessel process and a spherical process. Section 2.6 consists in building some smooth approximations of some indicator functions that behave well under the action of the generator $\mathcal{L}^X$. In Section 2.7, we make use of the Girsanov theorem to prove that when two sets of particles of a KS-process are not too close from each other, they behave as two independent smaller KS-processes. In Section 2.8, we study explosion and continuity (in the usual sense) at the explosion time. Section 2.9 is devoted to establish some parts of Theorem 2.5 for some particular ranges of values of N and θ. Using the results of Section 2.7, we reduce the general study to the special cases of Section 2.9 and we prove, in Section 2.10, that the conclusions of Theorem 2.5 hold true quasi-everywhere. Finally, in Section 2.11, we remove the restriction *quasi-everywhere* and conclude the proofs of Propositions 2.2 and 2.3 and of Theorem 2.5.

Appendix 2.12 contains a few elementary computations : proof of Lemma 2.1, proof that μ is Radon on E_{k_0}, and study of a similar measure on a sphere. We end the paper with Appendix 2.13, that summarizes all the notions and results about Dirichlet spaces and Hunt processes we shall use.

2.2 Notation

We introduce the spaces

$$H = \Big\{x \in (\mathbb{R}^2)^N : S_{[\![1,N]\!]}(x) = 0\Big\}, \quad S = \Big\{x \in (\mathbb{R}^2)^N : \sum_{i=1}^N \|x^i\|^2 = 1\Big\} \quad \text{and} \quad \mathbb{S} = H \cap S.$$

For $u \in \mathbb{S}$, we have $S_{[\![1,N]\!]}(u) = 0$ and $R_{[\![1,N]\!]}(u) = 1$. We consider the (unnormalized) Lebesgue measure σ on $\mathbb{S}$, as well as, recall (2.4),

$$\beta(\mathrm{d}u) = \mathbf{m}(u)\sigma(\mathrm{d}u). \tag{2.8}$$

We define $\gamma : \mathbb{R}^2 \to (\mathbb{R}^2)^N$ by $\gamma(z) = (z, \dots, z)$ and $\Psi : \mathbb{R}^2 \times \mathbb{R}_+^* \times \mathbb{S} \to E_N \subset (\mathbb{R}^2)^N$ by

$$\Psi(z, r, u) = \gamma(z) + \sqrt{r}\, u, \qquad \text{i.e.} \qquad (\Psi(z, r, u))^i = z - \sqrt{r} u^i \quad \text{for} \quad i \in [\![1, N]\!]. \tag{2.9}$$

We have $S_{[\![1,N]\!]}(\Psi(z, r, u)) = z$ and $R_{[\![1,N]\!]}(\Psi(z, r, u)) = r$.

The orthogonal projection $\pi_H : (\mathbb{R}^2)^N \to H$ is given by

$$\pi_H(x) = x - \gamma(S_{[\![1,N]\!]}(x)), \qquad \text{i.e.} \qquad (\pi_H(x))^i = x^i - S_{[\![1,N]\!]}(x) \quad \text{for} \quad i \in [\![1, N]\!]$$

and we introduce $\Phi_{\mathbb{S}} : E_N \to \mathbb{S}$ defined by

$$\Phi_{\mathbb{S}}(x) = \frac{\pi_H x}{||\pi_H x||}, \qquad \text{i.e.} \qquad (\Phi_{\mathbb{S}}(x))^i = \frac{x^i - S_{[\![1,N]\!]}(x)}{\sqrt{R_{[\![1,N]\!]}(x)}} \quad \text{for} \quad i \in [\![1, N]\!]. \tag{2.10}$$

For $x \in (\mathbb{R}^2)^N \setminus \{0\}$, the projections $\pi_{x^\perp} : (\mathbb{R}^2)^N \to x^\perp$ and $\pi_x : (\mathbb{R}^2)^N \to \text{span}(x)$ are given by

$$\pi_{x^\perp}(y) = y - \frac{x \cdot y}{||x||^2} x \qquad \text{and} \qquad \pi_x(y) = \frac{x \cdot y}{||x||^2} x,$$

where $x \cdot y = \sum_{i=1}^N x^i \cdot y^i$.

We denote by $b : E_2 \to (\mathbb{R}^2)^N$ the drift coefficient of (2.1) : for $x = (x^1, \dots, x^N) \in E_2$,

$$b(x) = \frac{\nabla \mathbf{m}(x)}{2\mathbf{m}(x)} = \frac{\nabla \log \mathbf{m}(x)}{2} \in (\mathbb{R}^2)^N, \qquad \text{i.e.} \qquad b^i(x) = -\frac{\theta}{N} \sum_{j \neq i} \frac{x^i - x^j}{\|x^i - x^j\|^2} \in \mathbb{R}^2 \tag{2.11}$$

for $i \in [\![1, N]\!]$. Indeed, since $\log \mathbf{m}(x) = -\frac{\theta}{2N} \sum_{1 \le i \neq j \le N} \log ||x^i - x^j||^2$, we e.g. have

$$\frac{\nabla_{x^1} \log \mathbf{m}(x)}{2} = -\frac{\theta}{4N} \nabla_{x^1} \Big[\sum_{i=2}^N \log ||x^i - x^1||^2 + \sum_{j=2}^N \log ||x^1 - x^j||^2 \Big] = -\frac{\theta}{2N} \nabla_{x^1} \sum_{j=2}^N \log ||x^1 - x^j||^2,$$

whence

$$\frac{\nabla_{x^1} \log \mathbf{m}(x)}{2} = -\frac{\theta}{N} \sum_{j=2}^N \frac{x^1 - x^j}{\|x^1 - x^j\|^2}.$$

Finally, we introduce the natural operators defined for $\varphi \in C^1(\mathbb{S})$ and $u \in \mathbb{S}$ by

$$\nabla_{\mathbb{S}} \varphi(u) = \nabla[\varphi \circ \Phi_{\mathbb{S}}](u) \in (\mathbb{R}^2)^N \quad \text{and} \quad \Delta_{\mathbb{S}} \varphi(u) = \Delta[\varphi \circ \Phi_{\mathbb{S}}](u) \in \mathbb{R}, \tag{2.12}$$

where ∇ and Δ stand for the usual gradient and Laplacian in $(\mathbb{R}^2)^N$. Since $\mathbb{S} \subset E_N \subset (\mathbb{R}^2)^N$, with E_N open, and since $\Phi_{\mathbb{S}}$ is smooth on E_N, we can indeed define $\nabla[\varphi \circ \Phi_{\mathbb{S}}](u)$ and $\Delta[\varphi \circ \Phi_{\mathbb{S}}](u)$ for all $u \in \mathbb{S}$. Similarly, for $\varphi \in C^1(\mathbb{S}, (\mathbb{R}^2)^N)$ and $u \in \mathbb{S}$, we set

$$\text{div}_{\mathbb{S}} \varphi(u) = \text{div}[\varphi \circ \Phi_{\mathbb{S}}](u) \in \mathbb{R}. \tag{2.13}$$

To conclude this subsection, we note that for all $\varphi \in C^\infty((\mathbb{R}^2)^N)$, for all $u \in \mathbb{S}$,

$$\nabla_{\mathbb{S}}(\varphi|_{\mathbb{S}})(u) = \pi_H(\pi_{u^\perp}(\nabla \varphi(u))). \tag{2.14}$$

Indeed, it suffices to observe that setting $G(x) = x/||x||$ for all $x \in (\mathbb{R}^2)^N \setminus \{0\}$, we have $\Phi_{\mathbb{S}} = G \circ \pi_H$, $\mathrm{d}_x G = \pi_{x^\perp}/||x||$ and $\mathrm{d}_x \pi_H = \pi_H$ and that for $u \in \mathbb{S}$, we have $\pi_H(u) = u$ and $||\pi_H(u)|| = 1$.

2.3 Main ideas of the proofs

Here we explain the main ideas of the proofs of Proposition 2.3 and Theorem 2.5. The arguments below are completely informal. In particular, we do as if our $KS(\theta, N)$-process $(X_t)_{t \in [0, \zeta)}$ was a true solution to (2.1) until explosion and we apply Itô's formula without care. We always assume at least that $N \ge 2$, $\theta > 0$ and $N > \theta$, which implies that $k_0 = \lceil 2N/\theta \rceil \ge 3$.

2.3.1 Existence

The existence of the $KS(\theta, N)$-process $(X_t)_{t\in[0,\zeta)}$, with values in E_{k_0}, is an easy application of Fukushima-Oshima-Takeda [24, Theorem 7.2.1]. The only difficulty is to show that the invariant measure μ is a Radon on E_{k_0}, see Proposition 2.30. The process may explode, i.e. get out of any compact subset of E_{k_0} in finite time. Observe that a typical compact subset of E_{k_0} is of the form, for $\varepsilon > 0$,

$$\mathcal{K}_\varepsilon = \{x \in (\mathbb{R}^2)^N : ||x|| \leq 1/\varepsilon \text{ and for all } K \subset [\![1, N]\!] \text{ such that } |K| = k_0,\ R_K(x) \geq \varepsilon\}.$$

2.3.2 Center of mass and dispersion process

One can verify, using Itô's formula, that the center of mass $S_{[\![1,N]\!]}(X)$ is a 2-dimensional Brownian motion with diffusion constant $N^{-1/2}$, that the dispersion process $R_{[\![1,N]\!]}(X)$ is a squared Bessel process with dimension $d_{\theta,N}(N)$, recall (2.2), and that these two processes are independent.

Consequently, if $\zeta < \infty$, the limits $\lim_{t\to\zeta-} S_{[\![1,N]\!]}(X_t)$ and $\lim_{t\to\zeta-} R_{[\![1,N]\!]}(X_t)$ a.s. exist, and this implies that $\limsup_{t\to\zeta-} ||X_t|| < \infty$: the process cannot explode to infinity, it can only explode because it tends to the boundary of E_{k_0}. If moreover $k_0 > N$ (i.e. if $\theta < 2$), this is sufficient to show that $\zeta = \infty$, since then $E_{k_0} = (\mathbb{R}^2)^N$.

2.3.3 Behavior of distant subsets of particles

Consider a partition $K_1, \dots, K_p$ of $[\![1, N]\!]$. If we neglect interactions between particles of which the indexes are not in the same subset, we have, for each $\ell \in [\![1, p]\!]$, setting $\tilde{\theta}_\ell = \theta|K_\ell|/N$,

$$\mathrm{d}X_t^i = \mathrm{d}B_t^i - \frac{\tilde{\theta}_\ell}{|K_\ell|} \sum_{j\in K_\ell\setminus\{i\}} \frac{X_t^i - X_t^j}{\|X_t^i - X_t^j\|^2}\mathrm{d}t, \qquad i \in K_\ell$$

and we recognize a $KS(\tilde{\theta}_\ell, |K_\ell|)$-process.

During time intervals where particles indexed in different subsets are far enough from each other, we can indeed bound the interaction between those particles, so that the Girsanov theorem tells us that $(X_t^i)_{i\in K_1}, \dots, (X_t^i)_{i\in K_p}$ behave similarly, in the sense of trajectories, as independent $KS(\tilde{\theta}_1, |K_1|)$, ..., $KS(\tilde{\theta}_p, |K_p|)$-processes.

2.3.4 Brownian and Bessel behaviors of isolated subsets of particles

Consider $K \subset [\![1, N]\!]$. As seen just above, during time intervals where the particles indexed in K are far from all the other ones, the system $(X_t^i)_{i\in K}$ behaves, in the sense of trajectories, like a $KS(\theta|K|/N, |K|)$-process. Hence, as seen in Subsection 2.3.2, $S_K(X_t)$ behaves like a 2-dimensional Brownian motion with diffusion constant $|K|^{-1/2}$ and $R_K(X_t)$ behaves like a squared Bessel process of dimension $d_{\theta|K|/N,|K|}(|K|)$, which equals $d_{\theta,N}(|K|)$, recall (2.2).

2.3.5 Continuity at explosion

Here we assume that $N > \theta \geq 2$, so that $k_0 \in [\![2, N]\!]$ and we explain why a.s., $\zeta < \infty$ and $X_{\zeta-} = \lim_{t\to\zeta-} X_t$ exists, in the usual sense of $(\mathbb{R}^2)^N$.

(a) We first show that $\zeta < \infty$ a.s. On the event where $\zeta = \infty$, the squared Bessel process $R_{[\![1,N]\!]}(X)$ is defined for all times. Recall that $d_{\theta,N}(N) \le 0$ (because $\theta \ge 2$) and that a squared Bessel process with negative dimension can be defined on the whole time half-line and a.s. becomes negative in finite time. Since $R_{[\![1,N]\!]}(X) \ge 0$ by definition, this contradicts the fact that $\zeta = \infty$.

Similarly, one can show that a $KS(\theta, N)$-process has no chance to be defined after the first hitting time τ_K of 0 by $R_K(X_t)$, where $|K| = k_0$: this makes the choice of the space E_{k_0} very natural. Indeed, assume that X is defined during $[0, \zeta')$ with $\zeta' > \tau_K$. Consider the maximal subset L of $[\![1,N]\!]$ containing K and such that $R_L(X_{\tau_K}) = 0$. Then there is $\varepsilon > 0$ such that during $[\tau_K, \tau_K + \varepsilon] \subset [0, \zeta')$, the particles labeled in L are far from the ones labeled outside L. By Subsection 2.3.4, $(R_L(X_{\tau_K+t}))_{t\in[0,\varepsilon]}$ behaves like a squared Bessel process with dimension $d_{\theta,N}(|L|)$ issued from 0. But such a process is instantaneously negative, because $d_{\theta,N}(|L|) \le 0$ (since $|L| \ge k_0$). Since $R_L(X) \ge 0$, this contradicts the fact that $\tau_K \in [0, \zeta')$.

(b) We next show by reverse induction that a.s. for all $K \subset [\![1,N]\!]$ with $|K| \ge 2$, we have

$$\text{either} \quad \lim_{t\to\zeta-} R_K(X_t) = 0 \quad \text{or} \quad \liminf_{t\to\zeta-} R_K(X_t) > 0. \tag{2.15}$$

If $K = [\![1,N]\!]$, $\lim_{t\to\zeta-} R_K(X_t)$ exists by continuity of the (true) squared Bessel process $R_K(X_t)$ and this implies the result. We now fix $n \in [\![3,N]\!]$ and assume that (2.15) holds true for all K such that $|K| \ge n$. We consider $K \subset [\![1,N]\!]$ with $|K| = n-1$: by induction assumption, either there is $i \notin K$ such that $\lim_{t\to\zeta-} R_{K\cup\{i\}}(X_t) = 0$ and then $\lim_{t\to\zeta-} R_K(X_t) = 0$, or for all $i \in [\![1,N]\!] \setminus K$, $\liminf_{t\to\zeta-} R_{K\cup\{i\}}(X_t) > 0$. In this last case, and when $\limsup_{t\to\zeta-} R_K(X_t) > 0$ and $\liminf_{t\to\zeta-} R_K(X_t) = 0$ (which is the negation of (2.15)), there are $\alpha > 0$ and $\varepsilon > 0$ such that (i) $R_K(X_t)$ upcrosses $[\varepsilon/2, \varepsilon]$ infinitely often during $[\zeta - \alpha, \zeta)$ and (ii) for all $t \in [\zeta - \alpha, \zeta)$ such that $R_K(X_t) < \varepsilon$, the particles indexed in K are far from all the other ones (because then $R_K(X_t)$ is small and $R_{K\cup\{i\}}(X_t)$ is large for all $i \notin K$), so that $R_K(X_t)$ behaves like a squared Bessel process with dimension $d_{\theta,N}(|K|)$, see Subsection 2.3.4. Points (i) and (ii) are in contradiction, since a squared Bessel process is continuous and thus cannot upcross $[\varepsilon/2, \varepsilon]$ infinitely often during a finite time interval.

(c) We now show that $\lim_{t\to\zeta-} X_t$ exists. Using (b) and the (random) equivalence relation on $[\![1,N]\!]$ defined by $i \sim j$ if and only if $\lim_{t\to\zeta-} R_{\{i,j\}}(X_t) = 0$, one can build a (random) partition $\mathbf{K} = (K_p)_{p\in[\![1,\ell]\!]}$ of $[\![1,N]\!]$ such that for all $p \in [\![1,\ell]\!]$, $\lim_{t\to\zeta-} R_{K_p}(X_t) = 0$ and $\liminf_{t\to\zeta-} \min_{i\notin K_p} R_{K_p\cup\{i\}}(X_t) > 0$. Hence, there is $\alpha \in [0, \zeta)$ such that for all $p \ne q$, the particles labeled in K_p are far from the ones labeled in K_q during $[\alpha, \zeta)$. As seen in Subsection 2.3.4, we conclude that for all $p \in [\![1,\ell]\!]$, $S_{K_p}(X_t)$ behaves like a Brownian motion during $[\alpha, \zeta)$, and thus $M_p = \lim_{t\to\zeta-} S_{K_p}(X_t)$ exists. Since moreover $\lim_{t\to\zeta-} R_{K_p}(X_t) = 0$, we deduce that for all $i \in K_p$, $\lim_{t\to\zeta-} X^i_t = M_p$. As a conclusion $\lim_{t\to\zeta-} X^i_t$ exists for all $i \in [\![1,N]\!]$.

2.3.6 A spherical process

We recall that $\mathbb{S}$, π_H, $\pi_{u^\perp}$ and b were introduced in Section 2.2 and introduce the possibly exploding (with life-time ξ) process $(U_t)_{t\in[0,\xi)}$ with values in $\mathbb{S} \cap E_{k_0}$, informally solving (we will also use here the theory of Dirichlet spaces), for some given $U_0 \in \mathbb{S} \cap E_{k_0}$ and some $(\mathbb{R}^2)^N$-valued Brownian motion $(B_t)_{t\ge0}$,

$$U_t = U_0 + \int_0^t \pi_{U_s^\perp} \pi_H \mathrm{d}B_s + \int_0^t \pi_{U_s^\perp} \pi_H b(U_s) \mathrm{d}s - \frac{2N-3}{2} \int_0^t U_s \mathrm{d}s.$$

We call such a process a $SKS(\theta, N)$-process.

One can check that this process is β-symmetric, where β is defined in (2.8), and that β is Radon on $\mathbb{S} \cap E_{k_0}$, see Proposition 2.32. And we will see that if $k_0 \geq N$, then $\beta(\mathbb{S}) < \infty$, so that the process $(U_t)_{t\geq 0}$ is non-exploding and positive recurrent.

2.3.7 Decomposition of the process

We assume that $N \geq 2$ and $\theta > 0$ are such $d_{\theta,N}(N) < 2$ and, as usual, $N > \theta$. We consider a 2-dimensional Brownian $(M_t)_{t\geq 0}$ with diffusion constant $N^{-1/2}$, a squared Bessel process $(D_t)_{t\in[0,\tau_D)}$ with dimension $d_{\theta,N}(N)$ killed when it hits 0, with life-time τ_D, and a $SKS(\theta, N)$-process $(U_t)_{t\in[0,\xi)}$, these three processes being independent. We introduce the time-change

$$A_t = \int_0^t \frac{\mathrm{d}s}{D_s}, \quad t \in [0, \tau_D).$$

Since $\tau_D < \infty$ (because $d_{\theta,N}(N) < 2$), since $D_{\tau_D} = 0$ and since, roughly, the paths of $(\sqrt{D_t})_{t\in[0,\tau_D)}$ are 1/2-Hölder continuous, it holds that $A_{\tau_D} = \infty$ a.s. We introduce the inverse function $\rho : [0,\infty) \to [0, \tau_D)$ of $A : [0, \tau_D) \to [0,\infty)$.

We also set $\zeta' = \rho_\xi$ and observe that $\zeta' \leq \tau_D$, since ρ is $[0, \tau_D)$-valued, and that $\zeta' < \tau_D$ if and only if $\xi < \infty$. A fastidious but straightforward computation shows that, recalling (2.9),

$$X_t = \Psi(M_t, D_t, U_{A_t}), \qquad \text{i.e.} \quad X^i_t = M_t + \sqrt{D_t}U^i_{A_t}, \qquad i \in [\![1, N]\!],$$

which is well-defined during $[0, \zeta')$, solves (2.1).

This decomposition of the $KS(\theta, N)$-process, which is noticeable in that U satisfies an autonomous S.D.E. and thus is Markov, is at the basis of our analysis.

In other words, $(X_t)_{t\in[0,\zeta')}$ is the restriction to the time interval $[0, \zeta')$ of a $KS(\theta, N)$-process $(X_t)_{t\in[0,\zeta)}$. Moreover, we have $\zeta' = \zeta \wedge \tau_D$: if ξ is finite, then U gets out of $\mathbb{S} \cap E_{k_0}$ at time ξ, so that X gets out of E_{k_0} at time $\zeta' = \rho_\xi < \tau_D$, whence $\zeta = \zeta' = \zeta \wedge \tau_D$; if next $\xi = \infty$, then $\zeta' = \tau_D$ and U remains in E_{k_0} for all times, so that X remains in E_{k_0} during $[0, \tau_D)$, whence $\zeta \geq \tau_D$.

We have $S_{[\![1,N]\!]}(X_t) = M_t$ and $R_{[\![1,N]\!]}(X_t) = D_t$ for all $t \in [0, \zeta \wedge \tau_D)$, because U is $\mathbb{S}$-valued. By definition of $\mathbb{S}$, the process U cannot have any $[\![1, N]\!]$-collision. But for any $K \subset [\![1, N]\!]$ with cardinal at most $N - 1$,

$$U \text{ has a } K\text{-collision at } t \in [0, \xi) \text{ if and only if } X \text{ has a } K\text{-collision at } \rho_t \in [0, \zeta \wedge \tau_D). \tag{2.16}$$

Moreover, as seen a few lines above, $\xi < \infty$ is equivalent to $\zeta < \tau_D$. In other words, since $R_{[\![1,N]\!]}(X_t) = D_t$ for all $t \in [0, \zeta \wedge \tau_D)$ and since $\tau_D = \inf\{t > 0 : D_t = 0\}$, we have

$$\xi < \infty \qquad \text{if and only if} \qquad \inf_{t\in[0,\zeta)} R_{[\![1,N]\!]}(X_t) > 0. \tag{2.17}$$

2.3.8 Some special cases

Using the Girsanov theorem, see Subsection 2.3.4, we will manage to reduce a large part of the study to the special cases that we examine in the present subsection. Here we explain the following facts, for $N \geq 2$ and $\theta > 0$ with $N > \theta$:

(a) if $d_{\theta,N}(N-1) \in (0,2)$, then a.s., $\tau_D = \inf\{t > 0 : R_{[\![1,N]\!]}(X_t) = 0\} \leq \zeta$ and for all $r \in [0, \tau_D)$, all $K \subset [\![1,N]\!]$ with $|K| = N-1$, $(X_t)_{t\in[0,\zeta)}$ has infinitely many K-collisions during $[r, \tau_D)$;

(b) if $d_{\theta,N}(N-1) \leq 0$ (whence $k_0 \leq N-1$), then a.s., $\inf_{t\in[0,\zeta)} R_{[\![1,N]\!]}(X_t) > 0$.

We keep the same notation as in the previous subsection.

(i) We first verify that in (a), $\tau_D \leq \zeta$. Since $d_{\theta,N}(N-1) \in (0,2)$, it holds that $k_0 \geq N$. If first $k_0 > N$, then $\zeta = \infty$ by Subsection 2.3.2 and we are done. If next $k_0 = N$, then $\zeta < \infty$ and $X_{\zeta-}$ exists by Subsection 2.3.5. Moreover $X_{\zeta-}$ cannot belong to $E_{k_0} = E_N$ by definition of ζ and thus has its N particles at the same place, i.e. $R_{[\![1,N]\!]}(X_{\zeta-}) = 0$: we have $\zeta = \tau_D$.

(ii) In (b), $\zeta < \infty$ by Subsection 2.3.5 because $d_{\theta,N}(N-1) \leq 0$ implies that $\theta \geq 2$.

(iii) We consider, in any case, the spherical process $(U_t)_{t\in[0,\xi)}$ and assume that $\xi = \infty$. An Itô computation shows that for $K \subset [\![1,N]\!]$, for some 1-dimensional Brownian motion $(W_t)_{t\geq 0}$,

$$\begin{aligned} \mathrm{d}R_K(U_t) =& 2\sqrt{R_K(U_t)(1-R_K(U_t))}\mathrm{d}W_t + d_{\theta,N}(|K|)\mathrm{d}t - d_{\theta,N}(N)R_K(U_t)\mathrm{d}t \\ & - \frac{2\theta}{N}\sum_{i\in K, j\notin K} \frac{U_t^i - U_t^j}{||U_t^i - U_t^j||^2}\cdot(U_t^i - S_K(U_t))\mathrm{d}t. \end{aligned}$$

We fix $\varepsilon > 0$ to be chosen later. During time intervals where $\min_{i\in K, j\notin K} \|U_t^i - U_t^j\| \geq \varepsilon$, we thus have, for some constant C_ε,

$$\mathrm{d}R_K(U_t) \leq 2\sqrt{R_K(U_t)(1-R_K(U_t))}\mathrm{d}W_t + d_{\theta,N}(|K|)\mathrm{d}t + C_\varepsilon\sqrt{R_K(U_t)}\mathrm{d}t, \tag{2.18}$$

where we used the Cauchy-Schwarz inequality and that $R_K(U_t)$ is uniformly bounded (because U is $\mathbb{S}$-valued). Hence, still during time intervals where $\min_{i\in K, j\notin K} \|U_t^i - U_t^j\| \geq \varepsilon$, by comparison, $R_K(U_t)$ is smaller than S_t, the solution to

$$\mathrm{d}S_t = 2\sqrt{S_t(1-S_t)}\mathrm{d}W_t + d_{\theta,N}(|K|)\mathrm{d}t + C_\varepsilon\sqrt{S_t}\mathrm{d}t. \tag{2.19}$$

And a little study involving scale functions/speed measures shows that this process hits zero in finite time if and only if $d_{\theta,N}(|K|) < 2$, exactly as a squared Bessel process with dimension $d_{\theta,N}(|K|)$.

(iv) We end the proof of (a). In this case, $k_0 \geq N$, so that U is non-exploding, as seen in Subsection 2.3.6. Hence $\xi = \infty$ and we can use (iii). Moreover, U is recurrent, still by Subsection 2.3.6. We fix K with $|K| = N-1$ and we choose $\varepsilon > 0$ small enough so that we have

$$\beta\Big(\Big\{u \in \mathbb{S} : \min_{i\in K, j\notin K} \|u^i - u^j\| \geq \varepsilon\Big\}\Big) > 0,$$

where β is the invariant measure (2.8) of U. Hence the process $\min_{i\in K, j\notin K} \|U_t^i - U_t^j\|$ visits the zone (ε, ∞) infinitely often and each time, $R_K(U)$ has a (uniformly) positive probability to hit 0 by (iii) and since $d_{\theta,N}(|K|) = d_{\theta,N}(N-1) < 2$. Consequently, for any $s > 0$, $(U_t)_{t\geq 0}$ has infinitely many K-collisions during $[s, \infty)$. Recalling (2.16) and that $\zeta \wedge \tau_D = \tau_D$ by (i), we conclude that for any $r \in [0, \tau_D)$, $(X_t)_{t\in[0,\zeta)}$ has infinitely many K-collisions during $[r, \tau_D)$.

(v) We finally complete the proof of (b). By (2.17), it is sufficient to show that $\xi < \infty$ a.s.

Assume that U is recurrent (and thus non-exploding). Then we take $K = [\![2,N]\!]$ and apply the same reasoning as in (iv) : since $d_{\theta,N}(|K|) \leq 0 < 2$, $R_K(U)$ hits zero in finite time and this makes

U get out of E_{N-1} and thus explode, since U is $(E_{k_0} \cap \mathbb{S})$-valued and since $k_0 \le N-1$. We thus have a contradiction.

Hence U is transient and it eventually gets out of the compact of $E_{k_0} \cap \mathbb{S}$

$$\mathcal{K} = \{u \in \mathbb{S} : \forall K \subset [\![1,N]\!] \text{ such that } |K| = k_0, \text{ we have } R_K(u) \ge \varepsilon\},$$

for any fixed $\varepsilon > 0$. Hence on the event where $\xi = \infty$, $\lim_{t\to\infty} \min_{|K|=k_0} R_K(U_t) = 0$ a.s. Recalling now that $k_0 \le N-1$ and that U is $\mathbb{S}$-valued (whence $R_{[\![1,N]\!]}(U_t) = 1$) we can a.s. find K with $|K| \in [\![k_0, N-1]\!]$ such that $\liminf_{t\to\infty} R_K(U_t) = 0$ but $\liminf_{t\to\infty} \min_{i\notin K} R_{K\cup\{i\}}(U_t) > 0$. It is then not too hard to find $\alpha > 0$ and $\varepsilon > 0$ such that each time $R_K(U_t) < \alpha$ (which often happens), all the particles indexed in K are far from all the other ones with a distance greater than $\varepsilon > 0$. We conclude from (iii), since $d_{\theta,N}(|K|) \le 0$ (because $|K| \ge k_0$) that each time $R_K(U_t) < \alpha$, it has a (uniformly) positive probability to hit zero. On the event $\xi = \infty$, this will eventually happen, so that the process U will have a K-collision and thus will leave E_{k_0} in finite time. Hence U will explode, so that $\xi < \infty$.

2.3.9 Size of the cluster

We assume that $N > 3\theta \ge 6$. Hence $\zeta < \infty$ and $X_{\zeta-}$ exists, by Subsection 2.3.5. Moreover, by definition of ζ, we know that $X_{\zeta-} \notin E_{k_0}$. We want now to show that $X_{\zeta-} \in E_{k_0+1}$, i.e. that the cluster causing explosion is precisely composed of k_0 particles. If $k_0 = N$, there is nothing to do, since then $E_{k_0+1} = (\mathbb{R}^2)^N$. Now if $k_0 \le N-1$, we assume by contradiction, that there is $K \subset [\![1,N]\!]$ with $|K| \ge k_0 + 1$ such that $R_K(X_{\zeta-}) = 0$ and $\min_{i\notin K} R_{K\cup\{i\}}(X_{\zeta-}) > 0$. Then there is $\alpha > 0$ such that during $[\zeta-\alpha, \zeta)$, the particles indexed in K are far from the other ones, so that $(X_t^i)_{t\in[0,\zeta), i\in K}$ behaves like a $KS(\theta|K|/N, |K|)$-process by Subsection 2.3.3. Observe now that $d_{\theta|K|/N,|K|}(|K|-1) = d_{\theta,N}(|K|-1) \le 0$ because $|K|-1 \ge k_0$ and $|K| > \theta|K|/N$ because $N > \theta$. We thus know from the special case (b) of Subsection 2.3.8 that $\inf_{t\in[\zeta-\alpha,\zeta)} R_K(X_t) > 0$, which contradicts the fact that $R_K(X_{\zeta-}) = 0$.

2.3.10 Collisions before explosion

We fix again $N > 3\theta \ge 6$. We recall that $k_1 = k_0 - 1$ and we show that there are infinitely many k_1-ary collisions just before explosion. We know from the previous subsection that there exists $K_0 \subset [\![1,N]\!]$ such that $|K_0| = k_0$ and $R_{K_0}(X_{\zeta-}) = 0$ and $\min_{i\notin K_0} R_{K_0\cup\{i\}}(X_{\zeta-}) > 0$. Then there is $\alpha > 0$ such that during $[\zeta-\alpha, \zeta)$, the particles indexed in K_0 are far from the other ones, so that $(X_t^i)_{i\in K_0}$ behaves like a $KS(\theta k_0/N, k_0)$-process by Subsection 2.3.3. Observe now that $d_{\theta k_0/N, k_0}(k_0-1) = d_{\theta,N}(k_0-1) \in (0,2)$ thanks to Lemma 2.1 and that $k_0 > \theta k_0/N$ because $N > \theta$. We thus know from the special case (a) of Subsection 2.3.8 that $(X_t^i)_{i\in K_0}$ has infinitely many $(K_0 \setminus \{i\})$-collisions just before ζ, for all $i \in K_0$.

When $k_2 = k_1 - 1$, one can show in the very same way that for all K with $|K| = k_1$, for all $i \in K$, there are infinitely many $(K \setminus \{i\})$-collisions just before each K-collision. We may also use Subsection 2.3.8-(a), since $d_{\theta k_1/N, k_1}(k_1-1) = d_{\theta,N}(k_2) \in (0,2)$, see Lemma 2.1.

2.3.11 Absence of other collisions

We want to show that when $N > 3\theta \ge 6$, for $K \subset [\![1,N]\!]$ with $|K| \in [\![3, k_2-1]\!]$, there is no K-collision during $(0,\zeta)$. Suppose by contradiction that there is $K \subset [\![1,N]\!]$ with $|K| \in [\![3, k_2-1]\!]$

and $t \in (0,\zeta)$ such that $R_K(X_t) = 0$ and for all $i \notin K$, $R_{K\cup\{i\}}(X_t) > 0$. Then there is $\alpha > 0$ such that during $[t-\alpha, t]$, the particles indexed in K are far from the other ones, so that $R_K(X_t)$ behaves like a squared Bessel process with dimension $d_{\theta|K|/N,|K|}(|K|)$, see Subsection 2.3.4. Since $d_{\theta|K|/N,|K|}(|K|) = d_{\theta,N}(|K|) \geq 2$ because $|K| \in [\![3, k_2-1]\!]$, see Lemma 2.1, such a Bessel process cannot hit zero, whence a contradiction.

2.3.12 Binary collisions

We still assume that $N > 3\theta \geq 6$, we suppose that there is a K-collision for some $K \subset [\![1,N]\!]$ such that $|K| = k_2$ at some time $t \in (0,\zeta)$ and we want to show that there are infinitely many binary collisions just before t. There is $\alpha > 0$ such that the particles indexed in K are far from all the other ones during $[t-\alpha, t]$, so that Subsection 2.3.3 tells us that $(X_t^i)_{i\in K}$ behaves like a $KS(\theta k_2/N, k_2)$-process. We observe that $k_2 \geq 5$, that $d_{\theta k_2/N,k_2}(k_2-1) = d_{\theta,N}(k_2-1) \geq 2$ and that $d_{\theta k_2/N,k_2}(k_2) = d_{\theta,N}(k_2) \in (0,2)$ by Lemma 2.1.

We are reduced to show that a $KS(\theta,N)$-process, that we still denote by $(X_t^i)_{i\in[\![1,N]\!],t\geq 0}$, such that $N \geq 5$, $d_{\theta,N}(N-1) \geq 2$ and $d_{\theta,N}(N) \in (0,2)$, a.s. has infinitely many binary collisions before the first instant τ_D of $[\![1,N]\!]$-collision. Such a process does not explode, because $k_0 > N$ (since $d_{\theta,N}(N) > 0$), see Subsection 2.3.2. Hence using (2.16) (which is licit since $d_{\theta,N}(N) < 2$), we only have to show that e.g. U^1 collides infinitely often with U^2 during $[0,\infty)$.

First, one easily gets convinced that the probability that e.g. X^1 collides with X^2 before τ_D is positive, because the probability that all the particles are pairwise far from each other, except X^1 and X^2, during the time interval $[0,1]$, is positive. On this kind of event, by Subsection 2.3.4, $R_{\{1,2\}}(X_t)$ behaves like a squared Bessel process with dimension $d_{\theta,N}(2) \in (0,2)$ and thus hits zero during $[0,1]$ (and thus before τ_D) with positive probability.

Using again (2.16), we conclude that the probability that U^1 collides with U^2 in finite time is positive. Since now U is positive recurrent, recall Subsection 2.3.6 and that $k_0 > N$ (because $d_{\theta,N}(N) > 0$), we conclude that U^1 collides infinitely often with U^2 during $[0,\infty)$ as desired.

2.3.13 Non-integrability of the drift term

Here we check that when $d_{\theta,N}(k_1) \in (0,1)$, the S.D.E. (2.1) cannot have a solution in the classical sense, because the drift term is not integrable in time. More precisely, recall that there is some K-collision at some time τ strictly before explosion, for some $K \subset [\![1,N]\!]$ with cardinal k_1. We now show that a.s., for $a > 0$,

$$\int_{\tau-a}^{\tau+a} \sum_{i=1}^{N} \Big\| \sum_{j\neq i} \frac{X_s^i - X_s^j}{||X_s^i - X_s^j||^2} \Big\| \mathrm{d}s = \infty,$$

which indeed shows the non-integrability of the drift term. Since τ is an instant of K-collision, there exists $a > 0$ small enough so that during $[\tau-a, \tau+a] \subset [0,\zeta)$, the particles labeled in K are far from the particles labeled in K^c. It clearly suffices to show that $Z = \infty$ a.s., where

$$Z = \int_{\tau-a}^{\tau+a} \sum_{i\in K} \Big\| \sum_{j\in K, j\neq i} \frac{X_s^i - X_s^j}{||X_s^i - X_s^j||^2} \Big\| \mathrm{d}s.$$

But

$$Z = \int_{\tau-a}^{\tau+a} \frac{f(V_s)}{\sqrt{R_K(X_s)}} \mathrm{d}s, \quad \text{where} \quad V_s = (V_s^i)_{i \in K} \quad \text{is defined by} \quad V_s^i = \frac{X_s^i - S_K(X_s)}{\sqrt{R_K(X_s)}},$$

so that V_s a.s. belongs to $\mathbb{S}_K = \{(v^i)_{i\in K} \in (\mathbb{R}^2)^{|K|} : \sum_{i\in K} v^i = 0,\ \sum_{i\in K} ||v^i||^2 = 1\}$, and where

$$f(v) = \sum_{i\in K} \Big\| \sum_{j\in K, j\neq i} \frac{v^i - v^j}{||v^i - v^j||^2} \Big\|$$

for each $v \in \mathbb{S}_K$. Since the invariant measure $\mathbf{m}$ of X satisfies $\mathbf{m}(E_2^c) = 0$, it a.s. holds true that $X_s \in E_2$ for a.e. $s \in [0, \zeta)$ (at least for a.e. initial condition), so that a.s., $f(V_s)$ is well-defined for a.e. $s \in [0, \zeta)$. We now show that f is bounded from below on $\mathbb{S}_K$. We have

$$f(v) \geq \max_{i\in K} \Big\| \sum_{j\in K, j\neq i} \frac{v^i - v^j}{||v^i - v^j||^2} \Big\| \geq \sqrt{\frac{1}{|K|} \sum_{i\in K} \Big\| \sum_{j\in K, j\neq i} \frac{v^i - v^j}{||v^i - v^j||^2} \Big\|^2}.$$

Using now the Cauchy-Schwarz inequality and the fact that $\sum_{i\in K} ||v^i||^2 = 1$, we find that

$$f(v) \geq \frac{1}{\sqrt{|K|}} \sum_{i\in K} \sum_{j\in K, j\neq i} \frac{v^i - v^j}{||v^i - v^j||^2} \cdot v^i = \frac{1}{2\sqrt{|K|}} \sum_{i,j\in K, j\neq i} \frac{v^i - v^j}{||v^i - v^j||^2} \cdot (v^i - v^j) = \frac{|K|(|K|-1)}{2\sqrt{|K|}}.$$

To conclude that $Z = \infty$ a.s., it remains to verify that $\int_{\tau-a}^{\tau+a} [R_K(X_s)]^{-1/2} \mathrm{d}s = \infty$ a.s. By Subsection 2.3.4, $R_K(X)$ behaves like a squared Bessel process with dimension $d_{\theta,N}(k_1)$ during $[\tau-a, \tau+a]$. Since $d_{\theta,N}(k_1) \in (0,1)$ and $R_K(X_\tau) = 0$, we conclude that indeed, $\int_{\tau-a}^{\tau+a} [R_K(X_s)]^{-1/2} \mathrm{d}s = \infty$ a.s. : this can be shown by comparison with the 1-dimensional Brownian motion.

2.4 Construction of the Keller-Segel particle system

The aim of this section is to build a first version of the Keller-Segel particle system using the book of Fukushima-Oshima-Takeda [24]. We also build a $\mathbb{S}$-valued process for later use.

Proposition 2.6. *We fix $N \geq 2$ and $\theta > 0$ such that $N > \theta$, recall that $k_0 = \lceil 2N/\theta \rceil$ and that μ and β were defined in* (2.4) *and* (2.8). *We set $\mathcal{X} = E_{k_0}$ and $\mathcal{X}_\triangle = \mathcal{X} \cup \{\triangle\}$, as well as $\mathcal{U} = \mathbb{S} \cap E_{k_0}$ and $\mathcal{U}_\triangle = \mathcal{U} \cup \{\triangle\}$, where $\triangle$ is a cemetery point.*

(i) There exists a unique diffusion $\mathbb{X} = (\Omega^X, \mathcal{M}^X, (X_t)_{t\geq 0}, (\mathbb{P}_x^X)_{x\in\mathcal{X}_\triangle})$ with values in $\mathcal{X}_\triangle$, which is μ-symmetric, with regular Dirichlet space $(\mathcal{E}^X, \mathcal{F}^X)$ on $L^2((\mathbb{R}^2)^N, \mu)$ with core $C_c^\infty(\mathcal{X})$ defined by

$$\text{for all} \quad \varphi \in C_c^\infty(\mathcal{X}), \quad \mathcal{E}^X(\varphi, \varphi) = \frac{1}{2} \int_{(\mathbb{R}^2)^N} \|\nabla\varphi\|^2 \mathrm{d}\mu.$$

We call such a process a $QKS(\theta, N)$-process and denote by $\zeta = \inf\{t \geq 0 : X_t = \triangle\}$ its life-time.

(ii) There exists a unique diffusion $\mathbb{U} = (\Omega^U, \mathcal{M}^U, (U_t)_{t\geq 0}, (\mathbb{P}_u^U)_{u\in\mathcal{U}_\triangle})$ with values in $\mathcal{U}_\triangle$, which is β-symmetric, with regular Dirichlet space $(\mathcal{E}^U, \mathcal{F}^U)$ on $L^2(\mathbb{S}, \beta)$ with core $C_c^\infty(\mathcal{U})$ defined by

$$\text{for all} \quad \varphi \in C_c^\infty(\mathcal{U}), \quad \mathcal{E}^U(\varphi, \varphi) = \frac{1}{2} \int_{\mathbb{S}} \|\nabla_{\mathbb{S}}\varphi\|^2 \mathrm{d}\beta.$$

We call such a process a $QSKS(\theta, N)$ -process and denote by $\xi = \inf\{t \geq 0 : U_t = \triangle\}$ its life-time.

The proof that we can build a $KS(\theta, N)$-process, i.e. a $QKS(\theta, N)$-process such that $\mathbb{P}_x^X \circ X_t^{-1}$ has density for all $x \in E_2$ and all $t > 0$ will be handled in Section 2.11.

We refer to Subsection 2.13.1 for some explanations about the notions used in this proposition : link between a diffusion (i.e. a continuous Hunt process), its generator, semi-group and its Dirichlet space, definition of the one-point compactification topology, i.e. the topology endowing $\mathcal{X}_\triangle$ and $\mathcal{U}_\triangle$, and about the *quasi-everywhere* notion. The state $\triangle$ is absorbing, i.e. $X_t = \triangle$ for all $t \geq \zeta$ and $U_t = \triangle$ for all $t \geq \xi$.

Remark 2.7. *By definition of the one-point compactification topology, for any increasing sequence of compact subsets $(\mathcal{K}_n)_{n\geq 1}$ of $\mathcal{X}$ such that $\cup_{n\geq 1}\mathcal{K}_n = \mathcal{X}$, $\zeta = \lim_{n\to\infty} \inf\{t \geq 0 : X_t \notin \mathcal{K}_n\}$.*

Similarly, for any increasing sequence of compact subsets $(\mathcal{L}_n)_{n\geq 1}$ of $\mathcal{U}$ such that $\cup_{n\geq 1}\mathcal{L}_n = \mathcal{U}$, $\xi = \lim_{n\to\infty} \inf\{t \geq 0 : U_t \notin \mathcal{L}_n\}$.

The uniqueness stated e.g. in Proposition 2.6-(i) has to be understood in the following sense, see [24, Theorem 4.2.8 p 167] : if we have another diffusion $\mathbb{Y} = (\Omega^Y, \mathcal{M}^Y, (Y_t)_{t\geq 0}, (\mathbb{P}_x^Y)_{x\in\mathcal{X}})$ enjoying the same properties, then quasi-everywhere, the law of $(Y_t)_{t\geq 0}$ under $\mathbb{P}_x^Y$ equals the law of $(X_t)_{t\geq 0}$ under $\mathbb{P}_x^X$. The quasi-everywhere notion depends on the Hunt process under consideration but, as recalled in Subsection 2.13.1, two Hunt processes with the same Dirichlet space share the same quasi-everywhere notion.

Proof of Proposition 2.6. We start with (i). We consider the bilinear form $\mathcal{E}^X$ on $C_c^\infty(\mathcal{X})$ defined by $\mathcal{E}^X(\varphi, \varphi) = \frac{1}{2}\int_{(\mathbb{R}^2)^N} ||\nabla\varphi||^2 \mathrm{d}\mu$. It is well-defined, since μ is Radon on $\mathcal{X} = E_{k_0}$ by Proposition 2.30.

We first show that it is closable, see [24, page 2], i.e. that if $(\varphi_n)_{n\geq 1} \subset C_c^\infty(\mathcal{X})$ is such that $\lim_n \varphi_n = 0$ in $L^2((\mathbb{R}^2)^N, \mu)$ and $\lim_{n,m} \mathcal{E}^X(\varphi_n - \varphi_m, \varphi_n - \varphi_m) = 0$, then $\lim_n \mathcal{E}^X(\varphi_n, \varphi_n) = 0$: since $\nabla\varphi_n$ is a Cauchy sequence in $L^2((\mathbb{R}^2)^N, \mu)$, it converges to a limit g and it suffices to prove that $g = 0$ a.e. For $\psi \in C_c^\infty(E_2, (\mathbb{R}^2)^N)$, we have $\int_{(\mathbb{R}^2)^N} g \cdot \psi \mathrm{d}\mu = \lim_n \int_{(\mathbb{R}^2)^N} \nabla\varphi_n \cdot \psi \mathrm{d}\mu$. But, recalling (2.4),

$$\int_{(\mathbb{R}^2)^N} \nabla\varphi_n \cdot \psi \mathrm{d}\mu = \int_{(\mathbb{R}^2)^N} \nabla\varphi_n(x) \cdot \psi(x)\mathbf{m}(x)\mathrm{d}x = -\int_{(\mathbb{R}^2)^N} \varphi_n(x)\mathrm{div}(\mathbf{m}(x)\psi(x))\mathrm{d}x.$$

Thus by the Cauchy-Schwarz inequality,

$$\Big|\int_{(\mathbb{R}^2)^N} \nabla\varphi_n \cdot \psi \mathrm{d}\mu\Big| \leq \Big(\int_{(\mathbb{R}^2)^N} \varphi_n^2 \mathrm{d}\mu\Big)^{1/2}\Big(\int_{(\mathbb{R}^2)^N} \frac{|\mathrm{div}(\mathbf{m}(x)\psi(x))|^2}{\mathbf{m}(x)}\mathrm{d}x\Big)^{1/2},$$

which tends to 0 since $\lim_n \varphi_n = 0$ in $L^2((\mathbb{R}^2)^N, \mu)$, since $\psi \in C_c^\infty(E_2, (\mathbb{R}^2)^N)$ and since $\mathbf{m}$ is smooth and positive on E_2. Thus $\int_{(\mathbb{R}^2)^N} g \cdot \psi \mathrm{d}\mu = 0$ for all $\psi \in C_c^\infty(E_2, (\mathbb{R}^2)^N)$, so that $g = 0$ a.e.

We can thus consider the extension of $\mathcal{E}^X$ to $\mathcal{F}^X = \overline{C_c^\infty(\mathcal{X})}^{\mathcal{E}_1^X}$, where we have set $\mathcal{E}_1^X(\varphi, \varphi) = \int_{(\mathbb{R}^2)^N} (\varphi^2 + \frac{1}{2}||\nabla\varphi||^2)\mathrm{d}\mu$ for $\varphi \in C_c^\infty(\mathcal{X})$.

Next, $(\mathcal{E}^X, \mathcal{F}^X)$ is obviously regular with core $C_c^\infty(\mathcal{X})$, see [24, page 6], because $C_c^\infty(\mathcal{X})$ is dense in $\mathcal{F}^X$ for the norm associated to $\mathcal{E}_1^X$ by definition of $\mathcal{F}^X$ and $C_c^\infty(\mathcal{X})$ is dense, for the uniform norm, in $C_c(\mathcal{X})$. It is also strongly local, see [24, page 6], i.e. $\mathcal{E}^X(\varphi, \psi) = \frac{1}{2}\int_{(\mathbb{R}^2)^N} \nabla\varphi \cdot \nabla\psi \mathrm{d}\mu = 0$ if $\varphi, \psi \in C_c^\infty(\mathcal{X})$ and if φ is constant on a neighborhood of Supp ψ.

Then [24, Theorems 7.2.2 page 380 and 4.2.8 page 167] imply the existence and uniqueness of a Hunt process $\mathbb{X} = (\Omega^X, \mathcal{M}^X, (X_t)_{t\geq 0}, (\mathbb{P}_x^X)_{x\in\mathcal{X}_\triangle})$ with values in $\mathcal{X}_\triangle$, which is μ-symmetric, of which the Dirichlet space is $(\mathcal{E}^X, \mathcal{F}^X)$, and such that $t \mapsto X_t$ is $\mathbb{P}^X_x$-a.s. continuous on $[0, \zeta)$ for all $x \in \mathcal{X}$, where $\zeta = \inf\{t \geq 0 : X_t = \triangle\}$.

Furthermore, since $\mathcal{E}^X$ is strongly local, we know from [24, Theorem 4.5.3 page 186] that we can choose $\mathbb{X}$ (modifying $\mathbb{P}^X_x$ only on a properly exceptional set) such that $\mathbb{P}_x(\zeta < \infty, X_{\zeta-} = \triangle) = 1$ for all $x \in \mathcal{X}$. This implies that for all $x \in \mathcal{X}$, $\mathbb{P}_x$-a.s., the map $t \mapsto X_t$ is continuous from $[0, \infty)$ to $\mathcal{X}_\triangle$, endowed with the one-point compactification topology on $\mathcal{X}_\triangle$ recalled in Subsection 2.13.1. Hence $\mathbb{X}$ is a diffusion.

For (ii), the very same strategy applies. The only difference is the integration by parts to be used for the closability : for $\varphi \in C^1_c(\mathcal{U})$ and $\psi \in C^1_c(\mathbb{S} \cap E_2, (\mathbb{R}^2)^N)$, it classically holds that

$$\int_{\mathbb{S}} (\nabla_{\mathbb{S}} \varphi) \cdot \psi \mathrm{d}\beta = \int_{\mathbb{S}} (\nabla_{\mathbb{S}} \varphi(u)) \cdot \psi(u) \mathbf{m}(u) \sigma(\mathrm{d}u) = - \int_{\mathbb{S}} \varphi(u) \mathrm{div}_{\mathbb{S}}(\mathbf{m}(u)\psi(u))\sigma(\mathrm{d}u). \tag{2.20}$$

This can be shown naively using Lemma 2.31. □

We now make explicit the generators of $\mathbb{X}$ and $\mathbb{U}$ when applied to some functions enjoying a few properties. See Subsection 2.13.1 for a precise definition of the generator of a Hunt process. We have to introduce a few notation.

For $\varphi \in C^\infty((\mathbb{R}^2)^N)$, $\alpha \in (0, 1]$ and $x \in (\mathbb{R}^2)^N$, we set

$$\mathcal{L}^X_\alpha \varphi(x) = \frac{1}{2}\Delta\varphi(x) - \frac{\theta}{N} \sum_{1 \leq i \neq j \leq N} \frac{x^i - x^j}{\|x^i - x^j\|^2 + \alpha} \cdot (\nabla\varphi(x))^i = \frac{1}{2\mathbf{m}_\alpha(x)} \mathrm{div}[\mathbf{m}_\alpha(x)\nabla\varphi(x)], \tag{2.21}$$

where

$$\mathbf{m}_\alpha(x) = \prod_{1 \leq i \neq j \leq N} (\|x^i - x^j\|^2 + \alpha)^{-\theta/(2N)}.$$

This is in accordance with (2.4), in the sense that $\mathbf{m}_0 = \mathbf{m}$. The formula (2.21) makes sense for $x \in E_2$ when $\alpha = 0$ (with $\mathbf{m}_\alpha$ replaced by $\mathbf{m}$) and we recall that for $\varphi \in C^\infty((\mathbb{R}^2)^N)$ and $x \in E_2$, $\mathcal{L}^X\varphi(x)$ was defined in (2.5) by $\mathcal{L}^X\varphi(x) = \mathcal{L}^X_0\varphi(x)$. We will often use that for all $\varphi, \psi \in C^\infty((\mathbb{R}^2)^N)$, all $x \in (\mathbb{R}^2)^N$, all $\alpha \in (0, 1]$,

$$\mathcal{L}^X_\alpha(\varphi\psi)(x) = \varphi(x)\mathcal{L}^X_\alpha\psi(x) + \psi(x)\mathcal{L}^X_\alpha\varphi(x) + \nabla\varphi(x) \cdot \nabla\psi(x). \tag{2.22}$$

For $\varphi \in C^\infty(\mathbb{S})$, $\alpha \in (0, 1]$ and $u \in \mathbb{S}$, we set

$$\mathcal{L}^U_\alpha \varphi(u) = \frac{1}{2}\Delta_{\mathbb{S}}\varphi(u) - \frac{\theta}{N} \sum_{1 \leq i \neq j \leq N} \frac{u^i - u^j}{\|u^i - u^j\|^2 + \alpha} \cdot (\nabla_{\mathbb{S}}\varphi(u))^i = \frac{1}{2\mathbf{m}_\alpha(u)} \mathrm{div}_{\mathbb{S}}[\mathbf{m}_\alpha(u)\nabla_{\mathbb{S}}\varphi(u)]. \tag{2.23}$$

This formula makes sense for $u \in \mathbb{S} \cap E_2$ when $\alpha = 0$ (with $\mathbf{m}_\alpha$ replaced by $\mathbf{m}$) and we set, for $\varphi \in C^\infty(\mathbb{S})$ and $u \in \mathbb{S} \cap E_2$, $\mathcal{L}^U\varphi(u) = \mathcal{L}^U_0\varphi(u)$.

Remark 2.8. *(i) Denote by $(\mathcal{A}^X, \mathcal{D}_{A^X})$ the generator of the process $\mathbb{X}$ of Proposition 2.6-(i). If $\varphi \in C^\infty_c(\mathcal{X})$ satisfies $\sup_{\alpha \in (0,1]} \sup_{x \in (\mathbb{R}^2)^N} |\mathcal{L}^X_\alpha \varphi(x)| < \infty$, then $\varphi \in \mathcal{D}_{A^X}$ and $\mathcal{A}^X\varphi = \mathcal{L}^X\varphi$.*

(ii) Denote by $(\mathcal{A}^U, \mathcal{D}_{A^U})$ the generator of the process $\mathbb{U}$ of Proposition 2.6-(ii). If $\varphi \in C^\infty_c(\mathcal{U})$ satisfies $\sup_{\alpha \in (0,1]} \sup_{u \in \mathbb{S}} |\mathcal{L}^U_\alpha \varphi(u)| < \infty$, then $\varphi \in \mathcal{D}_{A^U}$ and $\mathcal{A}^U\varphi = \mathcal{L}^U\varphi$.

Démonstration. To check (i), it suffices by (2.70) to verify that (a) $\varphi \in \mathcal{F}^X$, (b) $\mathcal{L}^X\varphi \in L^2(\mathcal{X},\mu)$ and (c) for all $\psi \in \mathcal{F}^X$, we have $\mathcal{E}^X(\varphi,\psi) = -\int_{\mathcal{X}}(\mathcal{L}^X\varphi)\psi \mathrm{d}\mu$.

Point (a) is clear, since $\varphi \in C_c^\infty(\mathcal{X})$. Point (b) follows from the facts that μ is Radon on $\mathcal{X}$, that φ is compactly supported in $\mathcal{X}$ and that $\mathcal{L}^X\varphi \in L^\infty((\mathbb{R}^2)^N, \mathrm{d}x)$, because for all $x \in E_2$, $\mathcal{L}^X\varphi(x) = \lim_{\alpha\to 0}\mathcal{L}^X_\alpha\varphi(x)$. Concerning (c) it suffices, by definition of $(\mathcal{E}^X,\mathcal{F}^X)$ and since $\mathcal{L}^X\varphi \in L^2(\mathcal{X},\mu)$, to show that for all $\psi \in C_c^\infty(\mathcal{X})$, we have $\frac{1}{2}\int_{(\mathbb{R}^2)^N}\nabla\varphi\cdot\nabla\psi\mathrm{d}\mu = -\int_{(\mathbb{R}^2)^N}(\mathcal{L}^X\varphi)\psi\mathrm{d}\mu$. But for $\alpha \in (0,1]$, by a standard integration by parts, since φ,ψ and $\mathbf{m}_\alpha$ are smooth,

$$\frac{1}{2}\int_{(\mathbb{R}^2)^N}\nabla\varphi(x)\cdot\nabla\psi(x)\mathbf{m}_\alpha(x)\mathrm{d}x = -\frac{1}{2}\int_{(\mathbb{R}^2)^N}\mathrm{div}(\mathbf{m}_\alpha(x)\nabla\varphi(x))\psi(x)\mathrm{d}x$$
$$= -\int_{(\mathbb{R}^2)^N}[\mathcal{L}^X_\alpha\varphi(x)]\psi(x)\mathbf{m}_\alpha(x)\mathrm{d}x.$$

We conclude letting $\alpha \to 0$ by dominated convergence, since $\mathbf{m}_\alpha \to \mathbf{m}$ and $\mathcal{L}^X_\alpha\varphi \to \mathcal{L}^X\varphi$ a.e., since by assumption, $|\nabla\varphi(x)\cdot\nabla\psi(x)\mathbf{m}_\alpha(x)| + |[\mathcal{L}^X_\alpha\varphi(x)]\psi(x)\mathbf{m}_\alpha(x)| \le C\mathbb{1}_{\{x\in\mathcal{K}\}}\mathbf{m}(x)$ for some constant C and for $\mathcal{K}$ =Supp ψ which is compact in $\mathcal{X}$, and since $\mu(\mathcal{K}) = \int_{\mathcal{K}}\mathbf{m}(x)\mathrm{d}x < \infty$.

The proof of (ii) is exactly the same, using that if $\varphi,\psi \in C^\infty(\mathbb{S})$, it holds that

$$\frac{1}{2}\int_{\mathbb{S}}\nabla_{\mathbb{S}}\varphi\cdot\nabla_{\mathbb{S}}\psi\ \mathbf{m}_\alpha\mathrm{d}\sigma = -\frac{1}{2}\int_{\mathbb{S}}\mathrm{div}_{\mathbb{S}}(\mathbf{m}_\alpha\nabla_{\mathbb{S}}\varphi)\psi\mathrm{d}\sigma = -\int_{\mathbb{S}}[\mathcal{L}^U_\alpha\varphi]\psi\mathbf{m}_\alpha\mathrm{d}\sigma,$$

which can be shown naively using the projection $\Phi_{\mathbb{S}}$, see (2.10), and Lemma 2.31. ☐

We end the section with a quick irreducibility/recurrence/transience study of the spherical process, see Subsection 2.13.1 again for definitions.

Lemma 2.9. *We fix $N \ge 2$ and $\theta > 0$ such that $N > \theta$ and consider the process $\mathbb{U}$ and its Dirichlet space $(\mathcal{E}^U,\mathcal{F}^U)$ as in Proposition 2.6-(ii).*

(i) $(\mathcal{E}^U,\mathcal{F}^U)$ is irreducible and we have the alternative :

• *either $(\mathcal{E}^U,\mathcal{F}^U)$ is recurrent and in particular it is non-exploding and for all measurable $A \subset \mathcal{U}$ such that $\beta(A) > 0$, $\mathbb{P}^U_u(\limsup_{t\to\infty}\{U_t \in A\}) = 1$ quasi-everywhere ;*

• *or $(\mathcal{E}^U,\mathcal{F}^U)$ is transient and in particular for all compact set $\mathcal{K}$ of $\mathcal{U}$, we have quasi-everywhere $\mathbb{P}^U_u(\liminf_{t\to\infty}\{U_t \in \mathcal{K}\}) = 0$.*

(ii) If $d_{\theta,N}(N-1) > 0$, then $(\mathcal{E}^U,\mathcal{F}^U)$ is recurrent.

In the transient case, one might also prove that $\mathbb{P}^U_u(\limsup_{t\to\infty}\{U_t \in \mathcal{K}\}) = 0$, but this would be useless for our purpose.

Démonstration. We start with (i). We first show that in any case, $(\mathcal{E}^U,\mathcal{F}^U)$ is irreducible. By [24, Corollary 4.6.4 page 195] and since $\mathcal{E}^U(\varphi,\varphi) = \frac{1}{2}\int_{\mathbb{S}}\|\nabla_{\mathbb{S}}\varphi\|^2\mathbf{m}\mathrm{d}\sigma$ with $\mathbf{m}$ bounded from below by a constant (on $\mathbb{S}$), it suffices to prove that the σ-symmetric Hunt process with regular Dirichlet space $(\mathcal{E},\mathcal{F})$ on $L^2(\mathcal{U},\sigma)$ with core $C_c^\infty(\mathcal{U})$ such that for all $\varphi \in C_c^\infty(\mathcal{U})$, $\mathcal{E}(\varphi,\varphi) = \frac{1}{2}\int_{\mathbb{S}}\|\nabla_{\mathbb{S}}\varphi\|^2\mathrm{d}\sigma$ is irreducible. But this Hunt process is nothing but a $\mathbb{S}$-valued Brownian motion. This Brownian motion is *a priori* killed when it gets out of $\mathcal{U}$, but this does a.s. never occur since such a Brownian motion never has two (bi-dimensional) coordinates equal. This $\mathbb{S}$-valued Brownian motion is of course irreducible. We conclude from [24, Lemma 1.6.4 page 55] that $(\mathcal{E}^U,\mathcal{F}^U)$ is either recurrent or transient.

• When $(\mathcal{E}^U, \mathcal{F}^U)$ is recurrent, [24, Theorem 4.7.1-(iii) page 202] gives us the result.

• When $(\mathcal{E}^U, \mathcal{F}^U)$ is transient, we fix a compact set $\mathcal{K}$ of $\mathcal{U}$ and we know from Lemma 2.32 that $\beta(\mathcal{K}) < \infty$, so that by definition of transience, for β-a.e $u \in \mathcal{U}$, $\mathbb{E}_u^U[\int_0^\infty \mathbb{1}_{\mathcal{K}}(U_s)\mathrm{d}s] < \infty$. Setting $\tau_{\mathcal{K}^c} = \inf\{t \geq 0 : U_t \notin \mathcal{K}\}$, we get in particular that for β-a.e $u \in \mathcal{U}$, $\mathbb{P}_u^U(\tau_{\mathcal{K}^c} < \infty) = 1$. But, by [24, (4.1.9) page 155], $u \mapsto \mathbb{P}_u^U(\tau_{\mathcal{K}^c} < \infty)$ is finely continuous. Using [24, Lemma 4.1.5 page 155], we deduce that $\mathbb{P}_u^U(\tau_{\mathcal{K}^c} < \infty) = 1$ quasi-everywhere. The Markov property allows us to conclude.

Concerning (ii), we recall from Proposition 2.32 that $\beta(\mathbb{S}) < \infty$, because $d_{\theta,N}(N-1) > 0$ implies that $k_0 \geq N$, see Lemma 2.1. Moreover, $k_0 \geq N$ implies that $E_{k_0} \supset E_N \supset \mathbb{S}$, whence $\mathcal{U} = E_{k_0} \cap \mathbb{S} = \mathbb{S}$ is compact : the process cannot explode, i.e. $\xi = \infty$. Consequently, $(\mathcal{E}^U, \mathcal{F}^U)$ is recurrent, since $\varphi \equiv 1$ belongs to $L^1(\mathcal{U}, \beta)$ and since $\mathbb{E}_u^U[\int_0^\infty \varphi(U_s)\mathrm{d}s] = \mathbb{E}_u^U[\xi] = \infty$. Indeed, as recalled Subsection 2.13.1, if $(\mathcal{E}^U, \mathcal{F}^U)$ was transient, we would have $\mathbb{E}_u^U[\int_0^\infty \varphi(U_s)\mathrm{d}s] < \infty$ for all $\varphi \in L^1(\mathcal{U}, \beta)$, with the convention that $\varphi(\triangle) = 0$. □

2.5 Decomposition

The goal of this section is to prove the following decomposition of the Keller-Segel particle system defined in Proposition 2.6-(i). This decomposition is noticeable and crucial for our purpose.

Proposition 2.10. *We fix $N \geq 2$ and $\theta > 0$ such that $N > \theta$, and we recall that $k_0 = \lceil 2N/\theta \rceil$, that $\mathcal{X} = E_{k_0}$ and that $\mathcal{U} = \mathbb{S} \cap E_{k_0}$.*

For $x \in E_N$, we set $r = R_{[\![1,N]\!]}(x) > 0$, $z = S_{[\![1,N]\!]}(x) \in \mathbb{R}^2$ and $u = (x - \gamma(z))/\sqrt{r} \in \mathbb{S}$ and we consider three independent processes :

• *$(M_t)_{t\geq 0}$, a 2-dimensional Brownian motion with diffusion constant $N^{-1/2}$ starting from z,*

• *$(D_t)_{t\geq 0}$ a squared Bessel process with dimension $d_{\theta,N}(N)$ starting from r and killed when it gets out of $(0,\infty)$, with life-time $\tau_D = \inf\{t \geq 0 : D_t = \triangle\}$,*

• *$(U_t)_{t\geq 0}$, a $QSKS(\theta, N)$ -process starting from u, with life-time $\xi = \inf\{t \geq 0 : U_t = \triangle\}$.*

We introduce $A_t = \int_0^{t\wedge\tau_D} D_s^{-1}\mathrm{d}s$, and its generalized inverse $\rho_t = \inf\{s > 0 : A_s > t\}$. We define $Y_t = \Psi(M_t, D_t, U_{A_t})$, where we recall from (2.9) that $\Psi(z, r, u) = \gamma(z) + \sqrt{r}u \in E_N$ when $(z, r, u) \in \mathbb{R}^2 \times (0,\infty) \times \mathbb{S}$ and where we set $\Psi(z, r, u) = \triangle$ when $r = \triangle$ or $u = \triangle$. Observe that the life-time of Y equals $\zeta' = \rho_\xi \wedge \tau_D$.

Consider also a $QKS(\theta, N)$-process $\mathbb{X} = (\Omega^X, \mathcal{M}^X, (X_t)_{t\geq 0}, (\mathbb{P}_x^X)_{x\in\mathcal{X}_\triangle})$, with life-time ζ, and $\mathbb{X}^* = (\Omega^X, \mathcal{M}^X, (X_t^*)_{t\geq 0}, (\mathbb{P}_x^X)_{x\in(\mathcal{X}\cap E_N)\cup\{\triangle\}})$, where $X_t^* = X_t\mathbb{1}_{\{t<\tau\}} + \triangle\mathbb{1}_{\{t\geq\tau\}}$ and where $\tau = \inf\{t \geq 0 : R_{[\![1,N]\!]}(X_t) \notin (0,\infty)\}$. In other words, $\mathbb{X}^*$ is the version of $\mathbb{X}$ killed when it gets out of E_N. The life-time of $\mathbb{X}^*$ is τ.

The law of $(Y_t)_{t\geq 0}$ is the same as that of $(X_t^*)_{t\geq 0}$ under $\mathbb{P}_x^X$, quasi-everywhere in $\mathcal{X} \cap E_N$.

We take the convention that $R_{[\![1,N]\!]}(\triangle) = 0$, so that $\tau \in [0, \zeta]$. Since $R_{[\![1,N]\!]}(Y_t) = D_t$ and $S_{[\![1,N]\!]}(Y_t) = M_t$ for all $t \in [0, \zeta')$, Proposition 2.10 in particular implies that $(R_{[\![1,N]\!]}(X_t))_{t\geq 0}$ and $(S_{[\![1,N]\!]}(X_t))_{t\geq 0}$ are some independent squared Bessel process and Brownian motion until the first time $(R_{[\![1,N]\!]}(X_t))_{t\geq 0}$ vanishes. This actually holds true until explosion, as shown in Lemma 2.11 below. The quasi-everywhere notion refers to the Hunt process $\mathbb{X}$. Observe that when $\theta \geq 2$, we have $k_0 \leq N$, so that $\mathcal{X} \cap E_N = \mathcal{X}$ and $\mathbb{X} = \mathbb{X}^*$.

Démonstration. We slice the proof in several steps. The two first steps are more or less classical, even if we give all the details : we determine the Dirichlet spaces of the three processes $(M_t)_{t\geq 0}$, $(D_t)_{t\geq 0}$ and $(U_t)_{t\geq 0}$ involved in the construction of $(Y_t)_{t\geq 0}$; then we compute the Dirichlet space of $(D_{\rho_t})_{t\geq 0}$; we next identify the Dirichlet space of $(D_{\rho_t}, U_t)_{t\geq 0}$, which allows us to find the one of $(D_t, U_{A_t})_{t\geq 0}$ by a second time-change ; by concatenation, we deduce the Dirichlet space of $(M_t, D_t, U_{A_t})_{t\geq 0}$. The main computations are handled in Steps 3 and 4, where we find the Dirichlet space of $(Y_t)_{t\geq 0}$, which allows us to conclude in Step 5 by uniqueness.

Step 1. First, take $\mathbb{U} = (\Omega^U, \mathcal{M}^U, (U_t)_{t\geq 0}, (\mathbb{P}_u^U)_{u\in\mathcal{U}_\triangle})$ as in Proposition 2.6-(ii).

Second, consider a 2-dimensional Brownian motion $\mathbb{M} = (\Omega^M, \mathcal{M}^M, (M_t)_{t\geq 0}, (\mathbb{P}_z^M)_{z\in\mathbb{R}^2})$ with diffusion constant $N^{-1/2}$. We know from [24, Example 4.2.1 page 167] that $\mathbb{M}$ is a dz-symmetric (here dz is the Lebesgue measure on $\mathbb{R}^2$) diffusion with regular Dirichlet space $(\mathcal{E}^M, \mathcal{F}^M)$ on $L^2(\mathbb{R}^2, \mathrm{d}z)$ with core $C_c^\infty(\mathbb{R}^2)$ and for all $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\mathcal{E}^M(\varphi, \varphi) = \frac{1}{2N}\int_{\mathbb{R}^2} \|\nabla_z \varphi(z)\|^2 \mathrm{d}z. \tag{2.24}$$

Finally, let $\mathbb{D} = (\Omega^D, \mathcal{M}^D, (D_t)_{t\geq 0}, (\mathbb{P}_r^D)_{r\in\mathbb{R}_+^*\cup\{\triangle\}})$ be a squared Bessel process of dimension $d_{\theta,N}(N)$ killed when it gets out of $\mathbb{R}_+^* = (0,\infty)$ and set $\nu = d_{\theta,N}(N)/2 - 1$, see Revuz-Yor [44, page 443]. Fukushima [23, Theorem 3.3] tells us that $\mathbb{D}$ is a $r^\nu \mathrm{d}r$-symmetric diffusion (here dr is the Lebesgue measure on $\mathbb{R}_+^*$) with regular Dirichlet space $(\mathcal{E}^D, \mathcal{F}^D)$ on $L^2(\mathbb{R}_+, r^\nu \mathrm{d}r)$ with core $C_c^\infty(\mathbb{R}_+^*)$ where for all $\varphi \in C_c^\infty(\mathbb{R}_+^*)$,

$$\mathcal{E}^D(\varphi, \varphi) = 2\int_{\mathbb{R}_+} |\varphi'(r)|^2 r^{\nu+1} \mathrm{d}r. \tag{2.25}$$

Together with [23, Theorem 3.3], this uses that the scale function and the speed measure of $(D_t)_{t\geq 0}$ are respectively $r \mapsto r^{-\nu}$ and $-[r^\nu/(2\nu)]\mathrm{d}r$. Actually, we don't take the speed measure as reference measure but $r^\nu \mathrm{d}r$ which is the same up to a constant.

Step 2. We apply Lemma 2.35 to $\mathbb{D}$ with $g(r) = 1/r$, i.e. with $A_t = \int_0^t D_s^{-1} \mathrm{d}s = \int_0^{t\wedge\tau_D} D_s^{-1} \mathrm{d}s$ thanks to the convention $\triangle^{-1} = 0$ and recall that ρ is its generalized inverse : we find that setting $D_{\rho_t} = D_{\rho_t} \mathbb{1}_{\{\rho_t<\infty\}} + \triangle \mathbb{1}_{\{\rho_t=\infty\}}$,

$$\mathbb{D}_\rho = (\Omega^D, \mathcal{M}^D, (D_{\rho_t})_{t\geq 0}, (\mathbb{P}_r^D)_{r\in\mathbb{R}_+^*})$$

is a $r^{\nu-1}\mathrm{d}r$-symmetric $(\mathbb{R}_+^* \cup \{\triangle\})$-valued diffusion with regular Dirichlet space $(\mathcal{E}^{D_\rho}, \mathcal{F}^{D_\rho})$ on $L^2(\mathbb{R}_+, r^{\nu-1}\mathrm{d}r)$ with core $C_c^\infty(\mathbb{R}_+^*)$ such that for all $\varphi \in C_c^\infty(\mathbb{R}_+^*)$,

$$\mathcal{E}^{D_\rho}(\varphi, \varphi) = \mathcal{E}^D(\varphi, \varphi) = 2\int_{\mathbb{R}_+} |\varphi'(r)|^2 r^{\nu+1} \mathrm{d}r = 2\int_{\mathbb{R}_+} |r\varphi'(r)|^2 r^{\nu-1} \mathrm{d}r. \tag{2.26}$$

We use Lemma 2.37 and the notation therein : recalling that $\mathcal{M}^{(D,U)} = \sigma((D_{\rho_t}, U_t) : t \geq 0)$, with the convention that $(r, \triangle) = (\triangle, u) = (\triangle, \triangle) = \triangle$, and that $\mathbb{P}_{(r,u)}^{(D,U)} = \mathbb{P}_r^D \otimes \mathbb{P}_u^U$ if $(r,u) \in \mathbb{R}_+^* \times \mathcal{U}$ and $\mathbb{P}_\triangle^{(D,U)} = \mathbb{P}_\triangle^D \otimes \mathbb{P}_\triangle^U$, it holds that

$$(\mathbb{D}_\rho, \mathbb{U}) = \Big(\Omega^D \times \Omega^U, \mathcal{M}^{(D,U)}, (D_{\rho_t}, U_t)_{t\geq 0}, (\mathbb{P}_{(r,u)}^{(D,U)})_{(r,u)\in(\mathbb{R}_+^*\times\mathcal{U})\cup\{\triangle\}}\Big)$$

is a $r^{\nu-1}\mathrm{d}r\beta(\mathrm{d}u)$-symmetric $(\mathbb{R}_+^*\times\mathcal{U})\cup\{\triangle\}$-valued diffusion with regular Dirichlet space given by $(\mathcal{E}^{(D_\rho,U)},\mathcal{F}^{(D_\rho,U)})$ on $L^2(\mathbb{R}_+\times\mathbb{S},r^{\nu-1}\mathrm{d}r\beta(\mathrm{d}u))$ with core $C_c^\infty(\mathbb{R}_+^*\times\mathcal{U})$, and for all $\varphi\in C_c^\infty(\mathbb{R}_+^*\times\mathcal{U})$,

$$\mathcal{E}^{(D_\rho,U)}(\varphi,\varphi)=\int_{\mathbb{R}_+}\mathcal{E}^U(\varphi(r,\cdot),\varphi(r,\cdot))r^{\nu-1}\mathrm{d}r+\int_{\mathbb{S}}\mathcal{E}^{D_\rho}(\varphi(\cdot,u),\varphi(\cdot,u))\beta(\mathrm{d}u).$$

We now apply Lemma 2.35 to $(\mathbb{D}_\rho,\mathbb{U})$ with $g(r,u)=r$ for all $r\in\mathbb{R}_+^*$ and all $u\in\mathcal{U}$. We consider the time-change $\alpha_t=\int_0^t g(D_{\rho_s},U_s)\mathrm{d}s$, with the convention that $g(r,u)=0$ as soon as $(r,u)=\triangle$. We also set $B_t=\inf\{s>0:\alpha_s>t\}$. As we will see in a few lines, it holds that

$$(D_{\rho_{B_t}},U_{B_t})=(D_t,U_{A_t})\qquad\text{for all } t\geq 0. \tag{2.27}$$

Hence Lemma 2.35 tells us that

$$(\mathbb{D},\mathbb{U}_A)=\Big(\Omega^D\times\Omega^U,\mathcal{M}^{(D,U)},(D_t,U_{A_t})_{t\geq0},(\mathbb{P}^{(D,U)}_{(r,u)})_{(r,u)\in(\mathbb{R}_+^*\times\mathcal{U})\cup\{\triangle\}}\Big)$$

is a $r^\nu\mathrm{d}r\beta(\mathrm{d}u)$-symmetric $(\mathbb{R}_+^*\times\mathcal{U})\cup\{\triangle\}$-valued diffusion with Dirichlet space $(\mathcal{E}^{(D,U_A)},\mathcal{F}^{(D,U_A)})$ on $L^2(\mathbb{R}_+\times\mathbb{S},r^\nu\mathrm{d}r\beta(\mathrm{d}u))$, regular with core $C_c^\infty(\mathbb{R}_+^*\times\mathcal{U})$ and for all $\varphi\in C_c^\infty(\mathbb{R}_+^*\times\mathcal{U})$,

$$\mathcal{E}^{(D,U_A)}(\varphi,\varphi)=\mathcal{E}^{(D_\rho,U)}(\varphi,\varphi)=\int_{\mathbb{R}_+}\mathcal{E}^U(\varphi(r,\cdot),\varphi(r,\cdot))r^{\nu-1}\mathrm{d}r+\int_{\mathbb{S}}\mathcal{E}^{D_\rho}(\varphi(\cdot,u),\varphi(\cdot,u))\beta(\mathrm{d}u). \tag{2.28}$$

We now check the claim (2.27). Recall that D explodes at time τ_D, that $A_t=\int_0^{t\wedge\tau_D}D_s^{-1}\mathrm{d}s$ and that ρ is the generalized inverse of A. Hence $(\rho_t)_{t\in[0,A_{\tau_D})}$ is the true inverse of $(A_t)_{t\in[0,\tau_D)}$ and we have $\rho'_t=D_{\rho_t}$, whence $\rho_t=\int_0^t D_{\rho_s}\mathrm{d}s$ for $t\in[0,A_{\tau_D})$. We also have $\rho_t=\infty$ for $t\geq A_{\tau_D}$. Next, $\alpha_t=\int_0^t D_{\rho_s}\mathrm{d}s=\rho_t$ for $t\in[0,A_{\tau_D}\wedge\xi)$, because $g(D_{\rho_s},U_s)=D_{\rho_s}$ if $(D_{\rho_s},U_s)\neq\triangle$, i.e. if $s<A_{\tau_D}\wedge\xi$. Hence B, the generalized inverse of α, equals A during $[0,\tau_D\wedge\rho_\xi)$, thus in particular $\rho_{B_t}=t$ for $t\in[0,A_{\tau_D}\wedge\xi)$. As conclusion, (2.27) holds true for $t\in[0,A_{\tau_D}\wedge\xi)$. If now $t\geq\tau_D\wedge\rho_\xi$, then $B_t=\infty$, because B is the generalized inverse of α and because for all $t\geq0$,

$$\alpha_t\leq\alpha_{A_{\tau_D}\wedge\xi}=\rho_{A_{\tau_D}\wedge\xi}=\tau_D\wedge\rho_\xi.$$

Hence, still if $t\geq\tau_D\wedge\rho_\xi$, we have $(D_{\rho_{B_t}},U_{B_t})=\triangle$, while $(D_t,U_{A_t})=\triangle$ because either $t\geq\tau_D$ and thus $D_t=\triangle$ or $t\geq\rho_\xi$ and thus $A_t\geq\xi$ so that $U_{A_t}=\triangle$. We have proved (2.27).

We finally conclude, thanks to Lemma 2.37 again, setting $\mathcal{M}^{(M,D,U)}=\sigma((M_t,D_t,U_{A_t}):t\geq0)$ with the convention that $(z,\triangle)=\triangle$ and setting $\mathbb{P}^{(M,D,U)}_{(z,r,u)}=\mathbb{P}^M_z\otimes\mathbb{P}^{(D,U)}_{(r,u)}$ in the case where $(z,r,u)\in\mathbb{R}^2\times\mathbb{R}_+^*\times\mathcal{U}$ and $\mathbb{P}^{(M,D,U)}_\triangle=\mathbb{P}^M_\triangle\otimes\mathbb{P}^{(D,U)}_\triangle$, that

$$(\mathbb{M},\mathbb{D},\mathbb{U}_{\mathbb{A}})=\Big(\Omega^M\times\Omega^D\times\Omega^U,\mathcal{M}^{(M,D,U)},(M_t,D_t,U_{A_t})_{t\geq0},(\mathbb{P}^{(M,D,U)}_{(z,r,u)})_{(z,r,u)\in(\mathbb{R}^2\times\mathbb{R}_+^*\times\mathcal{U})\cup\{\triangle\}}\Big)$$

is a $\mathrm{d}zr^\nu\mathrm{d}r\beta(\mathrm{d}u)$-symmetric $(\mathbb{R}^2\times\mathbb{R}_+^*\times\mathcal{U})\cup\{\triangle\}$-valued diffusion with regular Dirichlet space $(\mathcal{E}^{(M,D,U_A)},\mathcal{F}^{(M,D,U_A)})$ on $L^2(\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{S},\mathrm{d}zr^\nu\mathrm{d}r\beta(\mathrm{d}u))$, with core $C_c^\infty(\mathbb{R}^2\times\mathbb{R}_+^*\times\mathcal{U})$. Moreover, for all $\varphi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}_+^* \times \mathcal{U})$,

$$\begin{aligned}\mathcal{E}^{(M,D,U_A)}(\varphi,\varphi) &= \int_{\mathbb{R}_+\times\mathbb{S}} \mathcal{E}^M(\varphi(\cdot,r,u),\varphi(\cdot,r,u))r^\nu \mathrm{d}r\beta(\mathrm{d}u) + \int_{\mathbb{R}^2} \mathcal{E}^{(D,U_A)}(\varphi(z,\cdot,\cdot),\varphi(z,\cdot,\cdot))\mathrm{d}z \\ &= \int_{\mathbb{R}_+\times\mathbb{S}} \mathcal{E}^M(\varphi(\cdot,r,u),\varphi(\cdot,r,u))r^\nu \mathrm{d}r\beta(\mathrm{d}u) + \int_{\mathbb{R}^2\times\mathbb{S}} \mathcal{E}^{D_\rho}(\varphi(z,\cdot,u),\varphi(z,\cdot,u))\mathrm{d}z\beta(\mathrm{d}u) \\ &\qquad + \int_{\mathbb{R}^2\times\mathbb{R}_+} \mathcal{E}^U(\varphi(z,r,\cdot),\varphi(z,r,\cdot))\mathrm{d}zr^{\nu-1}\mathrm{d}r \\ &= \int_{\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{S}} \Big[\frac{1}{2N}||\nabla_z\varphi(z,r,u)||^2 + 2r|\partial_r\varphi(z,r,u)|^2 + \frac{1}{2r}||\nabla_{\mathbb{S}}\varphi(z,r,u)||^2\Big]\mathrm{d}zr^\nu\mathrm{d}r\beta(\mathrm{d}u).\end{aligned} \tag{2.29}$$

For the second line, we used (2.28). For the last line, we used (2.24), (2.26) and the expression of $\mathcal{E}^U$, see Proposition 2.6-(ii).

Step 3. We recall that $Y_t = \Psi(M_t, D_t, U_{A_t})$, where $\Psi(z,r,u) = \gamma(z) + \sqrt{r}u$ for $(z,r,u) \in \mathbb{R}^2 \times \mathbb{R}_+^* \times \mathcal{U}$ and $\Psi(z,r,u) = \triangle$ for $(z,r,u) = \triangle$. One easily checks that Ψ is a bijection from $(\mathbb{R}^2 \times \mathbb{R}_+^* \times \mathcal{U}) \cup \{\triangle\}$ to $(\mathcal{X} \cap E_N) \cup \{\triangle\}$, recall that $\mathcal{X} = E_{k_0}$ and $\mathcal{U} = E_{k_0} \cap \mathbb{S}$.

We now study

$$\mathbb{Y} = (\Omega^Y, \mathcal{M}^Y, (Y_t)_{t\geq 0}, (\mathbb{P}_y^Y)_{y\in(\mathcal{X}\cap E_N)\cup\{\triangle\}}),$$

where $\Omega^Y = \Omega^M \times \Omega^D \times \Omega^U$, $\mathcal{M}^Y = \mathcal{M}^{(M,D,U)}$ and $\mathbb{P}_y^Y = \mathbb{P}_{(z,r,u)}^{(M,D,U)}$ for $(z,r,u) = \Psi^{-1}(y)$.

First, $\mathbb{Y}$ is a $(\mathcal{X}\cap E_N)\cup\{\triangle\}$-valued diffusion, because the bijection Ψ from $(\mathbb{R}^2\times\mathbb{R}_+^*\times\mathcal{U})\cup\{\triangle\}$ to $(\mathcal{X} \cap E_N) \cup \{\triangle\}$ is continuous, both sets being endowed with the one-point compactification topology, see Subsection 2.13.1.

Next, we prove that $\mathbb{Y}$ is μ-symmetric : if φ, ψ are nonnegative measurable functions on $\mathcal{X} \cap E_N$ and $t \geq 0$, we have, thanks to Lemma 2.31 (recall that $\nu = d_{\theta,N}(N)/2 - 1$),

$$\int_{(\mathbb{R}^2)^N} [P_t^Y\varphi(y)]\psi(y)\mu(\mathrm{d}y) = \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{S}} [(P_t^Y\varphi)(\Psi(z,r,u))]\psi(\Psi(z,r,u))r^\nu\mathrm{d}z\mathrm{d}r\beta(\mathrm{d}u).$$

But $(P_t^Y\varphi)(\Psi(z,r,u)) = \mathbb{E}_{(z,r,u)}[\varphi(\Psi(M_t,D_t,U_{A_t}))] = P_t^{(M,D,U_A)}(\varphi\circ\Psi)(z,r,u)$, so that

$$\int_{(\mathbb{R}^2)^N} [P_t^Y\varphi(y)]\psi(y)\mu(\mathrm{d}y) = \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{S}} [P_t^{(M,D,U_A)}(\varphi\circ\Psi)(z,r,u)][(\psi\circ\Psi)(z,r,u)]r^\nu\mathrm{d}z\mathrm{d}r\beta(\mathrm{d}u).$$

Using that $(\mathbb{M},\mathbb{D},\mathbb{U}_{\mathbb{A}})$ is $\mathrm{d}zr^\nu\mathrm{d}r\beta(\mathrm{d}u)$-symmetric and then the same computation in reverse order, one concludes that $\int_{(\mathbb{R}^2)^N}[P_t^Y\varphi]\psi\mathrm{d}\mu = \int_{(\mathbb{R}^2)^N}\varphi[P_t^Y\psi]\mathrm{d}\mu$ as desired.

Thus $\mathbb{Y}$ has a Dirichlet space $(\mathcal{E}^Y, \mathcal{F}^Y)$ on $L^2((\mathbb{R}^2)^N, \mu)$ that we now determine. For $\varphi \in L^2((\mathbb{R}^2)^N, \mu)$, using as above Lemma 2.31 and that $(P_t^Y\varphi)(\Psi(z,r,u)) = P_t^{(M,D,U_A)}(\varphi\circ\Psi)(z,r,u)$,

$$\begin{aligned}&\frac{1}{t}\int_{(\mathbb{R}^2)^N}(P_t^Y\varphi - \varphi)\varphi\mathrm{d}\mu \\ =&\frac{1}{2t}\int_{\mathbb{R}^2\times\mathbb{R}_+^*\times\mathbb{S}}[P_t^{(M,D,U_A)}(\varphi\circ\Psi)(z,r,u) - (\varphi\circ\Psi)(z,r,u)][\varphi\circ\Psi(z,r,u)]r^\nu\mathrm{d}z\mathrm{d}r\beta(\mathrm{d}u).\end{aligned}$$

Since Ψ is bijective, we deduce, see [24, Lemma 1.3.4 page 23], that

$$\mathcal{F}^Y = \Big\{\varphi \in L^2((\mathbb{R}^2)^N, \mu) : \varphi \circ \Psi \in \mathcal{F}^{(M,D,U_A)}\Big\} \tag{2.30}$$

$$\text{and for } \varphi \in \mathcal{F}^Y, \quad \mathcal{E}^Y(\varphi,\varphi) = \frac{1}{2}\mathcal{E}^{(M,D,U_A)}(\varphi\circ\Psi, \varphi\circ\Psi). \tag{2.31}$$

Step 4. We now compute $\mathcal{E}^Y(\varphi,\varphi)$ for $\varphi \in C_c^\infty(\mathcal{X} \cap E_N)$, so that $\varphi \circ \Psi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}_+^* \times \mathcal{U})$. Thanks to (2.29) and (2.31), we have

$$\mathcal{E}^Y(\varphi,\varphi) = \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{S}} I(z,r,u)\mathrm{d}z r^\nu \mathrm{d}r \beta(\mathrm{d}u), \tag{2.32}$$

where

$$I(z,r,u) = \frac{1}{2N}||\nabla_z(\varphi\circ\Psi)(z,r,u)||^2 + 2r|\partial_r(\varphi\circ\Psi)(z,r,u)|^2 + \frac{1}{2r}||\nabla_{\mathbb{S}}(\varphi\circ\Psi)(z,r,u)||^2.$$

We recall that for $\varphi : (\mathbb{R}^2)^N \to \mathbb{R}$, we call $\nabla\varphi(x) = ((\nabla\varphi(x))^1, \dots, (\nabla\varphi(x))^N) \in (\mathbb{R}^2)^N$ the total gradient of φ at $x \in (\mathbb{R}^2)^N$, and we have $(\nabla\varphi(x))^i \in \mathbb{R}^2$ for each $i \in [\![1,N]\!]$. And for $\phi : O \to \mathbb{R}^p$, where O is open in $\mathbb{R}^n$, we denote by $\mathrm{d}_z\phi$ the differential of ϕ at $z \in O$.

We start with the study of $\Psi(z,r,u) = \gamma(z) + \sqrt{r}u$, where we recall that γ was introduced in Section 2.2 and that $\Phi_{\mathbb{S}}(x) = \pi_H x/||\pi_H x||$ is defined on a neighborhood of $\mathbb{S}$ in $(\mathbb{R}^2)^N$, see (2.10). It holds that for all $(z,r,u) \in \mathbb{R}^2 \times \mathbb{R}_+^* \times \mathbb{S}$ and all $h \in \mathbb{R}^2$, $k \in \mathbb{R}$ and $\ell \in (\mathbb{R}^2)^N$,

$$\mathrm{d}_z\Psi(\cdot,r,u)(h) = \gamma(h), \qquad \mathrm{d}_r\Psi(z,\cdot,u)(k) = \frac{k}{2\sqrt{r}}u, \qquad \mathrm{d}_u[\Psi(z,r,\Phi_{\mathbb{S}}(\cdot))](\ell) = \sqrt{r}\pi_{u^\perp}(\pi_H(\ell)),$$

For the first equality, it suffices to use that γ is linear, so that $\mathrm{d}_z\Psi(\cdot,r,u)(h) = \mathrm{d}_z\gamma(h) = \gamma(h)$. The second equality is obvious. For the third equality, which is the differential at $u \in \mathbb{S}$ of the function $F(x) = \gamma(z) + \sqrt{r}\Phi_{\mathbb{S}}(x)$ defined for $x \in E_N$ (which is open in $(\mathbb{R}^2)^N$ and contains $\mathbb{S}$), we write $\mathrm{d}_uF = \sqrt{r}\mathrm{d}_u\Phi_{\mathbb{S}}$. But $\Phi_S = G \circ \pi_H$, where $G(x) = x/||x||$, and we have $\mathrm{d}_u\pi_H = \pi_H$ and $\mathrm{d}_{\pi_H(u)}G = \mathrm{d}_uG = \pi_{u^\perp}$ for $u \in \mathbb{S}$. All in all, $\mathrm{d}_uF = \sqrt{r}\pi_{u^\perp}\circ\pi_H$.

First, we have $\nabla_z(\varphi\circ\Psi)(z,r,u) = \sum_{i=1}^N[\nabla\varphi(\Psi(z,r,u))]^i$. Indeed, for all $h \in \mathbb{R}^2$, it holds that

$$\mathrm{d}_z(\varphi\circ\Psi(\cdot,r,u))(h) = (\mathrm{d}_{\Psi(z,r,u)}\varphi)[(\mathrm{d}_z\Psi(\cdot,r,u))(h)] = (\mathrm{d}_{\Psi(z,r,u)}\varphi)(\gamma(h)) = \nabla\varphi(\Psi(z,r,u))\cdot\gamma(h),$$

which, by definition of γ, equals $h \cdot \sum_{i=1}^N[\nabla\varphi(\Psi(z,r,u))]^i$.

This implies that

$$\frac{1}{2N}\|\nabla_z(\varphi\circ\Psi(z,r,u))\|^2 = \frac{1}{2N}\Big\|\sum_{i=1}^N[\nabla\varphi(\Psi(z,r,u))]^i\Big\|^2 = \frac{1}{2}\|\pi_{H^\perp}(\nabla\varphi(\Psi(z,r,u)))\|^2. \tag{2.33}$$

Indeed, recalling the expression of π_H, see Section 2.2, it suffices to note that for all $x \in (\mathbb{R}^2)^N$, $\|\pi_{H^\perp}(x)\|^2 = \|\gamma(S_{[\![1,N]\!]}(x))\|^2 = N\|S_{[\![1,N]\!]}(x)\|^2 = N^{-1}\|\sum_{i=1}^N x^i\|^2$.

Next, $\partial_r(\varphi\circ\Psi)(z,r,u) = (\nabla\varphi)(\Psi(z,r,u))\cdot u/(2\sqrt{r})$. Indeed, for $k \in \mathbb{R}$,

$$\mathrm{d}_r(\varphi\circ\Psi(z,\cdot,u))(k) = (\mathrm{d}_{\Psi(z,r,u)}\varphi)[(\mathrm{d}_r\Psi(z,\cdot,u))(k)] = (\mathrm{d}_{\Psi(z,r,u)}\varphi)(u) \times \frac{k}{2\sqrt{r}},$$

which is nothing but $(\nabla\varphi)(\Psi(z,r,u)) \cdot u \times k/(2\sqrt{r})$.

This implies, recalling that π_u is the orthogonal projection on $\mathrm{Span}(u) \subset (\mathbb{R}^2)^N$, that

$$2r|\partial_r(\varphi \circ \Psi)(z,r,u)|^2 = \frac{1}{2}\|\pi_u((\nabla\varphi)(\Psi(z,r,u)))\|^2 = \frac{1}{2}\|\pi_H(\pi_u((\nabla\varphi)(\Psi(z,r,u))))\|^2 \qquad (2.34)$$

since $u \in \mathbb{S}$, so that $||u|| = 1$ and $u \in H$.

Finally, $\nabla_{\mathbb{S}}(\varphi \circ \Psi)(z,r,u) = \sqrt{r}\pi_H(\pi_{u^\perp}(\nabla\varphi(\Psi(z,r,u))))$. Indeed, for all $\ell \in (\mathbb{R}^2)^N$,

$$\begin{aligned} d_u((\varphi \circ \Psi)(z,r,\Phi_{\mathbb{S}}(\cdot)))(\ell) =& (d_{\Psi(z,r,u)}\varphi)(d_u[\Psi(z,r,\Phi_{\mathbb{S}}(\cdot))](\ell)) \\ =& \sqrt{r}(d_{\Psi(z,r,u)}\varphi)(\pi_{u^\perp}(\pi_H(\ell))) \\ =& \sqrt{r}\nabla\varphi(\Psi(z,r,u)) \cdot \pi_{u^\perp}(\pi_H(\ell)) \\ =& \sqrt{r}\pi_H(\pi_{u^\perp}(\nabla\varphi(\Psi(z,r,u)))) \cdot \ell, \end{aligned}$$

and we conclude since $\nabla_{\mathbb{S}}(\varphi\circ\Psi)(z,r,u) = \nabla_x((\varphi\circ\Psi)(z,r,\Phi_{\mathbb{S}}(\cdot)))(u)$ by definition of $\nabla_{\mathbb{S}}$, see (2.12).

This implies that

$$\frac{1}{2r}||\nabla_{\mathbb{S}}(\varphi \circ \Psi)(z,r,u)||^2 = \frac{1}{2}\|\pi_H(\pi_{u^\perp}(\nabla\varphi(\Psi(z,r,u))))\|^2. \qquad (2.35)$$

Gathering (2.33), (2.34) and (2.35), we see that $I(z,r,u) = \frac{1}{2}\|\nabla\varphi(\Psi(z,r,u))\|^2$, since for $x \in (\mathbb{R}^2)^N$,

$$\|\pi_{H^\perp}(x)\|^2 + \|\pi_H(\pi_u(x))\|^2 + \|\pi_H(\pi_{u^\perp}(x))\|^2 = \|x\|^2$$

because $u \in \mathbb{S} \subset H$.

Injecting the value of I in (2.32) and using Lemma 2.31, we obtain

$$\mathcal{E}^Y(\varphi,\varphi) = \frac{1}{4}\int_{\mathbb{R}^2\times\mathbb{R}_+^*\times\mathbb{S}} \|\nabla\varphi(\Psi(z,r,u))\|^2 \mathrm{d}z r^\nu \mathrm{d}r \beta(\mathrm{d}u) = \frac{1}{2}\int_{(\mathbb{R}^2)^N} \|\nabla\varphi\|^2 \mathrm{d}\mu.$$

Step 5. As a last technical step, we verify that $(\mathcal{E}^Y, \mathcal{F}^Y)$ is a regular Dirichlet space on $L^2((\mathbb{R}^2)^N, \mu)$ with core $C_c^\infty(\mathcal{X} \cap E_N)$, i.e. that for all $\varphi \in \mathcal{F}^Y$, there is $\varphi_n \in C_c^\infty(\mathcal{X} \cap E_N)$ such that $\lim_n ||\varphi_n - \varphi||_{L^2((\mathbb{R}^2)^N,\mu)} + \mathcal{E}^Y(\varphi_n - \varphi, \varphi_n - \varphi) = 0$.

Recalling (2.30) and using that $(\mathcal{E}^{(M,D,U_A)}, \mathcal{F}^{(M,D,U_A)})$ on $L^2(\mathbb{R}^2 \times \mathbb{R}_+ \times \mathbb{S}, \mathrm{d}z r^\nu \mathrm{d}r \beta(\mathrm{d}u))$ is regular with core $C_c^\infty(\mathbb{R}^2 \times \mathbb{R}_+^* \times \mathcal{U})$, there is $g_n \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}_+^* \times \mathcal{U})$ such that

$$||g_n - \varphi\circ\Psi||_{L^2(\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{S},\mathrm{d}z r^\nu \mathrm{d}r\beta(\mathrm{d}u))} + \mathcal{E}^{(M,D,U_A)}(g_n - \varphi\circ\Psi, g_n - \varphi\circ\Psi) \to 0.$$

Setting $\varphi_n = g_n \circ \Psi^{-1}$, it holds that $\varphi_n \in C_c^\infty(\mathcal{X} \cap E_N)$ and we have, by (2.31),

$$\mathcal{E}^Y(\varphi_n - \varphi, \varphi_n - \varphi) = \frac{1}{2}\mathcal{E}^{(M,D,U_A)}(g_n - \varphi\circ\Psi, g_n - \varphi\circ\Psi) \to 0,$$

as well as, by Lemma 2.31,

$$||\varphi_n - \varphi||_{L^2((\mathbb{R}^2)^N,\mu)} = \frac{1}{2}||g_n - \varphi\circ\Psi||_{L^2(\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{S},\mathrm{d}z r^\nu \mathrm{d}r\beta(\mathrm{d}u))} \to 0.$$

Step 6. By Steps 3, 4 and 5, we know that $\mathbb{Y}$ is a μ-symmetric $(\mathcal{X} \cap E_N) \cup \{\triangle\}$-valued diffusion with regular Dirichlet space $(\mathcal{E}^Y, \mathcal{F}^Y)$ with core $C_c^\infty(\mathcal{X} \cap E_N)$ and with $\mathcal{E}^Y(\varphi, \varphi) = \frac{1}{2} \int_{(\mathbb{R}^2)^N} ||\nabla \varphi||^2 \mathrm{d}\mu$ for $\varphi \in C_c^\infty(\mathcal{X} \cap E_N)$.

Now, applying Lemma 2.38 to $\mathbb{X}$ defined in Proposition 2.6-(i) with the open set $\mathcal{X} \cap E_N$, we see that $\mathbb{X}^*$, i.e. $\mathbb{X}$ killed when getting outside $\mathcal{X} \cap E_N$, is a μ-symmetric $(\mathcal{X} \cap E_N) \cup \{\triangle\}$-valued diffusion process with regular Dirichlet space $(\mathcal{E}^{X^*}, \mathcal{F}^{X^*})$ with core $C_c^\infty(\mathcal{X} \cap E_N)$ and with $\mathcal{E}^{X^*}(\varphi, \varphi) = \frac{1}{2} \int_{(\mathbb{R}^2)^N} ||\nabla \varphi||^2 \mathrm{d}\mu$ for $\varphi \in C_c^\infty(\mathcal{X} \cap E_N)$.

This implies, as recalled in Subsection 2.13.1, that $(\mathcal{E}^{X^*}, \mathcal{F}^{X^*}) = (\mathcal{E}^Y, \mathcal{F}^Y)$. The conclusion follows by uniqueness, see [24, Theorem 4.2.8 p 167]. □

Actually, $(R_{[\![1,N]\!]}(X_t))_{t \geq 0}$ and $(S_{[\![1,N]\!]}(X_t))_{t \geq 0}$ are some independent squared Bessel process and Brownian motion *until explosion* (and not only until the first time where $R_{[\![1,N]\!]}(X_t) = 0$, as shown in Proposition 2.10), a fact that we shall often use.

Lemma 2.11. *We fix $N \geq 2$ and $\theta > 0$ such that $N > \theta$ and we consider a $QKS(\theta, N)$-process $\mathbb{X} = (\Omega^X, \mathcal{M}^X, (X_t)_{t \geq 0}, (\mathbb{P}_x^X)_{x \in \mathcal{X}_\triangle})$. Quasi-everywhere, there are a 2D-Brownian motion $(M_t)_{t \geq 0}$ with diffusion constant $N^{-1/2}$ issued from $S_{[\![1,N]\!]}(x)$ and a squared Bessel process $(D_t)_{t \geq 0}$ with dimension $d_{\theta,N}(N)$ issued from $R_{[\![1,N]\!]}(x)$ (killed when it gets out of $(0, \infty)$ if $d_{\theta,N}(N) \leq 0$) independent of $(M_t)_{t \geq 0}$ such that $\mathbb{P}_x^X$-a.s., $S_{[\![1,N]\!]}(X_t) = M_t$ and $R_{[\![1,N]\!]}(X_t) = D_t$ for all $t \in [0, \zeta)$.*

Démonstration. If $\theta \geq 2$, this follows from Proposition 2.10 : setting $\tau = \inf\{t > 0 : R_{[\![1,N]\!]}(X_t) \notin (0, \infty)\}$, we have $\tau = \zeta$. Indeed, on $\{\tau < \zeta\}$, we have $X_\tau \notin E_N$, whence $X_\tau \notin \mathcal{X}$ since $\mathcal{X} = E_{k_0}$ with $k_0 \leq N$ (because $\theta \geq 2$), which contradicts the fact that $\tau < \zeta$.

We now suppose that $\theta < 2$, so that $k_0 > N$ and thus $\mathcal{X} = (\mathbb{R}^2)^N$. We introduce the shortened notation $R(x) = R_{[\![1,N]\!]}(x)$, $S(x) = S_{[\![1,N]\!]}(x)$ and split the proof in three parts.

Step 1. First, one can show similarly (but much more easily) as in the proof of Proposition 2.10 that there exists a $2D$-Brownian motion $(M_t)_{t \geq 0}$ independent of $(X_t - \gamma(S(X_t)))_{t \geq 0}$, such that $S(X_t) = M_t$ for all $t \in [0, \zeta)$. This moreover shows that $(M_t)_{t \geq 0}$ is independent of $(R(X_t))_{t \geq 0}$, because $R(X_t) = \|X_t - \gamma(S(X_t))\|^2$.

Step 2. We consider some function $g_m \in C_c^\infty((\mathbb{R}^2)^N)$ such that $g_m = 1$ on $B(0, m)$ and $\sup_{\alpha \in (0,1]} \sup_{x \in (\mathbb{R}^2)^N} |\mathcal{L}_\alpha^X g_m(x)| < \infty$. Such a function exists by Remark 2.14. For $\varphi \in C_c^\infty(\mathbb{R}_+)$, we set $\psi(x) = \varphi(R(x))$ and show that $\psi g_m \in \mathcal{D}_{\mathcal{A}^X}$ and that for all $x \in B(0, m)$,

$$\mathcal{A}^X(\psi g_m)(x) = 2R(x)\varphi''(R(x)) + d_{\theta,N}(N)\varphi'(R(x)). \tag{2.36}$$

To this end, we apply Remark 2.8. Since $\psi g_m \in C_c^\infty((\mathbb{R}^2)^N)$ and since $\mathcal{X} = (\mathbb{R}^2)^N$, we have to show that $\sup_{\alpha \in (0,1]} \sup_{x \in (\mathbb{R}^2)^N} |\mathcal{L}_\alpha^X(\psi g_m)(x)| < \infty$, and we will deduce that $\mathcal{A}^X(\psi g_m) = \mathcal{L}^X(\psi g_m)$. By (2.22), we have $\mathcal{L}_\alpha^X(\psi g_m) = g_m \mathcal{L}_\alpha^X \psi + \psi \mathcal{L}_\alpha^X g_m + \nabla \psi \cdot \nabla g_m$. The only difficulty consists in showing that $\sup_{\alpha \in (0,1]} \sup_{x \in (\mathbb{R}^2)^N} |\mathcal{L}_\alpha^X \psi(x)| < \infty$. Using that $\nabla_{x^i} R(x) = 2(x^i - S(x))$, we find $\nabla_{x^i} \psi(x) = 2(x^i - S(x))\varphi'(R(x))$. Hence by symmetry,

$$\begin{aligned}\frac{\theta}{N} \sum_{1 \leq i \neq j \leq N} \frac{x^i - x^j}{\|x^i - x^j\|^2 + \alpha} \cdot \nabla_{x^i} \psi(x) &= \frac{2\theta}{N} \varphi'(R(x)) \sum_{1 \leq i \neq j \leq N} \frac{x^i - x^j}{\|x^i - x^j\|^2 + \alpha} \cdot x^i \\ &= \frac{\theta}{N} \varphi'(R(x)) \sum_{1 \leq i \neq j \leq N} \frac{\|x^i - x^j\|^2}{\|x^i - x^j\|^2 + \alpha}.\end{aligned} \tag{2.37}$$

Besides, $\Delta_{x^i}\psi(x) = 4(1-1/N)\varphi'(R(x)) + 4\|x^i - S(x)\|^2\varphi''(R(x))$, whence

$$\Delta\psi(x) = 4(N-1)\varphi'(R(x)) + 4R(x)\varphi''(R(x)). \tag{2.38}$$

We conclude by combining (2.37) and (2.38) that

$$\mathcal{L}_\alpha^X\psi(x) = 2R(x)\varphi''(R(x)) + \Big(2(N-1) - \frac{\theta}{N}\sum_{1\le i\ne j\le N}\frac{\|x^i-x^j\|^2}{\|x^i-x^j\|^2+\alpha}\Big)\varphi'(R(x)).$$

We immediately deduce, since φ is compactly supported, that $\sup_{\alpha\in(0,1]}\sup_{x\in(\mathbb{R}^2)^N}|\mathcal{L}_\alpha^X\psi(x)| < \infty$, whence $\sup_{\alpha\in(0,1]}\sup_{x\in(\mathbb{R}^2)^N}|\mathcal{L}_\alpha^X(\psi g_m)(x)| < \infty$. Hence $\psi g_m \in \mathcal{D}_{\mathcal{A}^X}$ and $\mathcal{A}^X(\psi g_m) = \mathcal{L}^X(\psi g_m)$. Moreover, recalling that $\mathcal{L}^X\psi = \mathcal{L}_\alpha^X\psi$ with $\alpha = 0$ and that $g_m = 1$ on $B(0,m)$, we conclude that $\mathcal{A}^X(\psi g_m)(x) = \mathcal{L}_0^X\psi(x)$ for $x \in B(0,m)$, whence (2.36), because $2(N-1) - \theta(N-1) = d_{\theta,N}(N)$.

Step 3. We define $\zeta_m = \inf\{t > 0 : X_t \notin B(0,m)\}$. By Lemma 2.34 and Step 1, for all $\varphi \in C_c^\infty(\mathbb{R}_+)$, quasi-everywhere in $B(0,m)$, $\varphi(R(X_{t\wedge\zeta_m})) - \varphi(R(x)) - \int_0^{t\wedge\zeta_m}\mathcal{L}^X\varphi(R(X_s))\mathrm{d}s$ is a $\mathbb{P}_x^X$-martingale. Recalling (2.36), we classically conclude that there is a Brownian motion W such that $R(X_t) = R(x) + 2\int_0^t\sqrt{R(X_s)}\mathrm{d}W_s + d_{\theta,N}(N)t$ during $[0,\zeta_n]$. We recognize the S.D.E. of a squared Bessel process with dimension $d_{\theta,N}(N)$, see Revuz-Yor [44, Chapter XI]. Since we know from Remark 2.7 that $\zeta = \lim_m \zeta_m$, the proof is complete. □

2.6 Some cutoff functions

We will need several times to approximate some indicator functions by some smooth functions, on which the generator $\mathcal{L}^X$ (or $\mathcal{L}^U$) is bounded. This does not seem obvious, due to the singularity of $\mathcal{L}^X$. We recall that $\mathcal{L}_\alpha^X$ and $\mathcal{L}_\alpha^U$ were defined in (2.21) and (2.23).

Lemma 2.12. *Fix $N \ge 2$, $\theta > 0$, recall that $k_0 = \lceil 2N/\theta\rceil$ and that $\mathcal{X} = E_{k_0}$. Consider a partition $\mathbf{K} = (K_p)_{p\in[\![1,\ell]\!]}$ and define, for $\varepsilon \in [0,1]$, (with the convention that $B(0,1/0) = (\mathbb{R}^2)^N$),*

$$G_{\mathbf{K},\varepsilon} = \Big\{x \in \mathcal{X} : \min_{1\le p\ne q\le \ell}\ \min_{i\in K_p, j\in K_q}||x^i - x^j||^2 > \varepsilon\Big\} \cap B\Big(0, \frac{1}{\varepsilon}\Big).$$

(i) For all $\varepsilon \in (0,1]$, there is a family of open relatively compact subsets $G^n_{\mathbf{K},\varepsilon}$ of $G_{\mathbf{K},0}$ such that

$$\bigcup_{n\ge1} G^n_{\mathbf{K},\varepsilon} \supset G_{\mathbf{K},\varepsilon} \quad \textit{and for each } n \ge 1,\ G^n_{\mathbf{K},\varepsilon} \subset G^{n+1}_{\mathbf{K},\varepsilon},$$

and some of $[0,1]$-valued functions $\Gamma^n_{\mathbf{K},\varepsilon} \in C_c^\infty(G_{\mathbf{K},0})$ such that for some $\eta \in (0,1]$, for all $n \ge 1$,

$$\mathrm{Supp}\,\Gamma^n_{\mathbf{K},\varepsilon} \subset G_{\mathbf{K},\eta}, \quad \Gamma^n_{\mathbf{K},\varepsilon} = 1 \ \textit{ on } \ G^n_{\mathbf{K},\varepsilon} \quad \textit{and} \quad \sup_{\alpha\in(0,1]}\sup_{x\in(\mathbb{R}^2)^N}\Big|\mathcal{L}_\alpha^X\Gamma^n_{\mathbf{K},\varepsilon}(x)\Big| < \infty.$$

(ii) With the same sets $G^n_{\mathbf{K},\varepsilon}$ as in (i), there is a family of functions $\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon} \in C_c^\infty(\mathbb{S}\cap G_{\mathbf{K},0})$ with values in $[0,1]$ such that for all $n \ge 1$,

$$\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon} = 1 \ \textit{ on } \ \mathbb{S}\cap G^n_{\mathbf{K},\varepsilon} \quad \textit{and} \quad \sup_{\alpha\in(0,1]}\sup_{u\in\mathbb{S}}\Big|\mathcal{L}_\alpha^U\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon}(u)\Big| < \infty.$$

The section is devoted to the proof of this lemma. We start with the following technical result.

Lemma 2.13. *We define the family* $(c_\ell)_{\ell\in[\![1,N]\!]}$ *by* $c_0 = 1$ *and for all* $\ell \in [\![1, N-1]\!]$, $c_{\ell+1} = (2+4\ell)c_\ell$. *For all* $K \subsetneq [\![1,N]\!]$, *all* $\varepsilon \in (0,1]$, *all* $x \in (\mathbb{R}^2)^N$ *such that*

$$R_K(x) \le 2c_{|K|}\varepsilon \quad \text{and} \quad \min_{j\notin K} R_{K\cup\{j\}}(x) \ge c_{|K|+1}\varepsilon,$$

it holds that $\|x^i - x^j\|^2 \ge c_{|K|}\varepsilon$ *for all* $i \in K$, *all* $j \notin K$.

Démonstration. We fix $K \subsetneq [\![1,N]\!]$, $\varepsilon \in (0,1]$ and $x \in (\mathbb{R}^2)^N$ as in the statement and assume by contradiction that there are $i_0 \in K$, $j_0 \notin K$ such that $\|x^{i_0} - x^{j_0}\|^2 < c_{|K|}\varepsilon$. Then for all $i \in K$,

$$\|x^{j_0} - x^i\|^2 \le 2\|x^{i_0} - x^{j_0}\|^2 + 2\|x^{i_0} - x^i\|^2 \le 2\|x^{i_0} - x^{j_0}\|^2 + 2|K|R_K(x) < (2+4|K|)c_{|K|}\varepsilon.$$

This implies that

$$R_{K\cup\{j_0\}}(x) = \frac{1}{2(|K|+1)}\Big(2|K|R_K(x) + 2\sum_{i\in K}\|x^{j_0} - x^i\|^2\Big) \le R_K(x) + \frac{1}{|K|+1}\sum_{i\in K}\|x^{j_0} - x^i\|^2,$$

whence

$$R_{K\cup\{j_0\}}(x) < 2c_{|K|}\varepsilon + \frac{2+4|K|}{|K|+1}|K|c_{|K|}\varepsilon < (2+4|K|)c_{|K|}\varepsilon = c_{|K|+1}\varepsilon,$$

which is a contradiction. □

We are now ready to give the

Proof of Lemma 2.12. We introduce some nondecreasing C^∞ function $\varrho : \mathbb{R}_+ \to [0,1]$ such that $\varrho = 0$ on $[0, 1/2]$ and $\varrho = 1$ on $[1,\infty)$. We divide the proof in three steps.

Step 1. We fix $n \ge 1$ and define, for $K \subset [\![1,N]\!]$, using the family $(c_\ell)_{\ell\in[\![1,N]\!]}$ of Lemma 2.13,

$$\tilde{E}_{K,n} = \Big\{x \in (\mathbb{R}^2)^N : \forall\, L \supset K,\ R_L(x) > \frac{c_{|L|}}{n}\Big\} \qquad \text{and} \qquad \tilde{\Gamma}_{K,n}(x) = \prod_{L\supset K} \varrho\Big(\frac{nR_L(x)}{c_{|L|}}\Big),$$

where $\{L \supset K\} = \{L \subset [\![1,N]\!] : K \subset L\}$. We have

$$\tilde{\Gamma}_{K,n} \in C^\infty((\mathbb{R}^2)^N), \quad \operatorname{Supp}\tilde{\Gamma}_{K,n} \subset \tilde{E}_{K,2n} \quad \text{and} \quad \tilde{\Gamma}_{K,n} = 1 \quad \text{on} \quad \tilde{E}_{K,n}. \tag{2.39}$$

Since $R_K(x) > 0$ implies that $R_L(x) > 0$ for all $L \supset K$, we also have

$$\cup_{n\ge1}\tilde{E}_{K,n} = \tilde{E}_K, \quad \text{where} \quad \tilde{E}_K = \{x \in (\mathbb{R}^2)^N : R_K(x) > 0\}. \tag{2.40}$$

We now show, and this is the main difficulty of the step, that for all $A > 0$, all $K \subset [\![1,N]\!]$ with $|K| \ge 2$, we have $\sup_{\alpha\in(0,1]}\sup_{x\in B(0,A)}|\mathcal{L}^X_\alpha\tilde{\Gamma}_{K,n}(x)| < \infty$. Since $\sup_{x\in B(0,A)}|\Delta\tilde{\Gamma}_{K,n}(x)| < \infty$, we only have to verify that $\sup_{\alpha\in(0,1]}\sup_{x\in B(0,A)}|I_{K,n,\alpha}(x)| < \infty$, where

$$I_{K,n,\alpha}(x) = \sum_{1\le i\ne j\le N}\frac{x^i - x^j}{\|x^i - x^j\|^2}\cdot\nabla_{x^i}\tilde{\Gamma}_{K,n}(x) = \sum_{L\supset K} f_{K,L,n}(x)\sum_{1\le i\ne j\le N}\frac{x^i - x^j}{\|x^i - x^j\|^2}\cdot\nabla_{x^i}R_L(x),$$

with

$$f_{K,L,n}(x) = \frac{n}{c_{|L|}}\varrho'\Big(\frac{nR_L(x)}{c_{|L|}}\Big)\prod_{M\supset K, M\ne L}\varrho\Big(\frac{nR_M(x)}{c_{|M|}}\Big).$$

Using that $\nabla_{x^i} R_L(x) = 2(x^i - S_L(x))\mathbb{1}_{\{i\in L\}}$, we now write

$$I_{K,n,\alpha}(x) = 2\sum_{L\supset K} f_{K,L,n}(x)(A_{L,\alpha}(x) + B_{L,\alpha}(x)),$$

where,

$$A_{L,\alpha}(x) = \sum_{i,j\in L, i\neq j} \frac{(x^i - x^j)\cdot(x^i - S_L(x))}{\|x^i - x^j\|^2 + \alpha} \quad \text{and} \quad B_{L,\alpha}(x) = \sum_{i\in L, j\in L^c} \frac{(x^i - x^j)\cdot(x^i - S_L(x))}{\|x^i - x^j\|^2 + \alpha}.$$

We have $\sup_{\alpha\in(0,1]}\sup_{x\in B(0,A)} |f_{K,L,n}(x)A_{L,\alpha}(x)| < \infty$ because $f_{K,L,n}$ is bounded and because

$$A_{L,\alpha}(x) = \sum_{i,j\in L, i\neq j} \frac{(x^i - x^j)\cdot x^i}{\|x^i - x^j\|^2 + \alpha} = \frac{1}{2}\sum_{i,j\in L, i\neq j} \frac{\|x^i - x^j\|^2}{\|x^i - x^j\|^2 + \alpha} \in \Big[0, \frac{|L|(|L|-1)}{2}\Big].$$

Next, we assume that $L \subsetneq [\![1,N]\!]$ (else $B_{L,\alpha}(x) = 0$) and observe that $f_{K,L,n}(x) \neq 0$ implies that $R_L(x) < c_{|L|}/n$ (because $\varrho' = 0$ on $[1,\infty)$) and that $\min_{i\notin L} R_{L\cup\{i\}}(x) > c_{|L|+1}/(2n)$ (because $\varrho = 0$ on $[0, 1/2]$). By Lemma 2.13, this implies that $\min_{i\in L, j\in L^c} ||x^i - x^j||^2 \geq c_{|L|}/(2n)$. We immediately conclude that $\sup_{\alpha\in(0,1]}\sup_{x\in B(0,A)} |f_{K,L,n}(x)B_{L,\alpha}(x)| < \infty$.

Step 2. We can now prove (i). We fix $\varepsilon \in (0,1]$ and a partition $\mathbf{K} = (K_p)_{p\in[\![1,\ell]\!]}$ of $[\![1,N]\!]$. For some $m \geq 1$ to be chosen later (as a function of ε), for each $n \geq 1$, we set

$$G^n_{\mathbf{K},\varepsilon} = B(0,m) \cap \Big(\bigcap_{K\subset[\![1,N]\!]:|K|=k_0} \tilde{E}_{K,n}\Big) \cap \Big(\bigcap_{1\leq p\neq q\leq \ell}\ \bigcap_{i\in K_p, j\in K_q} \tilde{E}_{\{i,j\},m}\Big),$$
$$\Gamma^n_{\mathbf{K},\varepsilon}(x) = g_m(x)\Big(\prod_{K\subset[\![1,N]\!]:|K|=k_0} \tilde{\Gamma}_{K,n}(x)\Big)\Big(\prod_{1\leq p\neq q\leq \ell}\ \prod_{i\in K_p, j\in K_q} \tilde{\Gamma}_{\{i,j\},m}(x)\Big),$$

where $g_m(x) = \varrho(m/\|x\|)$ with the extension $g_m(0) = 1$.

First, $G^n_{\mathbf{K},\varepsilon}$ is clearly included in $G^{n+1}_{\mathbf{K},\varepsilon}$ and relatively compact in $G_{\mathbf{K},0}$. We deduce from (2.40) that, setting $H_{\mathbf{K},m} = B(0,m) \cap (\cap_{1\leq p\neq q\leq \ell}\ \cap_{i\in K_p, j\in K_q} \tilde{E}_{\{i,j\},m})$,

$$\bigcup_{n\geq 1} G^n_{\mathbf{K},\varepsilon} = \Big(\bigcap_{K\subset[\![1,N]\!]:|K|=k_0} \tilde{E}_K\Big) \cap H_{\mathbf{K},m} = E_{k_0} \cap H_{\mathbf{K},m} = \mathcal{X} \cap H_{\mathbf{K},m}.$$

By (2.40) again, we can choose m large enough so that $H_{\mathbf{K},m}$ contains $G_{\mathbf{K},\varepsilon}$. Next, by (2.39), it holds that $\Gamma^n_{\mathbf{K},\varepsilon} \in C^\infty((\mathbb{R}^2)^N)$, that $\Gamma^n_{\mathbf{K},\varepsilon} = 1$ on $G^n_{\mathbf{K},\varepsilon}$ and that

$$\operatorname{Supp}\Gamma^n_{\mathbf{K},\varepsilon} \subset B(0,2m) \cap \Big(\bigcap_{K\subset[\![1,N]\!]:|K|=k_0} \tilde{E}_{K,2n}\Big) \cap \Big(\bigcap_{1\leq p\neq q\leq \ell}\ \bigcap_{i\in K_p, j\in K_q} \tilde{E}_{\{i,j\},2m}\Big),$$

which is compact in $G_{\mathbf{K},0}$. Moreover, $\operatorname{Supp}\Gamma^n_{\mathbf{K},\varepsilon} \subset H_{\mathbf{K},2m}$. Since there exists $\eta \in (0,1]$ such that $H_{\mathbf{K},2m} \subset G_{\mathbf{K},\eta}$, we conclude that $\operatorname{Supp}\Gamma^n_{\mathbf{K},\varepsilon} \subset G_{\mathbf{K},\eta}$.

It remains to show that $\sup_{\alpha\in(0,1]}\sup_{x\in(\mathbb{R}^2)^N} |\mathcal{L}^X_\alpha \Gamma^n_{\mathbf{K},\varepsilon}(x)| < \infty$. Introducing

$$\chi^n_{\mathbf{K},\varepsilon}(x) = \Big(\prod_{K\subset[\![1,N]\!]:|K|=k_0} \tilde{\Gamma}_{K,n}(x)\Big)\Big(\prod_{1\leq p\neq q\leq \ell}\ \prod_{i\in K_p, j\in K_q} \tilde{\Gamma}_{\{i,j\},m}(x)\Big),$$

which belongs to $C^\infty((\mathbb{R}^2)^N)$ by Step 1, we have $\Gamma^n_{\mathbf{K},\varepsilon} = g_m \chi^n_{\mathbf{K},\varepsilon}(x)$ (with the chosen value of m) and thus by (2.22)

$$\mathcal{L}^X_\alpha \Gamma^n_{\mathbf{K},\varepsilon}(x) = g_m(x)\mathcal{L}^X_\alpha \chi^n_{\mathbf{K},\varepsilon}(x) + \chi^n_{\mathbf{K},\varepsilon}\mathcal{L}^X_\alpha g_m(x) + \nabla g_m(x)\cdot\nabla\chi^n_{\mathbf{K},\varepsilon}(x).$$

The first term is uniformly bounded because g_m is bounded and supported in $B(0,2m)$ and because $\sup_{\alpha\in(0,1]}\sup_{x\in B(0,2m)} |\mathcal{L}^X\chi^n_{\mathbf{K},\varepsilon}(x)| < \infty$ by Step 1 and (2.22). The third term is also uniformly bounded, since $\chi^n_{\mathbf{K},\varepsilon} \in C^\infty((\mathbb{R}^2)^N)$ and since ∇g_m is bounded and supported in $B(0,2m)$. Finally, the middle term is bounded because $\chi^n_{\mathbf{K},\varepsilon}$ is bounded by 1 and because $\mathcal{L}^X_\alpha g_m$ is uniformly bounded, as we now show : Δg_m is obviously bounded since $g_m \in C^\infty_c((\mathbb{R}^2)^N)$ and, since $\nabla_{x^i} g_m(x) = -m\varrho'(m/||x||)x^i/\|x\|^3$,

$$\begin{aligned}\sum_{1\le i,j\le N} \frac{x^i-x^j}{\|x^i-x^j\|^2+\alpha}\cdot\nabla_{x^i} g_m(x) &= -\frac{m\varrho'(m/||x||)}{\|x\|^3}\sum_{1\le i,j\le N}\frac{x^i-x^j}{\|x^i-x^j\|^2+\alpha}\cdot x^i\\ &= -\frac{m\varrho'(m/||x||)}{2\|x\|^3}\sum_{1\le i,j\le N}\frac{\|x^i-x^j\|^2}{\|x^i-x^j\|^2+\alpha}.\end{aligned}$$

This last quantity is uniformly bounded, since ϱ' is bounded and vanishes on $[1,\infty)$.

Step 3. We now prove (ii), by showing that the restriction $\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon} = \Gamma^n_{\mathbf{K},\varepsilon}|_{\mathbb{S}}$ satisfies the required conditions. We obviously have $\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon} \in C^\infty_c(\mathbb{S}\cap G_{\mathbf{K},0})$ and $\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon} = 1$ on $\mathbb{S}\cap G^n_{\mathbf{K},\varepsilon}$. It remains to show that $\sup_{\alpha\in(0,1]}\sup_{u\in\mathbb{S}} |\mathcal{L}^U_\alpha\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon}| < \infty$, recall (2.23). Since $\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon} \in C^\infty(\mathbb{S})$, $\Delta_{\mathbb{S}}\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon}$ is bounded. We thus only have to verify that $\sup_{\alpha\in(0,1]}\sup_{u\in\mathbb{S}} |T_\alpha(u)| < \infty$, where

$$T_\alpha(u) = -\frac{\theta}{N}\sum_{1\le i,j\le N}\frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}\cdot(\nabla_{\mathbb{S}}\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon}(u))^i$$

Setting $b^i_\alpha(u) = -\frac{\theta}{N}\sum_{j=1}^N \frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}$ and using (2.14),

$$T_\alpha(u) = b_\alpha(u)\cdot\nabla_{\mathbb{S}}\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon}(u) = b_\alpha(u)\cdot\pi_H(\pi_{u^\perp}(\nabla\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon}(u))).$$

Since now $b(u)\in H$ and since π_H and $\pi_{u^\perp}$ are self-adjoint, as every orthogonal projection, we get

$$T_\alpha(u) = \pi_{u^\perp}(b_\alpha(u))\cdot\nabla\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon}(u) = b_\alpha(u)\cdot\nabla\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon}(u) - (b_\alpha(u)\cdot u)(u\cdot\nabla\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon}(u)).$$

But $b_\alpha(u)\cdot\nabla\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon}(u) = \mathcal{L}^X_\alpha\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon}(u) - \frac{1}{2}\Delta\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon}(u)$ is uniformly bounded by point (i) and since $\Delta\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon}(u)$ is bounded on $\mathbb{S}$. Next, $u\cdot\nabla\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon}(u)$ is smooth and thus bounded on $\mathbb{S}$. Finally,

$$b_\alpha(u)\cdot u = -\frac{\theta}{N}\sum_{1\le i,j\le N}\frac{(u^i-u^j)\cdot u^i}{\|u^i-u^j\|^2+\alpha} = -\frac{\theta}{2N}\sum_{1\le i,j\le N}\frac{\|u^i-u^j\|^2}{\|u^i-u^j\|^2+\alpha}$$

is also uniformly bounded. □

Remark 2.14. *We have proved in Step 2 that for each $m>0$, $g_m \in C^\infty_c((\mathbb{R}^2)^N)$ satisfies $g_m = 1$ on $B(0,m)$ and $\sup_{\alpha\in(0,1]}\sup_{x\in(\mathbb{R}^2)^N} |\mathcal{L}^X_\alpha g_m(x)| < \infty$.*

2.7 A Girsanov theorem for the Keller-Segel particle system.

In this section, we prove a rigorous version of the intuitive argument presented in Subsection 2.3.4.

For $x \in (\mathbb{R}^2)^N$, all $K \subset [\![1, N]\!]$, we denote by $x|_K = (x^i)_{i\in K}$. For $\mathbf{K} = (K_p)_{p\in[\![1,\ell]\!]}$ a partition of $[\![1, N]\!]$, for $y_1 \in (\mathbb{R}^2)^{|K_1|}, \dots, y_\ell \in (\mathbb{R}^2)^{|K_\ell|}$, we abusively denote by $(y_p)_{p\in[\![1,\ell]\!]}$ the element y of $(\mathbb{R}^2)^N$ such that for all $i \in [\![1, \ell]\!]$, $y|_{K_i} = y_i$.

We adopt the convention that for any $\theta > 0$, a $QKS(\theta, 1)$-process is a 2-dimensional Brownian motion. This is natural in view of (2.1).

Proposition 2.15. *Let $N \geq 2$, $\theta > 0$ such that $N > \theta$ and set $k_0 = \lceil 2N/\theta\rceil$. Fix some partition $\mathbf{K} = (K_p)_{p\in[\![1,\ell]\!]}$ of $[\![1, N]\!]$ with $\ell \geq 2$. Consider the state spaces $\mathcal{X} = E_{k_0}$ and, for each $p \in [\![1, \ell]\!]$,*

$$\mathcal{Y}_p = \Big\{y \in (\mathbb{R}^2)^{|K_p|} : \forall K \subset [\![1, |K_p|]\!] \text{ with } |K| \geq k_0,\ \sum_{i,j=1}^{|K_p|} ||y^i - y^j||^2 > 0\Big\}.$$

Consider

- $\mathbb{X} = (\Omega^X, \mathcal{M}^X, (X_t)_{t\geq 0}, (\mathbb{P}^X_x)_{x\in\mathcal{X}_\triangle})$ *a $QKS(\theta, N)$-process,*
- *For all $p \in [\![1, \ell]\!]$, $\mathbb{Y}^p = (\Omega^p, \mathcal{M}^p, (Y_{p,t})_{t\geq 0}, (\mathbb{P}^p_y)_{y\in\mathcal{Y}^p_\triangle})$ a $QKS(\theta|K_p|/N, |K_p|)$-process.*

We set $\Omega^Y = \prod_{p=1}^{\ell} \Omega^p$ and $Y_t = (Y_{p,t})_{p\in[\![1,\ell]\!]}$, with the convention that $Y_t = \triangle$ as soon as $Y_{p,t} = \triangle$ for some $p \in [\![1, \ell]\!]$. We also introduce $\mathcal{M}^Y = \sigma(Y_t : t \geq 0)$, as well as $\mathbb{P}^Y_y = \otimes_{p=1}^{\ell}\mathbb{P}^p_{y_p}$ for all $y = (y_p)_{p\in[\![1,\ell]\!]} \in (\mathbb{R}^2)^N$.

We fix $\varepsilon \in (0, 1]$, recall that

$$G_{\mathbf{K},\varepsilon} = \Big\{x \in \mathcal{X} : \min_{1\leq p\neq q\leq \ell}\ \min_{i\in K_p, j\in K_q} \|x^i - x^j\|^2 > \varepsilon\Big\} \cap B\Big(0, \frac{1}{\varepsilon}\Big),$$

and set

$$\tau_{\mathbf{K},\varepsilon} = \big\{t \geq 0 : X_t \notin G_{\mathbf{K},\varepsilon}\big\} \quad \text{and} \quad \tilde{\tau}_{\mathbf{K},\varepsilon} = \big\{t \geq 0 : Y_t \notin G_{\mathbf{K},\varepsilon}\big\}.$$

Fix $T > 0$. Quasi-everywhere in $G_{\mathbf{K},\varepsilon}$, there is a probability measure $\mathbb{Q}^{T,\varepsilon,\mathbf{K}}_x$ on $(\Omega^X, \mathcal{M}^X)$, equivalent to $\mathbb{P}^X_x$, such that the law of the process $(X_{t\wedge T\wedge\tau_{\mathbf{K},\varepsilon}})_{t\geq 0}$ under $\mathbb{Q}^{T,\varepsilon,\mathbf{K}}_x$ is the same as that of $(Y_{t\wedge T\wedge\tilde{\tau}_{\mathbf{K},\varepsilon}})_{t\geq 0}$ on $(\Omega^Y, \mathcal{M}^Y)$ under $\mathbb{P}^Y_x$.

Furthermore, the Radon-Nikodym density $\frac{\mathrm{d}\mathbb{Q}^{T,\varepsilon,\mathbf{K}}_x}{\mathrm{d}\mathbb{P}^X_x}$ is $\mathcal{M}^X_{T\wedge\tau_{\mathbf{K},\varepsilon}}$-measurable, where as usual $\mathcal{M}^X_t = \sigma(X_s, s \leq t)$, and there is a deterministic constant $C_{T,\varepsilon,\mathbf{K}} > 0$ such that quasi-everywhere in $G_{\mathbf{K},\varepsilon}$,

$$C^{-1}_{T,\varepsilon,\mathbf{K}} \leq \frac{d\mathbb{Q}^{T,\varepsilon,\mathbf{K}}_x}{d\mathbb{P}^X_x} \leq C_{T,\varepsilon,\mathbf{K}}.$$

The quasi-everywhere notion refers to the process $\mathbb{X}$. Let us mention that for ζ the life-time of $\mathbb{X}$, we have $\tau_{\mathbf{K},\varepsilon} \in [0, \zeta]$ when $\zeta < \infty$ because $\triangle \notin G_{\mathbf{K},\varepsilon}$. Although this is not clear at this point of the paper, the event $\{\tau_{\mathbf{K},\varepsilon} = \zeta\}$ has a positive probability if $\max_{p=1,\dots,\ell} |K_p| \geq k_0$.

Démonstration. We only consider the case where $\ell = 2$. The general case is heavier in terms of notation but contains no additional difficulty. We fix $\mathbf{K} = (K_1, K_2)$ a non-trivial partition of $[\![1, N]\!]$. The main idea is to apply Lemma 2.39 to $\mathbb{X}$ with the function

$$\varrho(x) = \exp(u(x)), \quad \text{where} \quad u(x) = \frac{\theta}{N} \sum_{i \in K_1, j \in K_2} \log(\|x^i - x^j\|). \tag{2.41}$$

Unfortunately, this is not licit because $u \notin \mathcal{F}^X$.

Step 1. Set $\mathbb{Y} = (\Omega^Y, \mathcal{M}^Y, (Y_t)_{t \geq 0}, (\mathbb{P}_y^Y)_{y \in (\mathcal{Y}_1 \times \mathcal{Y}_2) \cup \{\triangle\}})$ and fix $\varepsilon \in (0, 1]$ and $n \geq 1$. We first compute the Dirichlet space of $\mathbb{Y}$ killed when it gets outside of $G^n_{\mathbf{K},\varepsilon}$, recall Lemma 2.12. Consider the measures

$$\mu_1(\mathrm{d}y) = \prod_{i,j \in K_1, i \neq j} ||y^i - y^j||^{-\theta/N} \mathrm{d}y \quad \text{and} \quad \mu_2(\mathrm{d}y) = \prod_{i,j \in K_2, i \neq j} ||y^i - y^j||^{-\theta/N} \mathrm{d}y$$

on $(\mathbb{R}^2)^{|K_1|}$ and $(\mathbb{R}^2)^{|K_2|}$, with $\mu_i(\mathrm{d}y) = \mathrm{d}y$ if $|K_i| = 1$. Recall that $\mu(\mathrm{d}x) = \mathbf{m}(x)\mathrm{d}x$, see (2.4) and that by definition, see (2.41), $\varrho(x) = \prod_{i \in K_1, j \in K_2} \|x^i - x^j\|^{\theta/N}$: we deduce that

$$\mu_1 \otimes \mu_2 = \varrho^2 \mu.$$

By Proposition 2.6, for $p = 1, 2$, $\mathbb{Y}^p$ is a $\mathcal{Y}^p_\triangle$-valued μ_p-symmetric (since $(\theta|K_p|/N)/|K_p| = \theta/N$) diffusion with regular Dirichlet space $(\mathcal{E}_p, \mathcal{F}_p)$ with core $C_c^\infty(\mathcal{Y}_p)$ and, for $\varphi \in C_c^\infty(\mathcal{Y}_p)$, $\mathcal{E}_p(\varphi, \varphi) = \frac{1}{2}\int_{(\mathbb{R}^2)^{|K_p|}} ||\nabla\varphi||^2 \mathrm{d}\mu_p$. This also holds true if e.g. $|K_1| = 1$, see [24, Example 4.2.1 page 167], since then μ_1 is nothing but the Lebesgue measure on $\mathbb{R}^2$. Since now $\mu_1 \otimes \mu_2 = \varrho^2\mu$, by Lemma 2.37, $\mathbb{Y}$ is a $\varrho^2\mu$-symmetric $\mathcal{X}_\triangle$-valued diffusion with regular Dirichlet space $(\mathcal{E}^Y, \mathcal{F}^Y)$ on $L^2(\mathcal{Y}_1 \times \mathcal{Y}_2, \varrho^2\mathrm{d}\mu)$ with core $C_c^\infty(\mathcal{Y}_1 \times \mathcal{Y}_2)$ and, for $\varphi \in C_c^\infty(\mathcal{Y}_1 \times \mathcal{Y}_2)$,

$$\mathcal{E}^Y(\varphi, \varphi) = \int_{(\mathbb{R}^2)^{|K_1|}} \mathcal{E}_2(\varphi(y, \cdot), \varphi(y, \cdot))\mu_1(\mathrm{d}y) + \int_{(\mathbb{R}^2)^{|K_2|}} \mathcal{E}_1(\varphi(\cdot, z), \varphi(\cdot, z))\mu_2(\mathrm{d}z) = \frac{1}{2}\int_{(\mathbb{R}^2)^N} \|\nabla\varphi\|^2 \varrho^2 \mathrm{d}\mu.$$

Finally, we apply Lemma 2.38 to $\mathbb{Y}$ with the open set $G^n_{\mathbf{K},\varepsilon} \subset \mathcal{X} \subset \mathcal{Y}_1 \times \mathcal{Y}_2$, to find that the resulting killed process

$$\mathbb{Y}^{n,\varepsilon} = \Big(\Omega^Y, \mathcal{M}^Y, (Y_t^{n,\varepsilon})_{t \geq 0}, (\mathbb{P}^Y_y)_{y \in G^n_{\mathbf{K},\varepsilon} \cup \{\triangle\}}\Big)$$

is a $\varrho^2\mu|_{G^n_{\mathbf{K},\varepsilon}}$-symmetric $G^n_{\mathbf{K},\varepsilon} \cup \{\triangle\}$-valued diffusion with regular Dirichlet space $(\mathcal{E}^{Y,n,\varepsilon}, \mathcal{F}^{Y,n,\varepsilon})$ with core $C_c^\infty(G^n_{\mathbf{K},\varepsilon})$ such that for all $\varphi \in C_c^\infty(G^n_{\mathbf{K},\varepsilon})$,

$$\mathcal{E}^{Y,n,\varepsilon}(\varphi, \varphi) = \frac{1}{2}\int_{G^n_{\mathbf{K},\varepsilon}} ||\nabla\varphi||^2 \varrho^2 \mathrm{d}\mu.$$

Step 2. We now fix $\varepsilon \in (0, 1]$ and introduce, for each $n \geq 1$, $u_{n,\varepsilon}(x) = u(x)\Gamma^n_{\mathbf{K},\varepsilon}(x)$, recall (2.41) and Lemma 2.12, and $\varrho_{n,\varepsilon} = \exp(u_{n,\varepsilon})$. We check here that the functions $u_{n,\varepsilon}$ and $\varrho_{n,\varepsilon}$ satisfy the assumptions of Lemma 2.39 (to be applied to $\mathbb{X}$), that $\mathcal{A}^X[\varrho_{n,\varepsilon} - 1] = \mathcal{L}^X \varrho_{n,\varepsilon}$ and that

$$\sup_{n \geq 1} \sup_{x \in \mathcal{X}} |u_{n,\varepsilon}(x)| < \infty \qquad \text{and} \qquad \sup_{n \geq 1} \sup_{x \in G^n_{\mathbf{K},\varepsilon}} |\mathcal{L}^X \varrho_{n,\varepsilon}(x)| < \infty. \tag{2.42}$$

First, $u_{n,\varepsilon} \in \mathcal{F}^X$ because $u_{n,\varepsilon} \in C_c^\infty(\mathcal{X})$, and $|u_{n,\varepsilon}|$ is bounded, uniformly in $n \geq 1$, because $\Gamma^n_{\mathbf{K},\varepsilon}$ is bounded by 1 and vanishes outside $G_{\mathbf{K},\eta}$ (see Lemma 2.12), while u is smooth on $G_{\mathbf{K},\eta}$. To show that $\mathcal{A}^X[\varrho_{n,\varepsilon} - 1] = \mathcal{L}^X \varrho_{n,\varepsilon}$, it suffices by Remark 2.8 to verify that $\varrho_{n,\varepsilon} - 1 \in C_c^\infty(\mathcal{X})$, which is clear, and that $\sup_{\alpha\in(0,1]} \sup_{x\in(\mathbb{R}^2)^N} |\mathcal{L}^X_\alpha \varrho_{n,\varepsilon}(x)| < \infty$. We have

$$\mathcal{L}^X_\alpha \varrho_{n,\varepsilon}(x) = e^{u_{n,\varepsilon}(x)} \mathcal{L}^X_\alpha u_{n,\varepsilon}(x) + \frac{1}{2} e^{u_{n,\varepsilon}(x)} \|\nabla u_{n,\varepsilon}(x)\|^2.$$

Since $u_{n,\varepsilon} \in C_c^\infty((\mathbb{R}^2)^N)$, the only difficulty is to check that $\sup_{\alpha\in(0,1]} \sup_{x\in(\mathbb{R}^2)^N} |\mathcal{L}^X_\alpha u_{n,\varepsilon}(x)| < \infty$. By (2.22),

$$\mathcal{L}^X_\alpha u_{n,\varepsilon}(x) = \Gamma^n_{\mathbf{K},\varepsilon}(x) \mathcal{L}^X_\alpha u(x) + u(x) \mathcal{L}^X_\alpha \Gamma^n_{\mathbf{K},\varepsilon}(x) + \nabla \Gamma^n_{\mathbf{K},\varepsilon}(x) \cdot \nabla u(x).$$

Again, the only difficulty consists of the first term, because $\mathcal{L}^X_\alpha \Gamma^n_{\mathbf{K},\varepsilon}$ is uniformly bounded by Lemma 2.12 and vanishes outside $G_{\mathbf{K},\eta}$, while u is smooth on $G_{\mathbf{K},\eta}$. Since Supp $\Gamma^n_{\mathbf{K},\varepsilon} \subset G_{\mathbf{K},\eta}$, we are reduced to show that $\sup_{\alpha\in(0,1]} \sup_{x\in G_{\mathbf{K},\eta}} |\mathcal{L}^X_\alpha u(x)| < \infty$. But

$$\mathcal{L}^X_\alpha u = \frac{1}{2}\Delta u - \frac{\theta}{N} S_\alpha, \quad \text{where} \quad S_\alpha(x) = \sum_{1\leq i,j\leq N} \frac{x^i - x^j}{\|x^i - x^j\|^2 + \alpha} \cdot \nabla_{x^i} u(x),$$

and we only have to verify that $\sup_{\alpha\in(0,1]} \sup_{x\in G_{\mathbf{K},\eta}} |S_\alpha(x)| < \infty$.

For $k \in K_1$ and $\ell \in K_2$, we have

$$\nabla_{x^k} u(x) = \sum_{j\in K_2} \frac{\theta}{N} \frac{x^k - x^j}{\|x^k - x^j\|^2} \quad \text{and} \quad \nabla_{x^\ell} u(x) = \sum_{i\in K_1} \frac{\theta}{N} \frac{x^\ell - x^i}{\|x^\ell - x^i\|^2}.$$

Hence $S_\alpha = S_{1,\alpha} + S_{2,\alpha} + S_{3,\alpha} + S_{4,\alpha}$, where

$$S_{1,\alpha}(x) = \frac{\theta}{N} \sum_{i,j\in K_1} \frac{x^i - x^j}{\|x^i - x^j\|^2 + \alpha} \cdot \sum_{k\in K_2} \frac{x^i - x^k}{\|x^i - x^k\|^2},$$

$$S_{2,\alpha}(x) = \frac{\theta}{N} \sum_{i\in K_2, j\in K_1} \frac{x^i - x^j}{\|x^i - x^j\|^2 + \alpha} \cdot \sum_{k\in K_1} \frac{x^i - x^k}{\|x^i - x^k\|^2},$$

and $S_{3,\alpha}$ (resp. $S_{4,\alpha}$) is defined as $S_{1,\alpha}$ (resp. $S_{2,\alpha}$) exchanging the roles of K_1 and K_2. First, $S_{2,\alpha}$ (and $S_{4,\alpha}$) is obviously uniformly bounded on $G_{\mathbf{K},\eta}$. Next, by symmetry,

$$S_{1,\alpha}(x) = \frac{\theta}{2N} \sum_{i,j\in K_1} \frac{x^i - x^j}{\|x^i - x^j\|^2 + \alpha} \sum_{k\in K_2} \Big(\frac{x^i - x^k}{\|x^i - x^k\|^2} - \frac{x^j - x^k}{\|x^j - x^k\|^2} \Big).$$

Moreover, there is $C_\eta > 0$ such that for all $x \in G_{\mathbf{K},\eta}$, all $i, j \in K_1$ such that $i \neq j$, all $k \in K_2$,

$$\Big\| \frac{x^i - x^k}{\|x^i - x^k\|^2} - \frac{x^j - x^k}{\|x^j - x^k\|^2} \Big\| \leq C_\eta \|x^i - x^j\|,$$

so that $S_{1,\alpha}$ (and $S_{3,\alpha}$) is bounded on $G_{\mathbf{K},\eta}$, uniformly in $\alpha \in (0, 1]$, as desired.

Finally, the above computations, together with the facts that $\Gamma^n_{\mathbf{K},\varepsilon} = 1$ on $G^n_{\mathbf{K},\varepsilon}$, also show that for $x \in G^n_{\mathbf{K},\varepsilon}$,

$$\mathcal{L}^X \varrho_{n,\varepsilon}(x) = e^{u(x)} \Big(\frac{1}{2}\Delta u(x) - \frac{\theta}{N} S_\alpha(x) \Big) + \frac{1}{2} e^{u(x)} ||\nabla u(x)||^2,$$

which is bounded on $G_{\mathbf{K},\eta}$. Since $G^n_{\mathbf{K},\varepsilon}\subset G_{\mathbf{K},\eta}$, this implies that $\sup_{n\geq 1}\sup_{x\in G^n_{\mathbf{K},\varepsilon}}|\mathcal{L}^X\varrho_{n,\varepsilon}(x)|$ and completes the step.

Step 3. We apply Lemma 2.39 to the process $\mathbb{X}$ with $u_{n,\varepsilon}$ and $\varrho_{n,\varepsilon}$ defined in Step 2. Recalling that $\mathcal{A}^X[\varrho_{n,\varepsilon}-1]=\mathcal{L}^X\varrho_{n,\varepsilon}$ and using the conventions $\varrho_{n,\varepsilon}(\triangle)=1$ and $\mathcal{L}^X\varrho_{n,\varepsilon}(\triangle)=0$, we set

$$L^{n,\varepsilon}_t=\frac{\varrho_{n,\varepsilon}(X_t)}{\varrho_{n,\varepsilon}(X_0)}\exp\Big(-\int_0^t\frac{\mathcal{L}^X\varrho_{n,\varepsilon}(X_s)}{\varrho_{n,\varepsilon}(X_s)}\mathrm{d}s\Big). \tag{2.43}$$

Set $\mathcal{M}^X_t=\sigma(\{X_s,s\leq t\})$. By Lemma 2.39, there is a family of probability measures $(\mathbb{Q}^{n,\varepsilon}_x)_{x\in\mathcal{X}\cup\{\triangle\}}$ such that

$$\mathbb{Q}^{n,\varepsilon}_x=L^{n,\varepsilon}_t\cdot\mathbb{P}^X_x\quad\text{on}\quad\mathcal{M}^X_t$$

for all $t\geq 0$ and quasi-everywhere in $\mathcal{X}\cup\{\triangle\}$, and such that

$$\mathbb{X}^{n,\varepsilon}=\Big(\Omega^X,\mathcal{M}^X,(X_t)_{t\geq 0},(\mathbb{Q}^{n,\varepsilon}_x)_{x\in\mathcal{X}_\triangle}\Big)$$

is a $\varrho^2_{n,\varepsilon}\mu$-symmetric $\mathcal{X}\cup\{\triangle\}$-valued diffusion with regular Dirichlet space $(\mathcal{E}^{n,\varepsilon},\mathcal{F}^{n,\varepsilon})$ with core $C^\infty_c(\mathcal{X})$ such that for all $\varphi\in C^\infty_c(\mathcal{X})$,

$$\mathcal{E}^{n,\varepsilon}(\varphi,\varphi)=\frac{1}{2}\int_{(\mathbb{R}^2)^N}||\nabla\varphi||^2\varrho^2_{n,\varepsilon}\mathrm{d}\mu.$$

Next, we apply Lemma 2.38 to $\mathbb{X}^{n,\varepsilon}$ with the open set $G^n_{\mathbf{K},\varepsilon}$: the resulting killed process

$$\mathbb{X}^{*,n,\varepsilon}=\Big(\Omega^X,\mathcal{M}^X,(X^{*,n,\varepsilon}_t)_{t\geq 0},(\mathbb{Q}^{n,\varepsilon}_x)_{x\in G^n_{\mathbf{K},\varepsilon}\cup\{\triangle\}}\Big)$$

is a $\varrho^2_{n,\varepsilon}\mu|_{G^n_{\mathbf{K},\varepsilon}}$-symmetric $G^n_{\mathbf{K},\varepsilon}\cup\{\triangle\}$-valued diffusion with regular Dirichlet space $(\mathcal{E}^{*,n,\varepsilon},\mathcal{F}^{*,n,\varepsilon})$ with core $C^\infty_c(G^n_{\mathbf{K},\varepsilon})$ such that for all $\varphi\in C^\infty_c(G^n_{\mathbf{K},\varepsilon})$,

$$\mathcal{E}^{*,n,\varepsilon}(\varphi,\varphi)=\frac{1}{2}\int_{G^n_{\mathbf{K},\varepsilon}}||\nabla\varphi||^2\varrho^2_{n,\varepsilon}\mathrm{d}\mu.$$

Comparing this Dirichlet space with the one found in Step 1, using that $\varrho_{n,\varepsilon}=\varrho$ on $G^n_{\mathbf{K},\varepsilon}$ and a uniqueness argument, see [24, Theorem 4.2.8 p 167], we conclude that quasi-everywhere in $G^n_{\mathbf{K},\varepsilon}$, the law of $X^{*,n,\varepsilon}$ under $\mathbb{Q}^{n,\varepsilon}_x$ equals the law of $Y^{n,\varepsilon}$ under $\mathbb{P}^Y_x$.

Step 4. We fix $T>0$ and $\varepsilon\in(0,1]$ and complete the proof. Since $\mathbb{Q}^{n,\varepsilon}_x=L^{n,\varepsilon}_T\cdot\mathbb{P}^X_x$ on $\mathcal{M}^X_T$, we know from Step 3 that for all $n\geq 1$, quasi-everywhere in $G^n_{\mathbf{K},\varepsilon}$, for all continuous bounded $\Phi:C([0,T],\mathcal{X}_\triangle)\to\mathbb{R}$, (observe that $\bar{G}^n_{\mathbf{K},\varepsilon}\subset\mathcal{X}\subset\mathcal{X}_\triangle$)

$$\mathbb{E}^X_x[\Phi(X_{\cdot\wedge\tau_{\mathbf{K},n,\varepsilon}\wedge T})L^{n,\varepsilon}_T]=\mathbb{E}^Y_x[\Phi(Y_{\cdot\wedge\tilde{\tau}_{\mathbf{K},n,\varepsilon}\wedge T})],$$

where $\tau_{\mathbf{K},n,\varepsilon}=\inf\{t>0:X_t\notin G^n_{\mathbf{K},\varepsilon}\}\wedge\tau_{\mathbf{K},\varepsilon}$ and $\tilde{\tau}_{\mathbf{K},n,\varepsilon}=\inf\{t>0:Y_t\notin G^n_{\mathbf{K},\varepsilon}\}\wedge\tilde{\tau}_{\mathbf{K},\varepsilon}$. Since $(L^{n,\varepsilon}_t)_{t\geq 0}$ is a $\mathbb{P}^X_x$-martingale by Lemma 2.39, we deduce that quasi-everywhere in $G^n_{\mathbf{K},\varepsilon}$,

$$\mathbb{E}^X_x[\Phi(X_{\cdot\wedge\tau_{\mathbf{K},n,\varepsilon}\wedge T})L^{n,\varepsilon}_{\tau_{\mathbf{K},n,\varepsilon}\wedge T}]=\mathbb{E}^Y_x[\Phi(Y_{\cdot\wedge\tilde{\tau}_{\mathbf{K},n,\varepsilon}\wedge T})]. \tag{2.44}$$

Recall that $G_{\mathbf{K},\varepsilon}\subset\cup_{n\geq 1}G^n_{\mathbf{K},\varepsilon}$, see Lemma 2.12. Hence $\lim_n\tau_{\mathbf{K},n,\varepsilon}=\tau_{\mathbf{K},\varepsilon}$, $\lim_n\tilde{\tau}_{\mathbf{K},n,\varepsilon}=\tilde{\tau}_{\mathbf{K},\varepsilon}$, and for each $x\in G_{\mathbf{K},\varepsilon}$, there is $n_x\geq 1$ such that $x\in G^n_{\mathbf{K},\varepsilon}$ for all $n\geq n_x$. We deduce from (2.44) that quasi-everywhere in $G_{\mathbf{K},\varepsilon}$, the process $(L^{n,\varepsilon}_{\tau_{\mathbf{K},n,\varepsilon}\wedge T})_{n\geq n_x}$ is a $(\mathcal{M}^X_{\tau_{\mathbf{K},n,\varepsilon}\wedge T})_{n\geq n_x}$-martingale under $\mathbb{P}^X_x$. Moreover, recalling the expression (2.43) of $L^{n,\varepsilon}$, that $\varrho_{n,\varepsilon} = \exp(u_{n,\varepsilon})$ and the bound (2.42), we conclude that there is a constant $C_{T,\varepsilon,\mathbf{K}} > 0$ such that quasi-everywhere in $G_{\mathbf{K},\varepsilon}$,

$$\mathbb{P}^X_x\text{-a.s., for all } n \geq n_x, \quad C^{-1}_{T,\varepsilon,\mathbf{K}} \leq L^{n,\varepsilon}_{\tau_{\mathbf{K},n,\varepsilon}\wedge T} \leq C_{T,\varepsilon,\mathbf{K}}.$$

Hence the martingale $(L^{n,\varepsilon}_{\tau_{\mathbf{K},n,\varepsilon}\wedge T})_{n\geq n_x}$ is closed by some $\mathcal{M}_{\tau_{\mathbf{K},\varepsilon}\wedge T}$-measurable random variable $J_{T,\varepsilon,\mathbf{K}}$ that satisfies $C^{-1}_{T,\varepsilon,\mathbf{K}} \leq J_{T,\varepsilon,\mathbf{K}} \leq C_{T,\varepsilon,\mathbf{K}}$, and (2.44) implies that for all $n \geq n_x$,

$$\mathbb{E}^X_x[\Phi(X_{\cdot\wedge\tau_{\mathbf{K},n,\varepsilon}\wedge T})J_{T,\varepsilon,\mathbf{K}}] = \mathbb{E}^Y_x[\Phi(Y_{\cdot\wedge\tilde{\tau}_{\mathbf{K},n,\varepsilon}\wedge T})].$$

Letting $n \to \infty$, we find that quasi-everywhere in $G_{\mathbf{K},\varepsilon}$, for $\Phi \in C_b(C([0,T],\mathcal{X}_\triangle),\mathbb{R})$,

$$\mathbb{E}^X_x[\Phi(X_{\cdot\wedge\tau_{\mathbf{K},\varepsilon}\wedge T})J_{T,\varepsilon,\mathbf{K}}] = \mathbb{E}^Y_x[\Phi(Y_{\cdot\wedge\tilde{\tau}_{\mathbf{K},\varepsilon}\wedge T})].$$

Setting $\mathbb{Q}^{T,\varepsilon,\mathbf{K}}_x = J_{T,\varepsilon,\mathbf{K}} \cdot \mathbb{P}^X_x$ completes the proof. □

2.8 Explosion and continuity at explosion

In this section we consider a $QKS(\theta, N)$-process $\mathbb{X}$ with life-time ζ. We show that $\zeta = \infty$ when $\theta \in (0,2)$ and that $\zeta < \infty$ when $\theta \geq 2$. In the latter case, we also prove that $\lim_{t\to\zeta-} X_t$ a.s. exists, *for the usual topology of* $(\mathbb{R}^2)^N$: the Keller-Segel process is continuous at explosion. This is not clear at all at first sight : we know that $\lim_{t\to\zeta-} X_t = \triangle$ a.s. for the one-point compactification topology, which means that the process escapes from every compact of $\mathcal{X}$, but it could either go to infinity, which is not difficult to exclude, or it could tend to the boundary of $\mathcal{X}$ without converging, e.g. because it could alternate very fast between having its particles labeled in $[\![1, k_0]\!]$ very close and having its particles labeled in $[\![2, k_0 + 1]\!]$ very close. The goal of the section is to prove the following result.

Proposition 2.16. *Fix $\theta > 0$ and $N \geq 2$ such that $N > \theta$, set $k_0 = \lceil 2N/\theta \rceil$ and $\mathcal{X} = E_{k_0}$ and consider a $QKS(\theta, N)$-process $\mathbb{X} = (\Omega^X, \mathcal{M}^X, (X_t)_{t\geq 0}, (\mathbb{P}^X_x)_{x\in\mathcal{X}\cup\{\triangle\}})$ with life-time ζ.*

(i) If $\theta < 2$, then quasi-everywhere, $\mathbb{P}^X_x(\zeta = \infty) = 1$.

(ii) If $\theta \geq 2$, then quasi-everywhere, $\mathbb{P}^X_x$-a.s., $\zeta < \infty$ and $X_{\zeta-} = \lim_{t\to\zeta} X_t$ exists for the usual topology of $(\mathbb{R}^2)^N$ and does not belong to E_{k_0}.

We first show that the process does not explode in the subcritical case and cannot go to infinity at explosion in the supercritical case.

Lemma 2.17. *(i) If $\theta < 2$ and $N \geq 2$, then quasi-everywhere, $\mathbb{P}^X_x(\zeta = \infty) = 1$.*

(ii) If $\theta \geq 2$ and $N > \theta$, then quasi-everywhere,

$$\mathbb{P}^X_x\Big(\zeta < \infty \text{ and } \sup_{t\in[0,\zeta)} \|X_t\| < \infty\Big) = 1.$$

Démonstration. The arguments below only apply quasi-everywhere, since we use Proposition 2.10. In both cases, we have for all $i \in [\![1, N]\!]$ and all $t \in [0, \zeta)$,

$$||X_t||^2 \leq 2 \sum_{i=1}^{N} (\|X_t^i - S_{[\![1,N]\!]}(X_t)\|^2 + \|S_{[\![1,N]\!]}(X_t)\|^2) = 2R_{[\![1,N]\!]}(X_t) + 2N\|S_{[\![1,N]\!]}(X_t)\|^2.$$

By Lemma 2.11, there are a Brownian motion $(M_t)_{t\geq 0}$ and a squared Bessel process $(D_t)_{t\geq 0}$ with dimension $d_{\theta,N}(N)$ (killed when it gets out of $(0, \infty)$ if $d_{\theta,N}(N) \leq 0$), such that $S_{[\![1,N]\!]}(X_t) = M_t$ and $R_{[\![1,N]\!]}(X_t) = D_t$ for all $t \in [0, \zeta)$. These processes being locally bounded, we conclude that

$$\text{a.s., for all } T > 0, \qquad \sup_{t\in[0,\zeta\wedge T)} \|X_t\| < \infty. \tag{2.45}$$

(i) When $\theta < 2$ and $N \geq 2$, we have $k_0 = \lceil 2N/\theta \rceil > N$, so that $\mathcal{X} = (\mathbb{R}^2)^N$. Hence on the event $\{\zeta < \infty\}$, we necessarily have $\limsup_{t\to\zeta-} ||X_t|| = \infty$, and this is incompatible with (2.45) with $T = \zeta$.

(ii) When $N > \theta \geq 2$, we have $d_{\theta,N}(N) \leq 0$, so that $(D_t)_{t\geq 0}$ is killed at some finite time τ. It holds that $\zeta \leq \tau$. Indeed, on the event where $\tau < \zeta$, we have $R_{[\![1,N]\!]}(X_\tau) = \lim_{t\to\tau-} R_{[\![1,N]\!]}(X_t) = \lim_{t\to\tau-} D_t = 0$, so that $X_\tau \notin E_{k_0}$ (since $k_0 \leq N$), which is not possible since $\tau < \zeta$. Hence ζ is also a.s. finite and it holds that $\sup_{t\in[0,\zeta)} \|X_t\| < \infty$ a.s. by (2.45) with the choice $T = \zeta$. □

To show the continuity at explosion in the supercritical case, we need to prove the following delicate lemma.

Lemma 2.18. *Assume that $N > \theta \geq 2$. Quasi-everywhere, for all $K \subset [\![1, N]\!]$ with $|K| \geq 2$,*

$$\mathbb{P}_x^X\text{- a.s.,} \qquad \lim_{t\to\zeta-} R_K(X_t) = 0 \quad \text{or} \quad \liminf_{t\to\zeta-} R_K(X_t) > 0.$$

Démonstration. We proceed by reverse induction on the cardinal of K. If first $K = [\![1, N]\!]$, the result is clear because $(R_{[\![1,N]\!]}(X_t))_{t\in[0,\zeta)}$ is a (killed) squared Bessel process on $[0, \zeta)$ by Lemma 2.11 (and since $\zeta \leq \tau$ exactly as in the proof of Lemma 2.17-(ii)), hence it has a limit in $\mathbb{R}_+$ as $t \to \zeta$. Then, we assume that the property is proved if $|K| \geq n$ where $n \in [\![3, N]\!]$, we take $K \subset [\![1, N]\!]$ such that $|K| = n - 1$ and we show in several steps that a.s., either $\lim_{t\to\zeta-} R_K(X_t) = 0$ or $\liminf_{t\to\zeta-} R_K(X_t) > 0$.

Step 1. We fix $\varepsilon \in (0, 1]$ and introduce $\tilde{\sigma}_0^\varepsilon = 0$ and, for $k \geq 1$,

$$\sigma_k^\varepsilon = \inf\{t \in (\tilde{\sigma}_{k-1}^\varepsilon, \zeta) : R_K(X_t) \leq \varepsilon\} \quad \text{and} \quad \tilde{\sigma}_k^\varepsilon = \inf\{t \in (\sigma_k^\varepsilon, \zeta) : R_K(X_t) \geq 2\varepsilon\},$$

with the convention that $\inf \emptyset = \zeta$. We show in this step that for all deterministic $A > 0$, there exists a constant $p_{A,\varepsilon} > 0$ such that for all $k \geq 1$, quasi-everywhere, on $\{\sigma_k^\varepsilon < \zeta\}$,

$$\mathbb{P}_x^X \Big(\{\tilde{\sigma}_k^\varepsilon \geq (\sigma_k^\varepsilon + A) \wedge \zeta\} \cup B_{k,\varepsilon} \Big| \mathcal{M}_{\sigma_k^\varepsilon}^X \Big) \geq p_{A,\varepsilon},$$

where $\mathcal{M}_t^X = \sigma(X_s : s \in [0, t])$, and where, setting $a_\varepsilon = c_{|K|+1}\varepsilon / c_{|K|}$ (recall Lemma 2.13),

$$B_{k,\varepsilon} = \Big\{ \sup_{t\in[\sigma_k^\varepsilon, \tilde{\sigma}_k^\varepsilon)} ||X_t|| \geq 1/\varepsilon \ \text{ or } \ \inf_{t\in[\sigma_k^\varepsilon, \tilde{\sigma}_k^\varepsilon)} \min_{i\notin K} R_{K\cup\{i\}}(X_t) \leq a_\varepsilon \Big\}.$$

By the strong Markov property of $\mathbb{X}$, on $\{\sigma_k^\varepsilon < \zeta\}$,

$$\mathbb{P}_x^X\Big(\{\tilde\sigma_k^\varepsilon \geq (\sigma_k^\varepsilon + A)\wedge\zeta\}\cup B_{k,\varepsilon}\Big|\mathcal{M}^X_{\sigma_k^\varepsilon}\Big) = g(X_{\sigma_k^\varepsilon}),$$

where

$$g(y) = \mathbb{P}_y^X\Big(\{\tilde\sigma_1^\varepsilon \geq (\sigma_1^\varepsilon + A)\wedge\zeta\}\cup B_{1,\varepsilon}\Big) = \mathbb{P}_y^X\Big(\{\tilde\sigma_1^\varepsilon \geq A\wedge\zeta\}\cup C_{1,\varepsilon}\Big)$$

and

$$C_{1,\varepsilon} = \Big\{\sup_{t\in[0,\tilde\sigma_1^\varepsilon)}||X_t||\geq 1/\varepsilon \quad\text{or}\quad \inf_{t\in[0,\tilde\sigma_1^\varepsilon)}\min_{i\notin K} R_{K\cup\{i\}}(X_t)\leq a_\varepsilon\Big\}.$$

We used that $R_K(X_{\sigma_k^\varepsilon})\leq\varepsilon$ on $\{\sigma_k^\varepsilon<\zeta\}$ by definition of σ_k^ε, so that $\sigma_1^\varepsilon=0$ under $\mathbb{P}^X_{X_{\sigma_k^\varepsilon}}$. Using again that $R_K(X_{\sigma_k^\varepsilon})\leq\varepsilon$ on $\{\sigma_k^\varepsilon<\zeta\}$, it suffices to show that there is a constant $p_{A,\varepsilon}>0$ such that $g(y)\geq p_{A,\varepsilon}$ quasi-everywhere in $\{y\in\mathcal{X} : R_K(y)\leq\varepsilon\}$.

If first $||y||\geq 1/\varepsilon$ or $\min_{i\notin K} R_{K\cup\{i\}}(y)\leq a_\varepsilon$, then clearly, $g(y)=1$.

Otherwise, $y\in G_{\mathbf{K},\varepsilon}$, where

$$G_{\mathbf{K},\varepsilon} = \{x\in\mathcal{X} :\ \text{for all } i\in K,\ \text{all } j\notin K,\ \|x^i-x^j\|^2>\varepsilon\}\cap B(0,1/\varepsilon)$$

as in Proposition 2.15 with $\mathbf{K}=(K,K^c)$, because $||y||<1/\varepsilon$ and because $R_K(y)\leq\varepsilon<2\varepsilon$ and $\min_{i\notin K}R_{K\cup\{i\}}(y)>a_\varepsilon=c_{|K|+1}\varepsilon/c_{|K|}$ imply that $||x^i-x^k||^2>\varepsilon$ for all $i\in K$, $j\notin K$ by Lemma 2.13. For the very same reasons and by definition of $\tilde\sigma_1^\varepsilon$, it holds that

$$C_{1,\varepsilon}^c\subset\{\text{for all } t\in[0,\tilde\sigma_1^\varepsilon),\ \ X_t\in G_{\mathbf{K},\varepsilon}\}. \tag{2.46}$$

We now apply Proposition 2.15 with $T=A$ (and ε) and we find that quasi-everywhere in $G_{\mathbf{K},\varepsilon}$,

$$\begin{aligned} g(y)\geq & C_{A,\varepsilon,\mathbf{K}}^{-1}\mathbb{Q}_y^{A,\varepsilon,\mathbf{K}}(\{\tilde\sigma_1^\varepsilon\geq A\wedge\zeta\}\cup C_{1,\varepsilon})\\ = & C_{A,\varepsilon,\mathbf{K}}^{-1}\mathbb{Q}_y^{A,\varepsilon,\mathbf{K}}(\{\tilde\sigma_1^\varepsilon\geq A\wedge\zeta\}\cap C_{1,\varepsilon}^c) + C_{A,\varepsilon,\mathbf{K}}^{-1}\mathbb{Q}_y^{A,\varepsilon,\mathbf{K}}(C_{1,\varepsilon}). \end{aligned} \tag{2.47}$$

But we know from Proposition 2.15 and Lemma 2.11 that under $\mathbb{Q}_y^{A,\varepsilon,\mathbf{K}}$, $(R_K(X_t))_{t\in[0,\tau_{K,\varepsilon}\wedge A]}$ is a squared Bessel process with dimension $d_{\theta|K|/N,|K|}(|K|)=d_{\theta,N}(|K|)$, issued from $R_K(y)\leq\varepsilon$, stopped at time $\tau_{\mathbf{K},\varepsilon}\wedge A$, where $\tau_{\mathbf{K},\varepsilon}=\inf\{t>0 : X_t\notin G_{\mathbf{K},\varepsilon}\}$. Hence there exists, under $\mathbb{Q}_y^{A,\varepsilon,\mathbf{K}}$, a squared Bessel process $(S_t)_{t\geq0}$ with dimension $d_{\theta,N}(|K|)$ such that $S_t=R_K(X_t)$ for all $t\in[0,\tau_{\mathbf{K},\varepsilon}\wedge A]$. We introduce $\kappa_\varepsilon=\inf\{t>0 : S_t\geq 2\varepsilon\}$ and we observe that

$$\{\kappa_\varepsilon\geq A\wedge\zeta\}\cap C_{1,\varepsilon}^c = \{\tilde\sigma_1^\varepsilon\geq A\}\cap C_{1,\varepsilon}^c.$$

Indeed, we used that on $C_{1,\varepsilon}^c$, we have $\tau_{\mathbf{K},\varepsilon}\geq\tilde\sigma_1^\varepsilon$ by (2.46) so that $R_K(X_t)=S_t$ for all $t\in[0,\tilde\sigma_1^\varepsilon\wedge A)$, from which we conclude that $\kappa_\varepsilon\geq A\wedge\zeta$ if and only $\tilde\sigma_1^\varepsilon\geq A\wedge\zeta$. Coming back to (2.47), we get

$$g(y)\geq C_{A,\varepsilon,\mathbf{K}}^{-1}\mathbb{Q}_y^{A,\varepsilon,\mathbf{K}}(\{\kappa_\varepsilon\geq A\wedge\zeta\}\cap C_{1,\varepsilon}^c) + C_{A,\varepsilon,\mathbf{K}}^{-1}\mathbb{Q}_y^{A,\varepsilon,\mathbf{K}}(C_{1,\varepsilon}) = C_{A,\varepsilon,\mathbf{K}}^{-1}\mathbb{Q}_y^{A,\varepsilon,\mathbf{K}}(\kappa_\varepsilon\geq A\wedge\zeta).$$

The step is complete, since $\mathbb{Q}_y^{A,\varepsilon,\mathbf{K}}(\kappa_\varepsilon\geq A)$ is the probability that a squared Bessel process with dimension $d_{\theta,N}(|K|)$ issued from $R_K(y)\leq\varepsilon$ remains below 2ε during $[0,A]$ and is thus strictly positive, uniformly in y (such that $y\in G_{\mathbf{K},\varepsilon}$ and $R_K(y)\leq\varepsilon$).

Step 2. We prove here that for all $\varepsilon\in(0,1]$, all $A>0$, quasi-everywhere,

$$\mathbb{P}_x^X\Big(\limsup_{t\to\zeta-}||X_t||\geq 1/\varepsilon \quad\text{or}\quad \liminf_{t\to\zeta-}\min_{i\notin K}R_{K\cup\{i\}}(X_t)\leq a_\varepsilon \quad\text{or}\quad \exists\, k\geq1,\ \sigma_k^\varepsilon\geq\zeta\wedge A\Big)=1.$$

All the arguments below only hold quasi-everywhere, even if we do not mention it explicitly during this step. For $k \geq 1$, we introduce, with $B_{k,\varepsilon}$ defined in Step 1,

$$\Omega_{k+1} = \{\sigma^\varepsilon_{k+1} < \zeta \wedge A\} \cap B^c_{k,\varepsilon}$$

and we first show that $\mathbb{P}^X_x(\liminf_k \Omega_k) = 0$. To this end, it suffices to check that for all $\ell \geq 1$, $\mathbb{P}^X_x(\cap_{k=\ell}^\infty \Omega_k) = 0$. Since Ω_k is $\mathcal{M}_{\sigma^\varepsilon_k}$-measurable, for all $m \geq \ell \geq 1$,

$$\mathbb{P}^X_x(\cap_{k=\ell}^{m+1}\Omega_k) = \mathbb{E}^X_x[\mathbb{1}_{\cap_{k=\ell}^m \Omega_k}\mathbb{P}^X_x(\Omega_{m+1}|\mathcal{M}_{\sigma^\varepsilon_m})].$$

Since moreover $\cap_{k=\ell}^m \Omega_k \subset \{\sigma^\varepsilon_m < \zeta\}$ and since $\sigma^\varepsilon_{m+1} \geq \tilde\sigma^\varepsilon_m \geq \tilde\sigma^\varepsilon_m - \sigma^\varepsilon_m$, we deduce that on $\cap_{k=\ell}^m \Omega_k$,

$$\begin{aligned}\mathbb{P}^X_x(\Omega_{m+1}|\mathcal{M}_{\sigma^\varepsilon_m}) =& 1 - \mathbb{P}^X_x(\{\sigma^\varepsilon_{m+1} \geq \zeta \wedge A\} \cup B_{m,\varepsilon}|\mathcal{M}_{\sigma^\varepsilon_m})\\ \leq& 1 - \mathbb{P}^X_x(\{\tilde\sigma^\varepsilon_m \geq (\sigma^\varepsilon_m + A) \wedge \zeta\} \cup B_{m,\varepsilon}|\mathcal{M}_{\sigma^\varepsilon_m}),\end{aligned}$$

so that $\mathbb{P}^X_x(\Omega_{m+1}|\mathcal{M}_{\sigma^\varepsilon_m}) \leq 1 - p_{A,\varepsilon}$ by Step 1. Hence we conclude that

$$\mathbb{P}^X_x(\cap_{k=\ell}^{m+1}\Omega_k) \leq (1 - p_{A,\varepsilon})\mathbb{P}^X_x(\cap_{k=\ell}^m \Omega_k)$$

for all $m \geq \ell \geq 1$, so that $\mathbb{P}^X_x(\cap_{k=\ell}^\infty \Omega_k) = 0$ as desired.

Hence $\mathbb{P}^X_x(\liminf_k \Omega_k) = 0$, so that a.s., an infinite number of Ω^c_k are realized. Recalling that

$$\Omega^c_{k+1} = \Big\{\sigma^\varepsilon_{k+1} \geq \zeta \wedge A \quad \text{or} \quad \inf_{t\in[\sigma^\varepsilon_k, \tilde\sigma^\varepsilon_k)} \min_{i\notin K} R_{K\cup\{i\}}(X_t) \leq a_\varepsilon \quad \text{or} \quad \sup_{t\in[\sigma^\varepsilon_k,\tilde\sigma^\varepsilon_k)} ||X_t|| \geq 1/\varepsilon\Big\},$$

we find the following alternative :

- either there is $k \geq 1$ such that $\sigma^\varepsilon_k \geq \zeta \wedge A$;
- or for all $k \geq 1$, $\sigma^\varepsilon_k < \zeta$ and $\inf_{t\in[\sigma^\varepsilon_k,\tilde\sigma^\varepsilon_k)} \min_{i\notin K} R_{K\cup\{i\}}(X_t) \leq a_\varepsilon$ for infinitely many k's, which implies that $\liminf_{t\to\zeta-} \min_{i\notin K} R_{K\cup\{i\}}(X_t) \leq a_\varepsilon$ because necessarily, $\lim_\infty \sigma^\varepsilon_k = \zeta$ by definition of the sequence $(\sigma^\varepsilon_k)_{k\geq1}$ and by continuity of $t \to R_K(X_t)$ on $[0, \zeta)$;
- or for all $k \geq 1$, $\sigma^\varepsilon_k < \zeta$ and there are infinitely many k's for which $\sup_{t\in[\sigma^\varepsilon_k,\tilde\sigma^\varepsilon_k)} ||X_t|| \geq 1/\varepsilon$ and this implies that $\limsup_{t\to\zeta-} ||X_t|| \geq 1/\varepsilon$, because $\lim_\infty \sigma^\varepsilon_k = \zeta$ as previously.

Step 3. We conclude. Applying Step 2, we find that quasi-everywhere, $\mathbb{P}^X_x$-a.s., for all $A \in \mathbb{N}$ and all $\varepsilon \in \mathbb{Q} \cap (0, 1]$,

$$\limsup_{t\to\zeta-} ||X_t|| \geq 1/\varepsilon \quad \text{or} \quad \liminf_{t\to\zeta-} \min_{i\notin K} R_{K\cup\{i\}}(X_t) \leq a_\varepsilon \quad \text{or} \quad \exists\, k \geq 1,\ \sigma^\varepsilon_k \geq \zeta \wedge A.$$

By Lemma 2.17-(ii), we know that $\zeta < \infty$, so that choosing $A = \lceil\zeta\rceil$, we conclude that quasi-everywhere, $\mathbb{P}^X_x$-a.s., for all $\varepsilon \in \mathbb{Q} \cap (0, 1]$

$$\limsup_{t\to\zeta-} ||X_t|| \geq 1/\varepsilon \quad \text{or} \quad \liminf_{t\to\zeta-} \min_{i\notin K} R_{K\cup\{i\}}(X_t) \leq a_\varepsilon \quad \text{or} \quad \exists\, k \geq 1,\ \sigma^\varepsilon_k = \zeta. \tag{2.48}$$

And by Lemma 2.17-(ii) again, $\limsup_{t\to\zeta-} ||X_t|| \leq 1/\varepsilon_0$ for some (random) $\varepsilon_0 \in (0, 1]$.

On the event where $\liminf_{t\to\zeta-} \min_{i\notin K} R_{K\cup\{i\}}(X_t) = 0$, there exists some (random) $i_0 \notin K$ such that $\liminf_{t\to\zeta-} R_{K\cup\{i_0\}}(X_t) = 0$, whence $\lim_{t\to\zeta-} R_{K\cup\{i_0\}}(X_t) = 0$ by induction assumption, and this obviously implies that $\lim_{t\to\zeta-} R_K(X_t) = 0$.

On the complementary event, we fix $\varepsilon_1 \in (0, \varepsilon_0]$ such that $\liminf_{t\to\zeta-} \min_{i\notin K} R_{K\cup\{i\}}(X_t) > a_{\varepsilon_1}$ and we conclude from (2.48) and the fact that $\limsup_{t\to\zeta-} ||X_t|| \le 1/\varepsilon_0$ that for all $\varepsilon \in \mathbb{Q}\cap(0, \varepsilon_1]$, there exists $k_\varepsilon \ge 1$ such that $\sigma^\varepsilon_{k_\varepsilon} = \zeta$. Recalling the definition of $(\sigma^\varepsilon_k)_{k\ge1}$, we deduce that for all $\varepsilon \in \mathbb{Q}\cap(0, \varepsilon_1]$, $R_K(X_t)$ upcrosses the segment $[\varepsilon, 2\varepsilon]$ a finite number of times during $[0, \zeta)$. Hence for all $\varepsilon \in (0, \varepsilon_1] \cap \mathbb{Q}$, there exists $t_\varepsilon \in [0, \zeta)$ such that either $R_K(X_t) > \varepsilon$ for all $t \in [t_\varepsilon, \zeta)$ or $R_K(X_t) < 2\varepsilon$ for all $t \in [t_\varepsilon, \zeta)$. If there is $\varepsilon \in \mathbb{Q}\cap(0, \varepsilon_1]$ such that $R_K(X_t) > \varepsilon$ for all $t \in [t_\varepsilon, \zeta)$, then $\liminf_{t\to\zeta-} R_K(X_t) \ge \varepsilon > 0$. If next for all $\varepsilon \in \mathbb{Q}\cap(0, \varepsilon_1]$, we have $R_K(X_t) < 2\varepsilon$ for all $t \in [t_\varepsilon, \zeta)$, then $\lim_{t\to\zeta-} R_K(X_t) = 0$.

Hence in any case, we have either $\lim_{t\to\zeta-} R_K(X_t) = 0$ or $\liminf_{t\to\zeta-} R_K(X_t) > 0$. □

We finally give the

Proof of Proposition 2.16. Point (i), which concerns the subcritical case, has already been checked in Lemma 2.17-(i). Concerning point (ii), which concerns the supercritical case $\theta \ge 2$, we already know that quasi-everywhere, $\mathbb{P}^X_x(\zeta < \infty) = 1$ by Lemma 2.17-(ii), and it remains to prove that $\mathbb{P}^X_x$-a.s., $\lim_{t\to\zeta-} X_t$ exists and does not belong to E_{k_0}. We divide the proof in four steps.

Step 1. For $\mathbf{K} = (K_p)_{p\in[\![1,\ell]\!]}$ a partition of $[\![1, N]\!]$ and $\varepsilon \in (0, 1]$, we consider as in Proposition 2.15

$$G_{\mathbf{K},\varepsilon} = \Big\{x \in \mathcal{X} : \min_{1\le p\ne q\le\ell} \min_{i\in K_p, j\in K_q} \|x^i - x^j\|^2 > \varepsilon\Big\} \cap B\Big(0, \frac{1}{\varepsilon}\Big)$$

and $\tau_{\mathbf{K},\varepsilon} = \inf\{t \ge 0 : X_t \notin G_{\mathbf{K},\varepsilon}\} \in [0, \zeta]$. We show here for each $T > 0$, quasi-everywhere in $G_{\mathbf{K},\varepsilon}$, $\mathbb{P}^X_x$-a.s., for all $T > 0$, all $p \in [\![1, \ell]\!]$, $S_{K_p}(X_t)$ has a limit in $\mathbb{R}^2$ as $t \to (\tau_{\mathbf{K},\varepsilon} \wedge T)-$.

If $\ell = 1$, the result is obvious since $S_{[\![1,N]\!]}(X_t)$ is a Brownian motion during $[0, \zeta)$ by Lemma 2.11. If next $\ell \ge 2$, Proposition 2.15 and Lemma 2.11 tell us that under $\mathbb{Q}^{T,\varepsilon,\mathbf{K}}_x$, which is equivalent to $\mathbb{P}^X_x$, the processes $S_{K_p}(X_t)$ are some Brownian motions on $[0, \tau_{\mathbf{K},\varepsilon} \wedge T)$, and thus have some limits as $t \to (\tau_{\mathbf{K},\varepsilon} \wedge T)-$.

Step 2. For $\varepsilon \in (0, 1]$ and $\mathbf{K} = (K_p)_{p\in[\![1,\ell]\!]}$ a partition of $[\![1, N]\!]$, we set $\tilde\eta^{\mathbf{K},\varepsilon}_0 = 0$ and, for $k \ge 0$,

$$\eta^{\mathbf{K},\varepsilon}_{k+1} = \inf\{t \ge \tilde\eta^{\mathbf{K},\varepsilon}_k : X_t \in G_{\mathbf{K},2\varepsilon}\} \quad \text{and} \quad \tilde\eta^{\mathbf{K},\varepsilon}_{k+1} = \inf\{t \ge \eta^{\mathbf{K},\varepsilon}_{k+1} : X_t \notin G_{\mathbf{K},\varepsilon}\},$$

with the convention that $\inf\emptyset = \zeta$. Using Step 1 and the strong Markov property, we conclude that quasi-everywhere, $\mathbb{P}^X_x$-a.s., for all $\varepsilon \in (0, 1]\cap\mathbb{Q}$, all $k \ge 1$, all $T \in \mathbb{N}_+$, on $\{\eta^{\mathbf{K},\varepsilon}_k < \zeta\}$, for all $p \in [\![1, \ell]\!]$, $S_{K_p}(X_t)$ admits a limit in $\mathbb{R}^2$ as t goes to $(\tilde\eta^{\mathbf{K},\varepsilon}_k \wedge T)-$. Choosing $T = \lceil\zeta\rceil$, we conclude that quasi-everywhere, $\mathbb{P}^X_x$-a.s., on $\{\eta^{\mathbf{K},\varepsilon}_k < \zeta\}$, for all $\varepsilon \in (0, 1]\cap\mathbb{Q}$, all $k \ge 1$, all $p \in [\![1, \ell]\!]$,

$$S_{K_p}(X_t) \text{ admits a limit in } \mathbb{R}^2 \text{ as } t \text{ goes to } \tilde\eta^{\mathbf{K},\varepsilon}_k - .$$

Step 3. We now check that quasi-everywhere, $\mathbb{P}^X_x$-a.s., there is a partition $\mathbf{K} = (K_p)_{p\in[\![1,\ell]\!]}$ of $[\![1, N]\!]$, some $\varepsilon \in (0, 1]\cap\mathbb{Q}$ and some $k \ge 1$ such that (i) $\eta^{\mathbf{K},\varepsilon}_k < \zeta$ and $\tilde\eta^{\mathbf{K},\varepsilon}_k = \zeta$ and (ii) for all $p \in [\![1, \ell]\!]$, $\lim_{t\to\zeta-} R_{K_p}(X_t) = 0$.

By Lemma 2.18, we know that for all $K \subset [\![1, N]\!]$, we have the alternative $\lim_{t\to\zeta-} R_K(X_t) = 0$ or $\liminf_{t\to\zeta-} R_K(X_t) > 0$. Hence the partition $\mathbf{K} = (K_p)_{p\in[\![1,\ell]\!]}$ of $[\![1, N]\!]$ consisting of the classes of the equivalence relation defined by $i \sim j$ if and only if $\lim_{t\to\zeta} R_{\{i,j\}}(X_t) = 0$ satisfies that for all $p \in [\![1, \ell]\!]$, $\lim_{t\to\zeta-} R_{K_p}(X_t) = 0$ and $\liminf_{t\to\zeta-} \min_{i\notin K_p} R_{K_p\cup\{i\}}(X_t) > 0$.

Using moreover that $\limsup_{t\to\zeta-} ||X_t|| < \infty$ according to Lemma 2.17, we deduce that there is $\alpha \in (0,\zeta)$ and $\varepsilon \in (0,1] \cap \mathbb{Q}$ such that for all $t \in [\alpha,\zeta)$, X_t belongs to $G_{\mathbf{K},2\varepsilon}$. Finally, we consider $k = \max\{m \geq 1 : \eta_m^{\mathbf{K},\varepsilon} \leq \alpha\}$, which is finite by continuity of $t \mapsto X_t$ on $[0,\alpha]$, and it holds that $\eta_k^{\mathbf{K},\varepsilon} \leq \alpha < \zeta$ and that $\tilde{\eta}_k^{\mathbf{K},\varepsilon} = \zeta$.

Step 4. We consider the (random) partition $\mathbf{K} = (K_p)_{p\in[\![1,\ell]\!]}$ introduced in Step 3. By Step 2 and since $\eta_k^{\mathbf{K},\varepsilon} < \zeta$ and $\tilde{\eta}_k^{\mathbf{K},\varepsilon} = \zeta$, we know that quasi-everywhere, $\mathbb{P}_x^X$-a.s., for all $p \in [\![1,\ell]\!]$, $M_p = \lim_{t\to\zeta-} S_{K_p}(X_t)$ exists in $\mathbb{R}^2$. By Step 3, we know that for all $p \in [\![1,\ell]\!]$, $\lim_{t\to\zeta} R_{K_p}(X_t) = 0$. We easily conclude that quasi-everywhere, $\mathbb{P}_x^X$-a.s., for all $p \in [\![1,\ell]\!]$, all $i \in K_p$, $\lim_{t\to\zeta-} X_t^i = M_p$. This shows that quasi-everywhere, $\mathbb{P}_x^X$-a.s., $X_{\zeta-} = \lim_{t\to\zeta-} X_t$ exists in $(\mathbb{R}^2)^N$. Moreover, $X_{\zeta-}$ cannot belong to $\mathcal{X} = E_{k_0}$, because $\lim_{t\to\zeta-} X_t = \triangle$ when $E_{k_0} \cup \{\triangle\}$ is endowed with the one-point compactification topology, see Subsection 2.13.1. □

2.9 Some special cases

During a K-collision, the particles labeled in K are isolated from the other ones. Thanks to Proposition 2.15, it will thus be possible to describe what happens in a neighborhood of the instant of this K-collision, by studying a $QKS(\theta|K|/N,|K|)$-process. In other words, we may assume that $|K| = N$, so that the following special cases, which are the purpose of this section, will be crucial.

Proposition 2.19. *Let $N \geq 4$ and $\theta > 0$ such that $N > \theta$. Consider a $QKS(\theta,N)$-process $\mathbb{X}$ as in Proposition 2.6. Recall that $\zeta = \inf\{t \geq 0 : X_t = \triangle\}$ and set $\tau = \inf\{t \geq 0 : R_{[\![1,N]\!]}(X_t) \notin (0,\infty)\}$ with the convention that $R_K(\triangle) = 0$, so that $\tau \in [0,\zeta]$.*

(i) If $d_{\theta,N}(N-1) \leq 0$ and $d_{\theta,N}(N) < 2$, then quasi-everywhere,

$$\mathbb{P}_x^X\Big(\inf_{t\in[0,\zeta)} R_{[\![1,N]\!]}(X_t) > 0\Big) = 1.$$

(ii) If $d_{\theta,N}(N-1) \in (0,2)$ and $d_{\theta,N}(N) < 2$, then quasi-everywhere, $\mathbb{P}_x^X$-a.s, for all $K \subset [\![1,N]\!]$ with cardinal $|K| = N-1$, there is $t \in [0,\tau)$ such that $R_K(X_t) = 0$.

(iii) If $0 < d_{\theta,N}(N) < 2 \leq d_{\theta,N}(N-1)$, then quasi-everywhere, $\mathbb{P}_x^X$-a.s, for all $K \subset [\![1,N]\!]$ with cardinal $|K| = 2$, there is $t \in [0,\tau)$ such that $R_K(X_t) = 0$.

The proof of this proposition is very long. First, we recall some notation about the decomposition of $\mathbb{X}$ obtained in Proposition 2.10 and we study the involved time-change. We then derive a formula describing $R_K(U_t)$, valid on certain time intervals, for any $K \subset [\![1,N]\!]$. This formula is of course not closed, but it allows us to compare $R_K(U_t)$, when it is close to 0, to some process resembling a squared Bessel process, of which one easily studies the behavior near 0. Finally, we prove Proposition 2.19, unifying a little points (i) and (ii) and treating separately point (iii).

2.9.1 Notation and preliminaries

We recall the decomposition of Proposition 2.10, which holds true quasi-everywhere in $\mathcal{X} \cap E_N$. Consider a Brownian motion $(M_t)_{t\geq0}$ with diffusion coefficient $N^{-1/2}$ starting from $S_{[\![1,N]\!]}(x)$, a squared Bessel process $(D_t)_{t\geq0}$ starting from $R_{[\![1,N]\!]}(x) > 0$ killed when leaving $(0,\infty)$ with life-time $\tau_D = \inf\{t \geq 0 : D_t = \triangle\}$ and a $QSKS(\theta,N)$ -process $(U_t)_{t\geq0}$ starting from $\Phi_{\mathbb{S}}(x)$ with life-time $\xi = \inf\{t \geq 0 : U_t = \triangle\}$, all these processes being independent. For $t \in [0,\tau_D)$, we put $A_t = \int_0^t \frac{\mathrm{d}s}{D_s}$. We also consider the inverse $\rho : [0,A_{\tau_D}) \to [0,\tau_D)$ of A.

Lemma 2.20. *If $d_{\theta,N}(N) < 2$, then $\tau_D < \infty$ and $A_{\tau_D} = \infty$ a.s.*

Démonstration. Since $(D_t)_{t\geq 0}$ is a (killed) squared Bessel process with dimension $d_{\theta,N}(N) < 2$, we have $\tau_D < \infty$ a.s according to Revuz-Yor [44, Chapter XI]. Moreover, there is a Brownian motion $(B_t)_{t\geq 0}$ such that $D_t = r + 2\int_0^t \sqrt{D_s}\mathrm{d}B_s + d_{\theta,N}(N)t$ for all $t \in [0, \tau_D)$, where $r = R_{[\![1,N]\!]}(x) > 0$. A simple computation shows the existence of a Brownian motion $(W_t)_{t\geq 0}$ such that for all $t \in [0, A_{\tau_D})$,

$$D_{\rho_t} = r + 2\int_0^t D_{\rho_s}\mathrm{d}W_s + d_{\theta,N}(N)\int_0^t D_{\rho_s}\mathrm{d}s.$$

Hence for all $t \in [0, A_{\tau_D})$, $D_{\rho_t} = r\exp(2W_t + (d_{\theta,N}(N) - 2)t)$. On the event where $A_{\tau_D} < \infty$, we have $0 = D_{\tau_D-} = \lim_{t\to A_{\tau_D}} D_{\rho_t} = \exp(2W_{A_{\tau_D}} + (d_{\theta,N}(N) - 2)A_{\tau_D}) > 0$. Hence $A_{\tau_D} = \infty$ a.s. $\square$

From now on, we assume that $d_{\theta,N}(N) < 2$. Hence $A : [0, \tau_D) \to [0, \infty)$ is an increasing bijection, as well as $\rho : [0, \infty) \to [0, \tau_D)$. By Proposition 2.10, quasi-everywhere in $\mathcal{X} \cap E_N$, we can find a triple $(M_t, D_t, U_t)_{t\geq 0}$ as above such that for $\mathbb{X}$ our $QKS(\theta, N)$ process starting from x, for all $t \in [0, \tau_D \wedge \rho_\xi)$, and actually for all $t \in [0, \rho_\xi)$ because $\rho_\xi \leq \tau_D$ since ρ is $[0, \tau_D)$-valued,

$$X_t = \Psi(M_t, D_t, U_{A_t}), \quad \text{i.e.} \quad M_t = S_{[\![1,N]\!]}(X_t), \quad D_t = R_{[\![1,N]\!]}(X_t) \quad \text{and} \quad U_{A_t} = \Phi_{\mathbb{S}}(X_t).$$

We recall that $\Psi(m, r, u) = \gamma(m) + \sqrt{r}u$ if $(m, r, u) \in \mathbb{R}^2 \times (0, \infty) \times \mathcal{U}$ and $\Psi(m, r, u) = \triangle$ if $(m, r, u) = \triangle$. Observe that $\tau = \tau_D \wedge \rho_\xi = \rho_\xi$, where $\tau = \inf\{t \geq 0 : R_{[\![1,N]\!]}(X_t) \notin (0, \infty)\} \in [0, \zeta]$.

We note that if $\xi < \infty$, then $\rho_\xi < \tau_D$, because ρ is an increasing bijection from $[0, \infty)$ into $[0, \tau_D)$. Hence, still if $\xi < \infty$, then X explodes at time ρ_ξ strictly before τ_D, whence

$$\{\xi < \infty\} \subset \Big\{\inf_{t\in[0,\zeta)} R_{[\![1,N]\!]}(X_t) > 0\Big\}. \tag{2.49}$$

Finally note that since U is $\mathbb{S}$-valued, it cannot have a $[\![1, N]\!]$-collision. But for any $K \subset [\![1, N]\!]$ with cardinal $|K| \leq N - 1$, it holds that

$$U \text{ has a } K\text{-collision at } t \in [0, \xi) \text{ if and only if } X \text{ has a } K\text{-collision at } \rho_t \in [0, \tau), \tag{2.50}$$

which follows from the facts that

- for all $(m, r, u) \in \mathbb{R}^2 \times (0, \infty) \times \mathcal{U}$, $R_K(\Psi(m, r, u)) = 0$ if and only if $R_K(u) = 0$;
- ρ is an increasing bijection from $[0, \xi)$ into $[0, \tau)$, because $\rho_\xi = \tau$.

We conclude this subsection with a remark about the quasi-everywhere notions of $\mathbb{X}$ and $\mathbb{U}$, in the case where they are related as above. See Subsection 2.13.1 for a short reminder on this notion.

Remark 2.21. *Fix $B \in \mathcal{M}^U$ such that $\mathbb{P}^U_u(B) = 1$ quasi-everywhere (here quasi-everywhere refers to the Hunt process $\mathbb{U}$). Then $\mathbb{P}^U_{\Phi_{\mathbb{S}}(x)}(B) = 1$ quasi-everywhere (here quasi-everywhere refers to the Hunt process $\mathbb{X}^*$, which is $\mathbb{X}$ killed when it gets outside E_N).*

Démonstration. By definition, there exists $\mathcal{N}^U$ a properly exceptional set relative to $\mathbb{U}$ such that for all $u \in \mathcal{U} \setminus \mathcal{N}^U$, $\mathbb{P}^U_u(B) = 1$. Thus for all $x \in \Phi_{\mathbb{S}}^{-1}(\mathcal{U} \setminus \mathcal{N}^U)$, $\mathbb{P}^U_{\Phi_{\mathbb{S}}(x)}(B) = 1$.

By Proposition 2.10, there exists $\mathcal{N}^X$ a properly exceptional set relative to $\mathbb{X}^*$, such that for all $x \in (\mathcal{X} \cap E_N) \setminus \mathcal{N}^X$, the law of $(X_t)_{t\geq 0}$ under $\mathbb{P}^X_x$ is equal to the the law of $(Y_t = \Psi(M_t, D_t, U_{A_t}))_{t\geq 0}$ under $\mathbb{Q}^Y_x = \mathbb{P}^M_{\pi_{H^\perp}(x)} \otimes \mathbb{P}^D_{\|\pi_H(x)\|^2} \otimes \mathbb{P}^U_{\Phi_{\mathbb{S}}(x)}$, with some obvious notation.

Hence we only have to prove that $\mathcal{N} = \Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U) \cup \mathcal{N}^X$ is properly exceptional for $\mathbb{X}^*$.

• First, we have $\mathbb{P}_x^X(X_t^* \notin \mathcal{N}$ for all $t \geq 0) = 1$ for all $x \in \mathcal{X} \setminus \mathcal{N}$. Indeed, since $x \in \mathcal{X} \setminus \mathcal{N}$, the law of $(X_t^*)_{t\geq 0}$ under $\mathbb{P}_x^X$ equals the law of $(Y_t)_{t\geq 0}$ under $\mathbb{Q}_x^Y$. Since $\mathbb{P}_u^U(U_t \notin \mathcal{N}^U$ for all $t \geq 0) = 1$ for all $u \in \mathcal{U} \setminus \mathcal{N}^U$ and since $\Phi_{\mathbb{S}}(Y_t) = U_{A_t}$, we have $\mathbb{Q}_x^Y(Y_t \notin \Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U)$ for all $t \geq 0) = 1$ for all $x \in \mathcal{X} \setminus \Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U)$. Hence $\mathbb{P}_x^X(X_t^* \notin \Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U)$ for all $t \geq 0) = 1$ for all $x \in \mathcal{X} \setminus (\Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U) \cup \mathcal{N}^X)$. Finally, $\mathbb{P}_x^X(X_t^* \notin \Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U) \cup \mathcal{N}^X$ for all $t \geq 0) = 1$ for all $x \in \mathcal{X} \setminus (\Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U) \cup \mathcal{N}^X)$ because $\mathcal{N}^X$ is properly exceptional for $\mathbb{X}^*$.

• We have $\mu(\mathcal{N}) = 0$. Indeed, $\mu(\mathcal{N}^X) = 0$ by definition and, using Lemma 2.31,

$$\mu(\Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U)) = \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}_+^*\times\mathbb{S}} \mathbb{1}_{\{\Psi(z,r,u)\in\Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U)\}} r^\nu \mathrm{d}z\mathrm{d}r\beta(\mathrm{d}u) = \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}_+^*} \beta(\mathcal{N}^U) r^\nu \mathrm{d}z\mathrm{d}r = 0,$$

because $\beta(\mathcal{N}^U) = 0$. We used that $\Psi(z,r,u) \in \Phi_{\mathbb{S}}^{-1}(\mathcal{N}^U) \Leftrightarrow u \in \mathcal{N}^U$, since $\Phi_{\mathbb{S}}(\Psi(z,r,u)) = u$. □

2.9.2 An expression of dispersion processes on the sphere

We now study the dispersion process $(R_K(U_t))_{t\geq 0}$, for $K \subset [\![1,N]\!]$. The equation below can be informally established if assuming that (2.1) rigorously holds true, after a time-change and several Itô computations.

Lemma 2.22. *Fix $N \geq 2$ and $\theta > 0$ such that $N > \theta$ and recall that $k_0 = \lceil 2N/\theta \rceil$. Consider a QSKS$(\theta, N)$ -process $\mathbb{U}$ with life-time ξ, fix $K \subset [\![1,N]\!]$ such that $|K| \geq 2$, and set $\mathbf{K} = (K, K^c)$. Recall that $G_{\mathbf{K},\varepsilon}$ was introduced in Lemma 2.12, and observe that*

$$G_{\mathbf{K},0} \cap \mathbb{S} = \Big\{ u \in \mathcal{U} : \min_{i\in K, j\notin K} ||u^i - u^j|| > 0 \Big\}.$$

Quasi-everywhere in $G_{\mathbf{K},0} \cap \mathbb{S}$, enlarging the filtered probability space $(\Omega^U, \mathcal{M}^U, (\mathcal{M}_t^U)_{t\geq 0}, \mathbb{P}_u^U)$ if necessary, there exists a 1-dimensional $(\mathcal{M}_t^U)_{t\geq 0}$-Brownian motion $(W_t)_{t\geq 0}$ under $\mathbb{P}_u^U$ such that

$$\begin{aligned} R_K(U_t) =& R_K(u) + 2\int_0^t \sqrt{R_K(U_s)(1-R_K(U_s))}\mathrm{d}W_s + d_{\theta,N}(|K|)t \\ & - d_{\theta,N}(N)\int_0^t R_K(U_s)\mathrm{d}s - \frac{2\theta}{N}\sum_{i\in K, j\notin K}\int_0^t \frac{U_s^i - U_s^j}{||U_s^i - U_s^j||^2}\cdot(U_s^i - S_K(U_s))\mathrm{d}s \end{aligned} \tag{2.51}$$

for all $t \in [0, \kappa_K)$, where $\kappa_K = \inf\{t \geq 0 : U_t \notin G_{\mathbf{K},0}\}$.

As usual, $\kappa_K \leq \xi$ because $\triangle \notin G_{\mathbf{K},0}$. Note also that if $K = [\![1,N]\!]$, then $R_K(U_t) = 1$ for all $t \in [0,\xi)$, and that the constant process 1 indeed solves (2.51).

Démonstration. We divide the proof in several steps. The main idea is to compute $\mathcal{L}^U R_K$ and $\mathcal{L}^U(R_K)^2$ and to use that $R_K(U_t) = R_K(u) + \int_0^t \mathcal{L}^U R_K(U_s)\mathrm{d}s + M_t$, for some martingale $(M_t)_{t\geq 0}$ of which we can compute the bracket. However, we need to regularize R_K and to localize space in a zone where the last term of (2.51) is bounded.

Step 1. We fix $n \geq 1$ and $\varepsilon \in (0,1]$ and recall $\Gamma_{\mathbf{K},\varepsilon}^{\mathbb{S},n} \in C^\infty(\mathbb{S})$, compactly supported in $G_{\mathbf{K},0} \cap \mathbb{S}$, was defined in Lemma 2.12. We want to apply Remark 2.8 to $R_K\Gamma_{\mathbf{K},\varepsilon}^{\mathbb{S},n}$ and $(R_K\Gamma_{\mathbf{K},\varepsilon}^{\mathbb{S},n})^2$. We thus have to show that $R_K\Gamma_{\mathbf{K},\varepsilon}^{\mathbb{S},n}$ and $(R_K\Gamma_{\mathbf{K},\varepsilon}^{\mathbb{S},n})^2$ belong to $C_c^\infty(\mathcal{U})$ for all $n \geq 1$, which is clear, and that

$$\sup_{\alpha\in(0,1]}\sup_{u\in\mathbb{S}} \Big(|\mathcal{L}_\alpha^U[R_K\Gamma_{\mathbf{K},\varepsilon}^{\mathbb{S},n}](u)| + |\mathcal{L}_\alpha^U[(R_K\Gamma_{\mathbf{K},\varepsilon}^{\mathbb{S},n})^2](u)| \Big) < \infty$$

for all $n \geq 1$. Since

$$\mathcal{L}_\alpha^U(fg) = f\mathcal{L}_\alpha^U g + g\mathcal{L}_\alpha^U f + \nabla_{\mathbb{S}} f \cdot \nabla_{\mathbb{S}} g \tag{2.52}$$

for all $f, g \in C^\infty(\mathbb{S})$ and recalling that $\sup_{\alpha\in(0,1]} \sup_{u\in\mathbb{S}} |\mathcal{L}_\alpha^U \Gamma_{\mathbf{K},\varepsilon}^{\mathbb{S},n}(u)| < \infty$ by Lemma 2.12 and that $\Gamma_{\mathbf{K},\varepsilon}^{\mathbb{S},n}$ is compactly supported in $G_{\mathbf{K},0} \cap \mathbb{S}$, the only issue is to verify that, for A compact in $G_{\mathbf{K},0} \cap \mathbb{S}$,

$$\sup_{\alpha\in(0,1]} \sup_{u\in A} |\mathcal{L}_\alpha^U R_K(u)| < \infty. \tag{2.53}$$

Step 2. Here we prove that

$$\begin{aligned}\mathcal{L}_\alpha^U R_K(u) =& 2(|K|-1) - 2(N-1)R_K(u) + \frac{\theta}{N} R_K(u) \sum_{1\leq i,j\leq N} \frac{\|u^i - u^j\|^2}{\|u^i-u^j\|^2+\alpha} \\ &- \frac{\theta}{N} \sum_{i\in K, j\in K} \frac{\|u^i - u^j\|^2}{\|u^i-u^j\|^2+\alpha} - \frac{2\theta}{N} \sum_{i\in K, j\notin K} \frac{u^i - u^j}{\|u^i-u^j\|^2+\alpha} \cdot (u^i - S_K(u)),\end{aligned} \tag{2.54}$$

and this will imply (2.53) : the first four terms are obviously uniformly bounded on $\mathbb{S}$, and the last one is uniformly bounded on A (because A is compact in $G_{\mathbf{K},0} \cap \mathbb{S}$).

This will also imply, taking $\alpha = 0$ and observing that $2(|K|-1) - \frac{\theta}{N}|K|(|K|-1) = d_{\theta,N}(|K|)$ and $2(N-1) - \frac{\theta}{N}N(N-1) = d_{\theta,N}(N)$, that for all $u \in \mathbb{S} \cap E_2$,

$$\mathcal{L}^U R_K(u) = d_{\theta,N}(|K|) - d_{\theta,N}(N) R_K(u) - \frac{2\theta}{N} \sum_{i\in K, j\notin K} \frac{u^i - u^j}{\|u^i - u^j\|^2} \cdot (u^i - S_K(u)). \tag{2.55}$$

Step 2.1. We first verify that for all $u \in \mathbb{S}$,

$$(\nabla_{\mathbb{S}} R_K(u))^i = 2(u^i - S_K(u))\mathbb{1}_{\{i\in K\}} - 2R_K(u)u^i, \qquad i \in [\![1, N]\!], \tag{2.56}$$

$$\Delta_{\mathbb{S}} R_K(u) = 4(|K|-1) - 4(N-1)R_K(u). \tag{2.57}$$

First, a simple computation shows that for $x \in (\mathbb{R}^2)^N$, for $i \in [\![1, N]\!]$,

$$\nabla_{x^i} R_K(x) = 2(x^i - S_K(x))\mathbb{1}_{\{i\in K\}} \quad \text{and} \quad \Delta_{x^i} R_K(x) = \frac{4(|K|-1)}{|K|} \mathbb{1}_{\{i\in K\}}, \tag{2.58}$$

so that in particular $\nabla R_K(x) \in H$ and

$$\nabla R_K(x) \cdot x = 2\sum_{i\in K} (x^i - S_K(x)) \cdot x^i = 2\sum_{i\in K} (x^i - S_K(x)) \cdot (x^i - S_K(x)) = 2R_K(x). \tag{2.59}$$

Next, proceeding as in (2.14), we get $\nabla[R_K \circ \Phi_{\mathbb{S}}](x) = ||\pi_H(x)||^{-1} \pi_H(\pi_{(\pi_H(x))^\perp}(\nabla R_K(\Phi_{\mathbb{S}}(x))))$ for all $x \in E_N$, so that

$$\nabla[R_K \circ \Phi_{\mathbb{S}}](x) = \frac{\pi_H\Big(\nabla R_K(\Phi_{\mathbb{S}}(x)) - \frac{\pi_H(x)\cdot\nabla R_K(\Phi_{\mathbb{S}}(x))}{||\pi_H(x)||^2}\pi_H(x)\Big)}{||\pi_H(x)||} = \frac{\nabla R_K(x) - 2R_K(x)\frac{\pi_H(x)}{||\pi_H(x)||^2}}{||\pi_H(x)||^2}.$$

We used that $\nabla R_K(\Phi_{\mathbb{S}}(x)) = \nabla R_K(x)/\|\pi_H(x)\|$ thanks to (2.58), that $\nabla R_K(x) \in H$ by (2.58) and that $\pi_H(x) \cdot \nabla R_K(x) = x \cdot \nabla R_K(x) = 2R_K(x)$ by (2.59).

We first conclude that for $u \in \mathbb{S}$, since $\pi_H(u) = u$ and $||u|| = 1$,

$$\nabla_{\mathbb{S}} R_K(u) = \nabla[R_K \circ \Phi_{\mathbb{S}}](u) = \nabla R_K(u) - 2R_K(u)u, \tag{2.60}$$

which implies (2.56) by (2.58).

Second, we deduce that for $x \in E_N$,

$$\begin{aligned}\Delta[R_K \circ \Phi_{\mathbb{S}}](x) =& \frac{1}{||\pi_H(x)||^2}\Big(\Delta R_K(x) - 2\nabla R_K(x) \cdot \frac{\pi_H(x)}{||\pi_H(x)||^2} - 2R_K(x)\frac{\mathrm{div}\pi_H(x)}{||\pi_H(x)||^2} + \frac{4R_K(x)}{||\pi_H(x)||^2}\Big) \\ &- \frac{2\pi_H(x)}{||\pi_H(x)||^4} \cdot \Big(\nabla R_K(x) - 2R_K(x)\frac{\pi_H(x)}{||\pi_H(x)||^2}\Big).\end{aligned}$$

Using that $\mathrm{div}\,\pi_H(x) = 2(N-1)$, we conclude that for $u \in \mathbb{S}$, since $\pi_H(u) = u$, $||u|| = 1$ and $u \cdot \nabla R_K(u) = 2R_K(u)$ by (2.59),

$$\Delta_{\mathbb{S}} R_K(u) = \Delta[R_K \circ \Phi_{\mathbb{S}}](u) = \Delta R_K(u) - 4R_K(u) - 4(N-1)R_K(u) + 4R_K(u).$$

Since finally $\Delta R_K(u) = 4(|K| - 1)$ by (2.58), this leads to (2.57).

Step 2.2. We fix $u \in \mathbb{S}$ and show that setting $I_\alpha(u) = -\frac{\theta}{N}\sum_{1\le i,j\le N} \frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha} \cdot (\nabla_{\mathbb{S}} R_K(u))^i$, it holds that

$$\begin{aligned}I_\alpha(u) =& -\frac{\theta}{N}\sum_{i\in K, j\in K} \frac{\|u^i-u^j\|^2}{\|u^i-u^j\|^2+\alpha} + \frac{\theta}{N}R_K(u)\sum_{1\le i,j\le N}\frac{\|u^i-u^j\|^2}{\|u^i-u^j\|^2+\alpha} \\ &- \frac{2\theta}{N}\sum_{i\in K, j\notin K}\frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}\cdot(u^i - S_K(u)).\end{aligned} \tag{2.61}$$

By (2.56), we may write $I_\alpha = I_{1,\alpha} + I_{2,\alpha}$, where

$$I_{1,\alpha}(u) = -\frac{2\theta}{N}\sum_{i\in K, j\in [\![1,N]\!]}\frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}\cdot(u^i - S_K(u)),$$

$$I_{2,\alpha}(u) = \frac{2\theta}{N}R_K(u)\sum_{1\le i,j\le N}\frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}\cdot u^i.$$

First, by symmetry,

$$\begin{aligned}I_{1,\alpha}(u) =& -\frac{2\theta}{N}\sum_{i\in K,j\in K}\frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}\cdot(u^i-S_K(u)) - \frac{2\theta}{N}\sum_{i\in K,j\notin K}\frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}\cdot(u^i-S_K(u)) \\ =& -\frac{2\theta}{N}\sum_{i\in K,j\in K}\frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}\cdot u^i - \frac{2\theta}{N}\sum_{i\in K,j\notin K}\frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}\cdot(u^i-S_K(u)) \\ =& -\frac{\theta}{N}\sum_{i\in K,j\in K}\frac{\|u^i-u^j\|^2}{\|u^i-u^j\|^2+\alpha} - \frac{2\theta}{N}\sum_{i\in K,j\notin K}\frac{u^i-u^j}{\|u^i-u^j\|^2+\alpha}\cdot(u^i-S_K(u)).\end{aligned}$$

Second, by symmetry,

$$I_{2,\alpha}(u) = \frac{\theta}{N}R_K(u)\sum_{1\le i,j\le N}\frac{\|u^i-u^j\|^2}{\|u^i-u^j\|^2+\alpha}.$$

Step 2.3. Since $\mathcal{L}^U_\alpha R_K(u) = \frac{1}{2}\Delta_{\mathbb{S}} R_K(u) + I_\alpha(u)$, (2.54) follows from (2.57) and (2.61).

Step 3. By Steps 1 and 2, we can apply Remark 2.8 and Lemma 2.34 : quasi-everywhere, for all $n \geq 1$, there exist two $(\mathcal{M}^U_t)_{t\geq 0}$-martingales $(M^{1,n,\varepsilon}_t)_{t\geq 0}$ and $(M^{2,n,\varepsilon}_t)_{t\geq 0}$ under $\mathbb{P}^U_u$, such that

$$(R_K \Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon})(U_t) = (R_K \Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon})(u) + M^{1,n,\varepsilon}_t + \int_0^t \mathcal{L}^U (R_K \Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon})(U_s)\mathrm{d}s,$$

$$(R_K \Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon})^2(U_t) = (R_K \Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon})^2(u) + M^{2,n,\varepsilon}_t + \int_0^t \mathcal{L}^U (R_K \Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon})^2(U_s)\mathrm{d}s$$

for all $t \geq 0$. We recall that $\kappa_K = \inf\{t \geq 0 : U_t \notin G^n_{\mathbf{K},0}\}$ and introduce

$$\kappa_{K,n,\varepsilon} = \inf\{t \geq 0 : U_t \notin G^n_{\mathbf{K},\varepsilon}\} \wedge \kappa_K.$$

Since $\cup_{n\geq 1} G^n_{\mathbf{K},\varepsilon} \supset G_{\mathbf{K},\varepsilon}$ and since $G_{\mathbf{K},\varepsilon}$ increases to $G_{\mathbf{K},0}$ as $\varepsilon \to 0$, see Lemma 2.12, we conclude that $\lim_{\varepsilon\to 0}\lim_{n\to\infty} \kappa_{K,n,\varepsilon} = \kappa_K$. Next, since $\Gamma^{\mathbb{S},n}_{\mathbf{K},\varepsilon} = 1$ on $G^n_{\mathbf{K},\varepsilon} \cap \mathbb{S}$, we have, for all $t \in [0, \kappa_{K,n,\varepsilon}]$,

$$R_K(U_t) = R_K(u) + M^{1,n,\varepsilon}_t + \int_0^t \mathcal{L}^U R_K(U_s)\mathrm{d}s, \tag{2.62}$$

$$(R_K(U_t))^2 = (R_K(u))^2 + M^{2,n,\varepsilon}_t + \int_0^t \mathcal{L}^U (R_K^2)(U_s)\mathrm{d}s. \tag{2.63}$$

Applying the Itô formula to compute $(R_K(U_t))^2$ from (2.62), recalling from (2.52) that $\mathcal{L}^U(R^2_K) = 2R_K\mathcal{L}^U R_K + ||\nabla_{\mathbb{S}} R_K||^2$ and comparing to (2.63), we obtain that for $t \in [0, \kappa_{K,n,\varepsilon}]$,

$$\langle M^{1,n,\varepsilon}\rangle_t = \int_0^t \|\nabla_{\mathbb{S}} R_K(U_s)\|^2 \mathrm{d}s.$$

Hence, enlarging the probability space if necessary, we can find a Brownian motion $(W_t)_{t\geq 0}$, which is defined by $W_t = \int_0^t \|\nabla_{\mathbb{S}} R_K(U_s)\|^{-1}\mathrm{d}M^{1,n,\varepsilon}_s$ for $t \in [0, \kappa_{K,n,\varepsilon}]$ and which is then extended to $\mathbb{R}_+$, such that $M^{1,n,\varepsilon}_t = \int_0^t \|\nabla_{\mathbb{S}} R_K(U_s)\|\mathrm{d}W_s$ during $[0, \kappa_{K,n,\varepsilon}]$. Hence, still for $t \in [0, \kappa_{K,n,\varepsilon}]$,

$$R_K(U_t) = R_K(u) + \int_0^t \|\nabla_{\mathbb{S}} R_K(U_s)\|\mathrm{d}W_s + \int_0^t \mathcal{L}^U R_K(U_s)\mathrm{d}s. \tag{2.64}$$

But $\nabla_{\mathbb{S}} R_K(u) = \nabla R_K(u) - 2R_K(u)u$ by (2.60), whence

$$\|\nabla_{\mathbb{S}} R_K(u)\|^2 = \|\nabla R_K(u)\|^2 - 4R_K(u)\nabla R_K(u)\cdot u + 4(R_K(u))^2.$$

Since $||\nabla R_K(u)||^2 = 4R_K(u)$ by (2.58) and $\nabla R_K(u)\cdot u = 2R_K(u)$ by (2.59),

$$\|\nabla_{\mathbb{S}} R_K(u)\|^2 = 4R_K(u) - 4(R_K(u))^2 = 4R_K(u)(1 - R_K(u)).$$

Inserting this, as well as the expression (2.55) of $\mathcal{L}^U R_K$, in (2.64), shows that $R_K(U_t)$ satisfies the desired equation on $[0, \kappa_{K,n,\varepsilon}]$. Since $\lim_{\varepsilon\to 0}\lim_{n\to\infty} \kappa_{K,n,\varepsilon} = \kappa_K$ a.s., the proof is complete. □

2.9.3 A squared Bessel-like process

The equation obtained in the previous lemma will be studied by comparison with the process we now introduce. This process behaves, near 0, like a squared Bessel processes.

Lemma 2.23. *Fix $\delta \in \mathbb{R}$, $a > 0$ and $b > 0$ such that $\delta + a\sqrt{b} < 2$. For $(W_t)_{t\geq 0}$ a 1-dimensional Brownian motion and for $x \in [0,1)$, consider the unique solution $(S_t)_{t\geq 0}$ of*

$$S_t = x + \int_0^t 2\sqrt{|S_s(1-S_s)|}\mathrm{d}W_s + \delta t + a\int_0^t \sqrt{b+|S_s|}\mathrm{d}s. \tag{2.65}$$

For $z \in \mathbb{R}$, set $\tau_z = \inf\{t > 0 : S_t = z\}$. For all $y \in (x,1)$, it holds that $\mathbb{P}(\tau_0 < \tau_y) > 0$.

Démonstration. This equation is classically well-posed, since the diffusion coefficient is 1/2-Hölder continuous and the drift coefficient is Lipschitz continuous, see Revuz-Yor [44, Theorem 3.5 page 390]. As in Karatzas-Shreve [34, (5.42) page 339], we introduce the scale function

$$f(z) = \int_{1/2}^z \exp\Big(-\int_{1/2}^u \frac{\delta + a\sqrt{b+|v|}}{2|v(1-v)|}\mathrm{d}v\Big)\mathrm{d}u.$$

This function is obviously continuous on $(0,1)$ and one gets convinced, for example approximating $(\delta+a\sqrt{b+|v|})/(2|v(1-v)|)$ by $(\delta+a\sqrt{b})/(2|v|)$, that it is also continuous at 0 because $\delta+a\sqrt{b} < 2$. By [34, (5.61) page 344], we have

$$\mathbb{P}(\tau_0 < \tau_y) = \frac{f(y)-f(x)}{f(y)-f(0)}. \tag{2.66}$$

for all $y \in (x,1)$. This last quantity is nonzero (which would not be the case if $\delta + a\sqrt{b} \geq 2$, since then $f(0) = -\infty$). ☐

2.9.4 Collisions of large clusters

We are now ready to give the

Proof of Proposition 2.19-(i)-(ii). We fix $N \geq 4$, $\theta > 0$ such that $N > \theta$. We always assume that $d_{\theta,N}(N) < 2$ and we use the notation of Subsection 2.9.1.

Step 1. We consider $\varepsilon \in (0,1]$ and $K \subset [\![1,N]\!]$ such that $|K| \in [\![2,N-1]\!]$ and $d_{\theta,N}(|K|) < 2$. We introduce the constant $a_K = c_{|K|+1}/(2c_{|K|})$ with $(c_\ell)_{\ell\in[\![1,N]\!]}$ defined in Lemma 2.13. We prove in this step that there are some constants $p_{K,\varepsilon} > 0$ and $T_{K,\varepsilon} > 0$ such that, setting

$$\tilde{\sigma}^{K,\varepsilon} = \inf\Big\{t > 0 : R_K(U_t) \geq \varepsilon \ \text{ or } \ \min_{i\notin K} R_{K\cup\{i\}}(U_t) \leq a_K\varepsilon\Big\} \wedge T_{K,\varepsilon},$$

with the convention that $\inf\emptyset = \xi$, it holds that quasi-everywhere on $\{u \in \mathcal{U} : R_K(u) \leq \varepsilon/2\}$,

$$\mathbb{P}_u^U\Big(\tilde{\sigma}^{K,\varepsilon} = \xi \ \text{ or } \inf_{t\in[0,\tilde{\sigma}^{K,\varepsilon})}\min_{i\notin K} R_{K\cup\{i\}}(U_t) \leq 2a_K\varepsilon \text{ or } R_K(U_t) = 0 \text{ for some } t \in [0,\tilde{\sigma}^{K,\varepsilon})\Big) \geq p_{K,\varepsilon}.$$

We introduce $Z_{K,\varepsilon} = \inf_{t\in[0,\tilde{\sigma}^{K,\varepsilon})}\min_{i\notin K} R_{K\cup\{i\}}(U_t)$. We note that for all $t \in [0,\tilde{\sigma}^{K,\varepsilon})$, $R_K(U_t) \leq \varepsilon$ and $Z_{K,\varepsilon} \geq a_K\varepsilon$ so that $\min_{i\in K,j\notin K}\|U_t^i - U_t^j\| \geq \varepsilon/2$ thanks to the definition of a_K

and to Lemma 2.13. This implies that $\tilde{\sigma}^{K,\varepsilon} \leq \kappa_K$, where we recall that $\kappa_K = \inf\{t \geq 0 : U_t \notin G_{\mathbf{K},0}\}$ was defined in Lemma 2.22, and that $G_{\mathbf{K},0} \cap \mathbb{S} = \{u \in \mathcal{U} : \min_{i\in K, j\notin K} ||u^i - u^j|| > 0\}$.

By the Cauchy-Schwarz inequality, and since R_K is bounded on $\mathcal{U}$, there is a deterministic constant $C_{K,\varepsilon} > 0$, allowed to change from line to line, such that for all $t \in [0, \tilde{\sigma}^{K,\varepsilon})$, we have

$$\begin{aligned}
&- d_{\theta,N}(N) R_K(U_t) - \frac{2\theta}{N} \sum_{i\in K, j\notin K} \frac{U_t^i - U_t^j}{\|U_t^i - U_t^j\|^2} \cdot (U_t^i - S_K(U_t)) \\
\leq & C_{K,\varepsilon} \sqrt{R_K(U_t)} + C_{K,\varepsilon} \Big(\sum_{i\in K} \|U_t^i - S_K(U_t)\|^2 \Big)^{1/2} \\
\leq & C_{K,\varepsilon} \sqrt{R_K(U_t)} \\
\leq & C_{K,\varepsilon} \sqrt{b + R_K(U_t)}
\end{aligned}$$

where $b > 0$ is chosen small enough so that $d_{\theta,N}(|K|) + C_{K,\varepsilon}\sqrt{b} < 2$. Actually, b is only introduced to make the drift coefficient of (2.65) Lipschitz continuous.

Recalling that $R_K(U_0) \leq \varepsilon/2$, the formula describing $R_K(U_t) \in [0,1]$ for $t \in [0, \kappa_K) \supset [0, \tilde{\sigma}^{K,\varepsilon})$, see Lemma 2.22, considering the process $(S_t)_{t\geq 0}$ solution to (2.65) with $x = \varepsilon/2$, $\delta = d_{\theta,N}(|K|)$, $a = C_{K,\varepsilon}$ and with b introduced a few lines above, driven by the same Brownian motion $(W_t)_{t\geq 0}$, and using the comparison theorem, we conclude that $R_K(U_t) \leq S_t$ for all $t \in [0, \tilde{\sigma}^{K,\varepsilon})$.

Setting $\tau_z = \inf\{t \geq 0 : S_t = z\}$ for $z \in \mathbb{R}$ and recalling the definition of $\tilde{\sigma}^{K,\varepsilon}$, we conclude that $\{Z_{K,\varepsilon} > 2a_K\varepsilon\} \subset \{\tilde{\sigma}^{K,\varepsilon} \geq \tau_\varepsilon \wedge T_{K,\varepsilon}\}$. Indeed, on $\{\inf_{t\in[0,\tilde{\sigma}^{K,\varepsilon})} \min_{i\notin K} R_{K\cup\{i\}}(U_t) > 2a_K\varepsilon\}$, either $\tilde{\sigma}^{K,\varepsilon} = T_{K,\varepsilon}$, or $(R_K(U_t))_{t\geq 0}$ reaches ε at time $\tilde{\sigma}^{K,\varepsilon}$ and we then have $\tau_\varepsilon \leq \tilde{\sigma}^{K,\varepsilon}$. In both cases, $\tilde{\sigma}^{K,\varepsilon} \geq \tau_\varepsilon \wedge T_{K,\varepsilon}$. Hence, using again that $R_K(U_t) \leq S_t$ for all $t \in [0, \tilde{\sigma}^{K,\varepsilon})$,

$$\begin{aligned}
&\Big\{ \tilde{\sigma}^{K,\varepsilon} < \xi \text{ and } Z_{K,\varepsilon} > 2a_K\varepsilon \text{ and } S_t = 0 \text{ for some } t \in [0, \tau_\varepsilon \wedge T_{K,\varepsilon}] \Big\} \\
\subset & \Big\{ \tilde{\sigma}^{K,\varepsilon} < \xi \text{ and } Z_{K,\varepsilon} > 2a_K\varepsilon \text{ and } R_K(U_t) = 0 \text{ for some } t \in [0, \tilde{\sigma}^{K,\varepsilon}) \Big\}.
\end{aligned}$$

But $A^c \cap B' \subset A^c \cap B$ gives $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(A^c \cap B) \geq \mathbb{P}(A) + \mathbb{P}(A^c \cap B') = \mathbb{P}(A \cup B')$. Hence

$$\begin{aligned}
&\mathbb{P}_u^U \Big(\tilde{\sigma}^{K,\varepsilon} = \xi \text{ or } Z_{K,\varepsilon} \leq 2a_K\varepsilon \text{ or } R_K(U_t) = 0 \text{ for some } t \in [0, \tilde{\sigma}^{K,\varepsilon}) \Big) \\
\geq & \mathbb{P}_u^U \Big(\tilde{\sigma}^{K,\varepsilon} = \xi \text{ or } Z_{K,\varepsilon} \leq 2a_K\varepsilon \text{ or } S_t = 0 \text{ for some } t \in [0, \tau_\varepsilon \wedge T_{K,\varepsilon}) \Big) \\
\geq & \mathbb{P}_u^U \Big(S_t = 0 \text{ for some } t \in [0, \tau_\varepsilon \wedge T_{K,\varepsilon}) \Big).
\end{aligned}$$

This last quantity equals $\mathbb{P}(\tau_0 < \tau_\varepsilon \wedge T_{K,\varepsilon})$ and does not depend on u such that $R_K(u) \leq \varepsilon/2$. But $\mathbb{P}(\tau_0 < \tau_\varepsilon) > 0$ by Lemma 2.23 and since $d_{\theta,N}(|K|) + C_{K,\varepsilon}\sqrt{b} < 2$. Hence there exists $T_{K,\varepsilon} > 0$ so that $\mathbb{P}(\tau_0 < \tau_\varepsilon \wedge T_{K,\varepsilon}) > 0$ and this completes the step.

Step 2. We prove (ii), i.e. that when $d_{\theta,N}(N-1) \in (0,2)$, for any $K \subset [\![1, N]\!]$ with cardinal $|K| = N-1$, quasi-everywhere, $\mathbb{P}_x^X$-a.s., $R_K(X_t)$ vanishes during $[0, \zeta)$. By (2.50) and Remark 2.21, and since $\mathbb{P}_u^U(\xi = \infty) = 1$ quasi-everywhere by Lemma 2.9-(ii), it suffices to check that quasi-everywhere, $\mathbb{P}_u^U$-a.s., $(R_K(U_t))_{t\geq 0}$ vanishes at least once during $[0, \infty)$.

We fix $K \subset [\![1, N]\!]$ with $|K| = N-1$, set $\varepsilon_0 = 1/(4a_K)$ and introduce $\tilde{\tau}_0^K = 0$ and for all $k \geq 0$,

$$\tau_{k+1}^K = \inf\{t \geq \tilde{\tau}_k^K : R_K(U_t) \leq \varepsilon_0/2\},$$
$$\tilde{\tau}_{k+1}^K = \inf\{t \geq \tau_{k+1}^K : R_K(U_t) \geq \varepsilon_0\} \wedge (\tau_{k+1}^K + T_{K,\varepsilon_0}).$$

with T_{K,ε_0} defined in Step 1. All these stopping times are finite since $(\mathcal{E}^U, \mathcal{F}^U)$ is recurrent by Lemma 2.9-(ii). We also put, for $k \geq 1$,

$$\Omega_k^K = \{R_K(U_t) = 0 \text{ for some } t \in [\tau_k^K, \tilde{\tau}_k^K]\}.$$

We now prove that $\mathbb{P}_u^U(\cap_{k\geq 1}(\Omega_k^K)^c) = 0$ quasi-everywhere, and this will complete the proof of (ii).

For $\ell \geq 1$, since $\cap_{k=1}^{\ell}(\Omega_k^K)^c$ is $\mathcal{M}^U_{\tau^K_{\ell+1}}$-measurable, the strong Markov property tells us that

$$\mathbb{P}_u^U\Big(\cap_{k=1}^{\ell+1}(\Omega_k^K)^c\Big) = \mathbb{E}_u^U\Big[\Big(\prod_{k=1}^{\ell} \mathbb{I}_{(\Omega_k^K)^c}\Big)\mathbb{P}^U_{U_{\tau^K_{\ell+1}}}((\Omega_1^K)^c)\Big].$$

We now prove that $\mathbb{P}_u^U(\Omega_1^K) \geq p_{K,\varepsilon_0}$ quasi-everywhere on $\{u \in \mathcal{U} : R_K(u) \leq \varepsilon_0/2\}$. For such a u, we have $\tau_1^K = 0$. Moreover, for all $i \notin K$, we have $R_{K\cup\{i\}}(u) = R_{[\![1,N]\!]}(u) = 1 > 2a_K\varepsilon_0$ thanks to our choice of ε_0. Hence $\tilde{\tau}_1^K = \tilde{\sigma}^{K,\varepsilon_0}$, recall Step 1. Since finally $\tilde{\sigma}^{K,\varepsilon_0} < \infty = \xi$ and since $R_{K\cup\{i\}}(U_t) = R_{[\![1,N]\!]}(U_t) = 1 > 2a_K\varepsilon_0$ for all $t \geq 0$ and all $i \notin K$,

$$\begin{aligned}\Omega_1^K =& \Big\{R_K(U_t) = 0 \text{ for some } t \in [0, \tilde{\sigma}^{K,\varepsilon_0}]\Big\}\\ =& \Big\{\tilde{\sigma}^{K,\varepsilon_0} = \xi \;\text{ or }\; \inf_{t\in[0,\tilde{\sigma}^{K,\varepsilon_0})} \min_{i\notin K} R_{K\cup\{i\}}(U_t) \leq 2a_K\varepsilon_0 \;\text{ or }\; R_K(U_t) = 0 \text{ for some } t \in [0, \tilde{\sigma}^{K,\varepsilon_0}]\Big\}.\end{aligned}$$

Hence Step 1 tells us that $\mathbb{P}_u^U(\Omega_1^K) \geq p_{K,\varepsilon_0}$ quasi-everywhere on $\{u \in \mathcal{U} : R_K(u) \leq \varepsilon_0/2\}$.

Since $R_K(U_{\tau^K_{\ell+1}}) \leq \varepsilon_0/2$, we have proved that for all $\ell \geq 1$,

$$\mathbb{P}_u^U\Big(\cap_{k=1}^{\ell+1}(\Omega_k^K)^c\Big) \leq (1 - p_{K,\varepsilon_0})\mathbb{P}_u^U\Big(\cap_{k=1}^{\ell}(\Omega_k^K)^c\Big).$$

This allows us to conclude that indeed, $\mathbb{P}_u^U(\cap_{k=1}^{\infty}(\Omega_k^K)^c) = 0$.

Step 3. We prove (i), i.e. that if $d_{\theta,N}(N-1) \leq 0$, then $\mathbb{P}_x^X(\inf_{[0,\zeta)} R_{[\![1,N]\!]}(X_t) > 0) = 1$ quasi-everywhere. By Remark 2.21 and (2.49), it suffices to show that quasi-everywhere, $\mathbb{P}_u^U(\xi < \infty) = 1$.

For all $K \subset [\![1, N]\!]$, all $\varepsilon \in (0, 1]$, we introduce $\tilde{\sigma}_0^{K,\varepsilon} = 0$ and for all $k \geq 0$,

$$\sigma_{k+1}^{K,\varepsilon} = \inf\Big\{t \geq \tilde{\sigma}_k^{K,\varepsilon} : R_K(U_t) \leq \varepsilon/2 \;\text{ and }\; \min_{i\notin K} R_{K\cup\{i\}}(U_t) \geq 2a_K\varepsilon\Big\},$$
$$\tilde{\sigma}_{k+1}^{K,\varepsilon} = \inf\Big\{t \geq \sigma_{k+1}^{K,\varepsilon} : R_K(U_t) \geq \varepsilon \;\text{ or }\; \min_{i\notin K} R_{K\cup\{i\}}(U_t) \leq a_K\varepsilon\Big\} \wedge (\sigma_{k+1}^{K,\varepsilon} + T_{K,\varepsilon}),$$

with $T_{K,\varepsilon}$ defined in Step 1 and with the convention that $\inf \emptyset = \xi$.

Step 3.1. We fix $\varepsilon \in (0, 1]$ and assume that $|K| \geq k_0$, so that $d_{\theta,N}(|K|) \leq 0$ by Lemma 2.1. We prove here that quasi-everywhere, $\mathbb{P}_u^U$-a.s., either there is $t \in [0, \xi)$ such that $R_K(U_t) = 0$ or there is $k \geq 1$ such that $\sigma_{k+1}^{K,\varepsilon} = \xi$ or there is $k \geq 1$ such that $\inf_{t\in[\sigma_k^{K,\varepsilon},\tilde{\sigma}_k^{K,\varepsilon})} \min_{i\notin K} R_{K\cup\{i\}}(U_t) \leq 2a_K\varepsilon$.

It suffices to prove that $\mathbb{P}_u^U(\cap_{k\geq 1}(\Omega_k^{K,\varepsilon})^c)=0$, where

$$\Omega_k^{K,\varepsilon}=\Big\{\sigma_{k+1}^{K,\varepsilon}=\xi \;\text{ or }\; \inf_{t\in[\sigma_k^{K,\varepsilon},\tilde\sigma_k^{K,\varepsilon})}\min_{i\notin K} R_{K\cup\{i\}}(U_t)\leq 2a_K\varepsilon \\ \text{ or }\; R_K(U_t)=0 \text{ for some } t\in[\sigma_k^{K,\varepsilon},\tilde\sigma_k^{K,\varepsilon})\Big\}.$$

But for all $\ell\geq 1$, $\cap_{k=1}^{\ell}(\Omega_k^{K,\varepsilon})^c$ is $\mathcal{M}^U_{\sigma_{\ell+1}^{K,\varepsilon}}$-measurable, whence, by the strong Markov property,

$$\mathbb{P}_u^U\Big(\cap_{k=1}^{\ell+1}(\Omega_k^{K,\varepsilon})^c\Big)=\mathbb{E}_u^U\Big[\Big(\prod_{k=1}^{\ell}\mathbb{1}_{(\Omega_k^{K,\varepsilon})^c}\Big)\mathbb{P}^U_{U_{\sigma_{\ell+1}^{K,\varepsilon}}}\Big((\Omega_1^{K,\varepsilon})^c\Big)\Big]\leq(1-p_{K,\varepsilon})\mathbb{P}_u^U\Big(\cap_{k=1}^{\ell}(\Omega_k^{K,\varepsilon})^c\Big).$$

We used Step 1, that $R_K(U_{\sigma_{\ell+1}^{K,\varepsilon}})\leq\varepsilon/2$ on the event $(\Omega_\ell^{K,\varepsilon})^c\subset\{\sigma_{\ell+1}^{K,\varepsilon}<\xi\}$, as well as the inclusion $\{\tilde\sigma_k^{K,\varepsilon}=\xi\}\subset\{\sigma_{k+1}^{K,\varepsilon}=\xi\}$. One easily concludes.

Step 3.2. For all $K\subset[\![1,N]\!]$ such that $|K|\geq k_0$, quasi-everywhere, $\mathbb{P}_u^U$-a.s., there is no $t\in[0,\xi)$ such that $R_K(U_t)=0$. Indeed, on the contrary event, there is $t\in[0,\xi)$ such that $U_t\notin E_{k_0}$, whence $U_t\notin\mathcal{U}$, which contradicts the fact that $t\in[0,\xi)$.

Step 3.3. We show by decreasing induction that

$$\mathcal{P}(n):\; \text{quasi-everywhere, } \mathbb{P}_u^U\text{-a.s. on the event } \{\xi=\infty\},\; b_n=\min_{\{|K|=n\}}\inf_{t\geq 0}R_K(U_t)>0$$

holds true for every $n\in[\![k_0,N]\!]$.

The result is clear when $n=N$, because for all $t\in[0,\xi)$, $R_{[\![1,N]\!]}(U_t)=1$.

We next assume $\mathcal{P}(n)$ for some $n\in[\![k_0+1,N]\!]$ and we show that $\mathcal{P}(n-1)$ is true. We fix $K\subset[\![1,N]\!]$ with cardinal $|K|=n-1$ and we apply Step 3.1 with K and with some $\varepsilon\in(0,b_n/(4a_K))$ (b_n is random but we may apply Step 3.1 simultaneously for all $\varepsilon\in\mathbb{Q}_+^*\cap(0,1]$) and Step 3.2, we find that on the event $\{\xi=\infty\}$, there either exists $k\geq 1$ such that $\sigma_{k+1}^{K,\varepsilon}=\infty$ or $k\geq 1$ such that $\inf_{t\in[\sigma_k^{K,\varepsilon},\tilde\sigma_k^{K,\varepsilon})}\min_{i\notin K}R_{K\cup\{i\}}(U_t)\leq 2a_K\varepsilon$. This second choice is not possible, since by induction assumption, $R_{K\cup\{i\}}(U_t)\geq b_n$ for all $t>0$ and all $i\notin K$. Hence there is $k\geq 1$ such that $\sigma_{k+1}^{K,\varepsilon}=\infty$.

By definition of $\sigma_{k+1}^{K,\varepsilon}$, this implies that, still on the event where $\xi=\infty$, there exists $t_0\geq 0$ such that for all $t\geq t_0$, either $R_K(U_t)\geq\varepsilon/2$ or $\min_{i\in K}R_{K\cup\{i\}}(U_t)\leq 2a_K\varepsilon$. Using again the induction assumption, we get that the second choice is never possible, so that actually, $R_K(U_t)\geq\varepsilon/2$ for all $t\geq t_0$. Since $(R_K(U_t))_{t\geq 0}$ is continuous and positive on $[0,t_0]$ according to Step 3.2, this completes the step.

Step 3.4. We conclude from Step 3.3 that quasi-everywhere, $\mathbb{P}_u^U$-a.s. on the event $\{\xi=\infty\}$, $U_t\in\mathcal{K}$ for all $t\geq 0$, where

$$\mathcal{K}=\{u\in\mathcal{U}: \text{for all } n\in[\![k_0,N]\!], \text{ all } K\subset[\![1,N]\!] \text{ with } |K|=n,\; R_K(u)\geq b_n\}.$$

This (random) set is compact in $\mathcal{U}$, so that Lemma 2.9-(i) tells us, both in the case where $(\mathcal{E}^U,\mathcal{F}^U)$ is recurrent and in the case where $(\mathcal{E}^U,\mathcal{F}^U)$ is transient, that this happens with probability 0. Hence quasi-everywhere, $\mathbb{P}_u^U(\xi=\infty)=0$ as desired. □

2.9.5 Binary collisions

We finally give the

Proof of Proposition 2.19-(iii). We assume that $N \geq 4$, that $0 < d_{\theta,N}(N) < 2 \leq d_{\theta,N}(N-1)$ and observe that $\theta < 2$ and $k_0 > N$, so that $\mathcal{X} = (\mathbb{R}^2)^N$ and $\mathcal{U} = \mathbb{S}$. The $QKS(\theta, N)$-process $\mathbb{X}$ is non-exploding by Proposition 2.16-(i), and the $QSKS(\theta, N)$-process $\mathbb{U}$ is irreducible recurrent by Lemma 2.9-(ii). In particular, $\zeta = \xi = \infty$ a.s. We divide the proof in 4 steps. First, we prove that $\mathbb{X}$ may have some binary collisions with positive probability. Then we check that this implies that $\mathbb{U}$ also may have some binary collisions with positive probability. Since $\mathbb{U}$ is recurrent, it will then necessarily be a.s. subjected to (infinitely many) binary collisions. Finally, we conclude using (2.50).

Step 1. We set $\mathbf{K} = (\{1,2\}, \{3\}, \dots, \{N\})$ and

$$\mathcal{K} = \Big\{x \in B(0,C) : \|x^1 - x^2\| < 1 \text{ and } \min_{i \in [\![1,N]\!], j \in [\![3,N]\!], i \neq j} \|x^i - x^j\| > 10\Big\},$$

with C large enough so that $\mu(\mathcal{K}) > 0$. We show in this step that $\mathbb{P}_x^X(A) > 0$ quasi-everywhere in $\mathcal{K}$, where

$$A = \Big\{X_t^1 = X_t^2 \text{ for some } t \in [0,1] \text{ and } \min_{t \in [0,1]} R_{[\![1,N]\!]}(X_t) > 0\Big\}.$$

To this end, we fix $x \in \mathcal{K}$ and introduce the set

$$O = \Big\{y \in (\mathbb{R}^2)^2 : R_{\{1,2\}}(y) < 2, \ \Big\|\frac{y^1 + y^2}{2} - \frac{x^1 + x^2}{2}\Big\| < 1\Big\},$$

and $B_i = \{y \in \mathbb{R}^2 : ||y - x^i||^2 < 1\}$ for $i \in [\![3, N]\!]$. Clearly, there is some $\varepsilon \in (0,1]$ such that

$$L = \Big\{y \in (\mathbb{R}^2)^N : (y^1, y^2) \in O \text{ and } y^i \in B_i \text{ for all } i \in [\![3, N]\!]\Big\} \subset G_{\mathbf{K},\varepsilon},$$

where as usual $G_{\mathbf{K},\varepsilon} = \{y \in B(0, 1/\varepsilon) : \forall\, i \in [\![1, N]\!],\ \forall\, j \in [\![3, N]\!] \setminus \{i\},\ ||y^i - y^j||^2 > \varepsilon\}$, recall that $\mathcal{X} = (\mathbb{R}^2)^N$ because $k_0 > N$.

Since $G_{\mathbf{K},\varepsilon}$ is obviously included in $\{y \in (\mathbb{R}^2)^N : R_{[\![1,N]\!]}(y) > 0\}$, we conclude that

$$\begin{aligned}\mathbb{P}_x^X(A) \geq & \mathbb{P}_x^X\Big(X_t^1 = X_t^2 \text{ for some } t \in [0,1] \text{ and } X_t \in L \text{ for all } t \in [0,1]\Big)\\ \geq & C_{1,\varepsilon,\mathbf{K}}^{-1} \mathbb{Q}_x^{1,\varepsilon,\mathbf{K}}\Big(X_t^1 = X_t^2 \text{ for some } t \in [0,1] \text{ and } X_t \in L \text{ for all } t \in [0,1]\Big)\end{aligned}$$

by Proposition 2.15 with $T = 1$. We now set $\tau_{\mathbf{K},\varepsilon} = \inf\{t > 0 : X_t \notin G_{\mathbf{K},\varepsilon}\}$. Proposition 2.15 tells us that, quasi-everywhere in $\mathcal{K} \subset G_{\mathbf{K},\varepsilon}$, the law of $(X_t)_{t \in [0, \tau_{\mathbf{K},\varepsilon}]}$ under $\mathbb{Q}_x^{1,\varepsilon,\mathbf{K}}$ equals the law of $Y_t = (Y_t^1, \dots, Y_t^N)_{t \in [0, \tilde{\tau}_{\mathbf{K},\varepsilon}]}$ where $(Y_t^1, Y_t^2)_{t \geq 0}$ is a $QKS(2\theta/N, 2)$-process issued from (x^1, x^2), where for all $i \in [\![3, N]\!]$, $(Y_t^i)_{t \geq 0}$ is a $QKS(\theta/N, 1)$-process, i.e. a 2-dimensional Brownian motion, issued from x^i, and where all these processes are independent. We have set $\tilde{\tau}_{\mathbf{K},\varepsilon} = \inf\{t > 0 : Y_t \notin G_{\mathbf{K},\varepsilon}\}$. This implies, together with the fact that $\{X_t \in L \text{ for all } t \in [0,1]\} \subset \{\tau_{\mathbf{K},\varepsilon} > 1\}$, that

$$\mathbb{P}_x^X(A) \geq C_{1,\varepsilon,\mathbf{K}}^{-1}\, p \prod_{i=3}^N q_i$$

quasi-everywhere in $\mathcal{K}$, where

$$p = \mathbb{P}\Big(\min_{s\in[0,1]} R_{\{1,2\}}((Y_s^1, Y_s^2)) = 0 \text{ and } (Y_t^1, Y_t^2) \in O \text{ for all } t \in [0,1]\Big),$$

and where $q_i = \mathbb{P}(Y_t^i \in B_i \text{ for all } t \in [0,1])$. Of course, $q_i > 0$ for all $i \in [\![3, N]\!]$, since $(Y_t^i)_{t\geq 0}$ is a Brownian motion issued from x^i. Moreover, we know from Lemma 2.11 that $(M_t = (Y_t^1+Y_t^2)/2)_{t\geq 0}$ is a 2-dimensional Brownian motion with diffusion coefficient $2^{-1/2}$ issued from $m = (x^1 + x^2)/2$, that $(R_t = R_{\{1,2\}}((Y_t^1, Y_t^2)))_{t\geq 0}$ is a squared Bessel process of dimension $d_{2\theta/N,2}(2) = d_{\theta,N}(2)$ issued from $r = ||x^1 - x^2||^2/2 \in (0, 1/2)$, and that these processes are independent. Hence, recalling the definition of O,

$$p = \mathbb{P}\Big(\min_{s\in[0,1]} R_s = 0 \text{ and } \max_{s\in[0,1]} R_s < 2\Big)\mathbb{P}\Big(\max_{s\in[0,1]} ||M_t - m|| < 1\Big).$$

This last quantity is clearly positive, because a squared Bessel process with dimension $d_{\theta,N}(2) \in (0, 2)$, see Lemma 2.1, does hit zero, see Revuz-Yor [44, Chapter XI].

Step 2. We now deduce from Step 1 that the set $F = \{u \in \mathcal{U} : u^1 = u^2\}$ is not exceptional for $\mathbb{U}$. Indeed, if it was exceptional, we would have $\mathbb{P}_u^U(\exists\, t \geq 0 : U_t \in F) = 0$ quasi-everywhere. By (2.50) and Remark 2.21, this would imply that quasi-everywhere, $\mathbb{P}_x^X(\exists\, t \in [0, \tau) : X_t \in G) = 0$, where $G = \{x \in \mathcal{X} : x^1 = x^2\}$ and $\tau = \inf\{t > 0 : R_{[\![1,N]\!]}(X_t) = 0\}$. But on the event A defined in Step 1, there is $t \in [0, 1]$ such that $X_t \in G$ and it holds that $\tau > 1$. As a conclusion, $\mathbb{P}_x^X(\exists\, t \in [0, \tau) : X_t \in G) > 0$ quasi-everywhere in $\mathcal{K}$, whence a contradiction, since $\mu(\mathcal{K}) > 0$.

Step 3. Since $(\mathcal{E}^U, \mathcal{F}^U)$ is irreducible-recurrent and since F is not exceptional, we know from Fukushima-Oshima-Takeda [24, Theorem 4.7.1-(iii) page 202] that quasi-everywhere,

$$\mathbb{P}_u^U(\forall\, r > 0, \exists\, t \geq r \;:\; U_t \in F) = 1.$$

Step 4. Using again (2.50) and Remark 2.21 and recalling that $\xi = \infty$ and that ρ is an increasing bijection from $[0, \infty)$ to $[0, \tau)$, we conclude that quasi-everywhere, $\mathbb{P}_x^X$-a.s., X_t visits F (an infinite number of times) during $[0, \tau)$. Of course, the same arguments apply when replacing $\{1, 2\}$ by any subset of $[\![1, N]\!]$ with cardinal 2, and the proof is complete. □

2.10 Quasi-everywhere conclusion

Here we prove that the conclusions of Theorem 2.5 hold quasi-everywhere.

Partial proof of Theorem 2.5. We assume that $\theta \geq 2$ and $N > 3\theta$, so that $k_0 = \lceil 2N/\theta \rceil \in [\![7, N]\!]$, and consider a $\mathcal{X}_\triangle$-valued $QKS(\theta, N)$-process $\mathbb{X}$ with life-time ζ as in Proposition 2.6, where $\mathcal{X} = E_{k_0}$.

Preliminaries. For $K \subset [\![1, N]\!]$ and $\varepsilon \in (0, 1]$, we write $\tau_{K,\varepsilon} = \inf\{t > 0 : X_t \notin G_{K,\varepsilon}\} \in [0, \zeta]$ and $G_{K,\varepsilon} = \{x \in \mathcal{X} : \min_{i\in K, j\notin K} ||x^i - x^j||^2 > \varepsilon\} \cap B(0, 1/\varepsilon)$ instead of $\tau_{\mathbf{K},\varepsilon}$ and $G_{\mathbf{K},\varepsilon}$ with $\mathbf{K} = (K, K^c)$ as in Proposition 2.15. We also write $\mathbb{Q}_x^{T,\varepsilon,K}$ instead of $\mathbb{Q}^{T,\varepsilon,\mathbf{K}}$ and recall that it is equivalent to $\mathbb{P}_x^X$ on $\mathcal{M}_T^X = \sigma(X_s : s \in [0, T])$.

Setting $X_t^K = (X_t^i)_{i\in K}$ and $X_t^{K^c} = (X_t^i)_{i\in K^c}$, we know that quasi-everywhere in $G_{K,\varepsilon}$, the law of $(X_t^K, X_t^{K^c})_{t\in[0,\tau_{K,\varepsilon}\wedge T]}$ under $\mathbb{Q}_x^{T,\varepsilon,K}$ is the same as the law of $(Y_t, Z_t)_{t\in[0,\tilde{\tau}_{K,\varepsilon}\wedge T]}$, where $(Y_t)_{t\geq 0}$ is a $QKS(|K|\theta/N, |K|)$-process issued from $x|_K$ and $(Z_t)_{t\geq 0}$ is a $QKS(|K^c|\theta/N, |K^c|)$-process issued from $x|_{K^c}$, these two processes being independent, and where $\tilde{\tau}_{K,\varepsilon} = \inf\{t > 0 : (Y_t, Z_t) \notin G_{K,\varepsilon}\}$. We denote by ζ^Y and ζ^Z the life-times of $(Y_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$. The life-time of $(Y_t, Z_t)_{t\geq 0}$ is given by $\zeta' = \zeta^Y \wedge \zeta^Z$ and it holds that $\tilde{\tau}_{K,\varepsilon} \in [0, \zeta']$.

No isolated points. Here we prove that for all $K \subset [\![1, N]\!]$ with $d_{\theta,N}(|K|) \in (0, 2)$, quasi-everywhere, we have $\mathbb{P}_x^X(A_K) = 0$, where $A_K = \{\mathcal{Z}_K$ has an isolated point$\}$ and

$$\mathcal{Z}_K = \{t \in (0, \zeta) : \text{ there is a } K\text{-collision in the configuration } X_t\}.$$

On A_K, we can find $u, v \in \mathbb{Q}_+$ such that $u < v < \zeta$ and such that there is a unique $t \in (u, v)$ with $R_K(X_t) = 0$ and $\min_{i\notin K} R_{K\cup\{i\}}(X_t) > 0$. By continuity, we deduce that on A_K, there exist $r, s \in \mathbb{Q}_+$ and $\varepsilon \in \mathbb{Q} \cap (0, 1]$ such that $r < s < \zeta$, $X_t \in G_{K,\varepsilon}$ for all $t \in [r, s]$ and such that $\{t \in (r, s) : R_K(X_t) = 0\}$ has an isolated point. It thus suffices that for all $r < s$ and all $\varepsilon \in (0, 1]$, that we all fix from now on, quasi-everywhere, $\mathbb{P}_x^X(A_{K,r,s,\varepsilon}) = 0$, where

$$A_{K,r,s,\varepsilon} = \Big\{X_t \in G_{K,\varepsilon} \text{ for all } t \in (r, s) \text{ and } \{t \in (r, s) : R_K(X_t) = 0\} \text{ has an isolated point}\Big\}.$$

By the Markov property, it suffices that $\mathbb{P}_x^X(A_{K,0,s,\varepsilon}) = 0$ quasi-everywhere in $G_{K,\varepsilon}$ and, by equivalence, that $\mathbb{Q}_x^{s,\varepsilon,K}(A_{K,0,s,\varepsilon}) = 0$ quasi-everywhere in $G_{K,\varepsilon}$. We write, recalling the preliminaries,

$$\begin{aligned}\mathbb{Q}_x^{s,\varepsilon,K}(A_{K,0,s,\varepsilon}) =&\mathbb{Q}_x^{s,\varepsilon,K}\Big(\tau_{K,\varepsilon} \geq s \text{ and } \{t \in (0, s) : R_K(X_t) = 0\} \text{ has an isolated point}\Big)\\ =&\mathbb{P}\Big(\tilde{\tau}_{K,\varepsilon} \geq s \text{ and } \{t \in (0, s) : R_K(Y_t) = 0\} \text{ has an isolated point}\Big)\\ \leq&\mathbb{P}\Big(\ \{t \in (0, s) : R_K(Y_t) = 0\} \text{ has an isolated point}\Big).\end{aligned}$$

But $(Y_t)_{t\geq 0}$ is a $QKS(|K|\theta/N, |K|)$-process, so that we know from Lemma 2.11 that $(R_K(Y_t))_{t\geq 0}$ is a squared Bessel process with dimension $d_{|K|\theta/N,|K|}(|K|) = d_{\theta,N}(|K|) \in (0, 2)$. Such a process has no isolated zero, see Revuz-Yor [44, Chapter XI].

Point (i). We have already seen in Proposition 2.16-(ii) that quasi-everywhere, $\mathbb{P}_x^X$-a.s., $\zeta < \infty$ and $X_{\zeta-} = \lim_{t\to\zeta-} X_t$ exists in $(\mathbb{R}^2)^N$ and does not belong to E_{k_0}.

Point (ii). We want to show that quasi-everywhere, $\mathbb{P}_x^X$-a.s., there is $K_0 \subset [\![1, N]\!]$ with $|K_0| = k_0$ such that there is a K_0-collision and no K-collision with $|K| > k_0$ in the configuration $X_{\zeta-}$. We already know that $X_{\zeta-} \notin E_{k_0}$, so that there is $K \subset [\![1, N]\!]$ with $|K| \geq k_0$ such that there is a K-collision in the configuration $X_{\zeta-}$. Hence the goal is to verify that quasi-everywhere, for all $K \subset [\![1, N]\!]$ with $|K| > k_0$, $\mathbb{P}_x^X(B_K) = 0$, where

$$B_K = \{\text{There is a } K\text{-collision in the configuration } X_{\zeta-}\}.$$

On B_K, there is $\varepsilon \in \mathbb{Q} \cap (0, 1]$ such that $X_{\zeta-} \in G_{K,2\varepsilon}$. By continuity, there also exists, still on B_K, some $r \in \mathbb{Q}_+ \cap [0, \zeta)$ such that $X_t \in G_{K,\varepsilon}$ for all $t \in [r, \zeta)$. Hence we only have to prove that for all $\varepsilon \in \mathbb{Q} \cap (0, 1]$, all $t \in \mathbb{Q}_+$, all $T \in \mathbb{Q}_+$ such that $T > r$, quasi-everywhere, $\mathbb{P}_x^X(B_{K,r,T,\varepsilon}) = 0$, where

$$B_{K,r,T,\varepsilon} = \{\zeta \in (r, T],\ X_t \in G_{K,\varepsilon} \text{ for all } t \in [r, \zeta) \text{ and } R_K(X_{\zeta-}) = 0\}.$$

By the Markov property, it suffices that $\mathbb{P}_x^X(B_{K,0,T,\varepsilon}) = 0$ quasi-everywhere in $G_{K,\varepsilon}$, for all $\varepsilon \in \mathbb{Q} \cap (0, 1]$ and all $T \in \mathbb{Q}_+^*$. We now fix $\varepsilon \in \mathbb{Q} \cap (0, 1]$ and $T \in \mathbb{Q}_+^*$. By equivalence, it suffices to prove that $\mathbb{Q}_x^{T,\varepsilon,K}(B_{K,0,T,\varepsilon}) = 0$. Using the notation introduced in the preliminaries, we write

$$\begin{aligned}\mathbb{Q}_x^{T,\varepsilon,K}(B_{K,0,T,\varepsilon}) =& \mathbb{Q}_x^{T,\varepsilon,K}\Big(\zeta \le T,\ \tau_{K,\varepsilon} = \zeta \text{ and } R_K(X_{\zeta-}) = 0\Big)\\ =& \mathbb{P}\Big(\zeta' \le T,\ \tilde{\tau}_{K,\varepsilon} = \zeta' \text{ and } R_K(Y_{\zeta'-}) = 0\Big)\\ \le& \mathbb{P}\Big(\inf_{t\in[0,\zeta^Y)} R_K(Y_t) = 0\Big).\end{aligned}$$

But $(Y_t)_{t\ge 0}$ is a $QKS(|K|\theta/N, |K|)$-process with $|K| > k_0 \ge 7$ and with $d_{|K|\theta/N,|K|}(|K|-1) = d_{\theta,N}(|K|-1) \le 0$ by Lemma 2.1 because $|K|-1 \ge k_0$. We also have $d_{|K|\theta/N,|K|}(|K|) = d_{\theta,N}(|K|) \le 0$. Hence Proposition 2.19-(i) tells us that $\mathbb{P}(\inf_{t\in[0,\zeta^Y)} R_K(Y_t) = 0) = 0$.

Point (iii). We recall that $k_1 = k_0 - 1$ and we fix $L \subset K \subset [\![1, N]\!]$ with $|K| = k_0$ and $|L| = k_1$. We want to prove that quasi-everywhere, $\mathbb{P}_x^X$-a.s., if $R_K(X_{\zeta-}) = 0$, then for all $t \in [0, \zeta)$, the set $\mathcal{Z}_L \cap (t, \zeta)$ is infinite and has no isolated point. But since $d_{\theta,N}(k_1) \in (0, 2)$, see Lemma 2.1, we already know that $\mathcal{Z}_L$ has no isolated point. It thus suffices to check that quasi-everywhere, for all $r \in \mathbb{Q}_+$, we have $\mathbb{P}_x^X(C_{K,L,r}) = 0$, where

$$C_{K,L,r} = \{\zeta > r,\ R_K(X_{\zeta-}) = 0, \text{ and } R_L(X_t) > 0 \text{ for all } t \in (r, \zeta)\}.$$

We used that since $|L| = k_1 = k_0 - 1$, for all $x \in \mathcal{X} = E_{k_0}$, there is a L collision in the configuration x if and only if $R_L(x) = 0$.

On $C_{K,L,r}$, thanks to point (ii) , there are $\varepsilon \in \mathbb{Q} \cap (0, 1]$, $T \in \mathbb{Q}_+$ and $s \in \mathbb{Q}_+^* \cap [r, \zeta)$ such that $\zeta \in (s, T]$ and $X_t \in G_{K,\varepsilon}$ for all $t \in [s, \zeta)$. Thus it suffices to prove that for all $s < T$ and all $\varepsilon \in (0, 1]$, that we now fix, quasi-everywhere, $\mathbb{P}_x^X(C_{K,L,s,T,\varepsilon}) = 0$, where

$$C_{K,L,s,T,\varepsilon} = \{\zeta \in (s, T],\ R_K(X_{\zeta-}) = 0,\ X_t \in G_{K,\varepsilon} \text{ and } R_L(X_t) > 0 \text{ for all } t \in [s, \zeta)\}.$$

By the Markov property, it suffices that $\mathbb{P}_x^X(C_{K,L,0,T,\varepsilon}) = 0$ quasi-everywhere in $G_{K,\varepsilon}$ and, by equivalence, that $\mathbb{Q}_x^{T,\varepsilon,K}(C_{K,L,0,T,\varepsilon}) = 0$. Recalling the preliminaries, we write

$$\begin{aligned}\mathbb{Q}_x^{T,\varepsilon,K}(C_{K,L,0,T,\varepsilon}) =& \mathbb{Q}_x^{T,\varepsilon,K}\Big(\zeta \le T,\ R_K(X_{\zeta-}) = 0,\ \tau_{K,\varepsilon} = \zeta \text{ and } R_L(X_t) > 0 \text{ for all } t \in [0, \zeta)\Big)\\ =& \mathbb{P}\Big(\zeta' \le T,\ R_K(Y_{\zeta'-}) = 0,\ \tilde{\tau}_{K,\varepsilon} = \zeta' \text{ and } R_L(Y_t) > 0 \text{ for all } t \in [0, \zeta')\Big).\end{aligned}$$

Setting $\sigma_K = \inf\{t > 0 : R_K(Y_t) = 0\}$, we observe that $\sigma_K = \zeta^Y$. Indeed, $|K| = k_0$ and $(Y_t)_{t\ge 0}$ is a $QKS(|K|\theta/N, |K|)$-process, of which the state space is given by $\mathcal{Y}_\triangle = \mathcal{Y} \cup \{\triangle\}$, where $\mathcal{Y} = \{y \in (\mathbb{R}^2)^{|K|} : R_M(y) > 0 \text{ for all } M \subset [\![1, N]\!] \text{ such that } |M| \ge k_0\}$, because $\lceil 2|K|/(|K|\theta/N)\rceil = \lceil 2N/\theta\rceil = k_0$. Hence $\{R_K(Y_{\zeta'-}) = 0\} \subset \{\zeta' = \sigma_K\}$, so that

$$\mathbb{Q}_x^{T,\varepsilon,K}(C_{K,L,0,T,\varepsilon}) \le \mathbb{P}(R_L(Y_t) > 0 \text{ for all } t \in [0, \sigma_K)).$$

This last quantity equals zero by Proposition 2.19-(ii), since $d_{|K|\theta/N,|K|}(|K|-1) = d_{\theta,N}(|K|-1) = d_{\theta,N}(k_0 - 1) \in (0, 2)$ by Lemma 2.1 and since $|L| = k_1 = |K| - 1$ and since $d_{|K|\theta/N,|K|}(|K|) = d_{\theta,N}(|K|) = d_{\theta,N}(k_0) \le 0 < 2$.

Point (iv). We assume that $k_2 = k_0 - 2$, i.e. that $d_{\theta,N}(k_0 - 2) \in (0, 2)$. We fix $L \subset K \subset [\![1, N]\!]$ with $|K| = k_1$ and $|L| = k_2$. We want to prove that quasi-everywhere, $\mathbb{P}_x^X$-a.s., for all $t \in [0, \zeta)$, if there is a K-collision in the configuration X_t, then for all $r \in [0, t)$, the set $\mathcal{Z}_L \cap (r, t)$ is infinite and has no isolated point. We already know that $\mathcal{Z}_L$ has no isolated point. It thus suffices to check that quasi-everywhere, for all $r \in \mathbb{Q}_+$, we have $\mathbb{P}_x^X(D_{K,L,r}) = 0$, where

$$D_{K,L,r} = \{\zeta > r \text{ and there is } t \in (r,\zeta) \text{ such that there is a } K\text{-collision at time } t \\ \text{but no } L\text{-collision during } (r,t)\}.$$

We set $\sigma_{K,r} = \inf\{t > r : \text{there is a } K\text{-collision in the configuration } X_t\}$. It holds that

$$D_{K,L,r} = \{\zeta > r,\ \sigma_{K,r} < \zeta \text{ and there is no } L\text{-collision during } u \in [r, \sigma_{K,r})\}.$$

On $D_{K,L,r}$, there exists $\varepsilon \in \mathbb{Q} \cap (0,1]$ such that $X_{\sigma_{K,r}} \in G_{K,2\varepsilon}$, so that by continuity, there exists $v \in \mathbb{Q}_+ \cap [r, \sigma_{K,r})$ such that $X_u \in G_{K,\varepsilon}$ for all $u \in [v, \sigma_{K,r}]$. Observe that $\sigma_{K,v} = \sigma_{K,r}$ and that for all $t \in [v, \sigma_{K,v})$, there is a L-collision at time t if and only if $R_L(X_t) = 0$, by definition of $\sigma_{K,v}$ and since $X_t \in G_{K,\varepsilon}$. All in all, it suffices to prove that for all $v \in \mathbb{Q}_+$, all $\varepsilon \in \mathbb{Q} \cap (0,1]$, all $T \in \mathbb{Q}_+^*$, $\mathbb{P}_x^X(D_{K,L,v,T,\varepsilon}) = 0$ quasi-everywhere, where

$$D_{K,L,v,T,\varepsilon} = \{\zeta \in (v,T],\ \sigma_{K,v} < \zeta,\ X_u \in G_{K,\varepsilon} \text{ and } R_L(X_u) > 0 \text{ for all } u \in [v, \sigma_{K,v})\}.$$

By the Markov property, it suffices to prove that $\mathbb{P}_x^X(D_{K,L,0,T,\varepsilon}) = 0$ quasi-everywhere in $G_{K,\varepsilon}$ and, by equivalence, we may use $\mathbb{Q}_x^{T,\varepsilon,K}$ instead of $\mathbb{P}_x^X$. But recalling the preliminaries,

$$\begin{aligned}\mathbb{Q}_x^{T,\varepsilon,K}(D_{K,L,0,T,\varepsilon}) =& \mathbb{Q}_x^{T,\varepsilon,K}\Big(\zeta \le T,\ \sigma_{K,0} < \zeta,\ \tau_{K,\varepsilon} \ge \sigma_{K,0} \text{ and } R_L(X_t) > 0 \text{ for all } t \in [0, \sigma_{K,0})\Big) \\ =& \mathbb{P}\Big(\zeta' \le T,\ \tilde\sigma_{K,0} < \zeta',\ \tilde\tau_{K,\varepsilon} \ge \tilde\sigma_{K,0} \text{ and } R_L(Y_t) > 0 \text{ for all } t \in [0, \tilde\sigma_{K,0})\Big) \\ \le& \mathbb{P}\Big(R_L(Y_t) > 0 \text{ for all } t \in [0, \tilde\sigma_{K,0})\Big),\end{aligned}$$

where we have set $\tilde\sigma_{K,0} = \inf\{t > 0 : R_K(Y_t) = 0\}$. Finally, $\mathbb{P}(R_L(Y_t) > 0 \text{ for all } t \in [0, \tilde\sigma_{K,0})) = 0$ by Proposition 2.19-(ii), because $(Y_t)_{t\ge 0}$ is a $QKS(|K|\theta/N, |K|)$-process, because $|L| = k_2 = |K| - 1$, because $d_{|K|\theta/N,|K|}(|K|-1) = d_{\theta,N}(|K|-1) = d_{\theta,N}(k_2) \in (0,2)$ and because $d_{|K|\theta/N,|K|}(|K|) = d_{\theta,N}(|K|) = d_{\theta,N}(k_1) \in (0,2)$.

Point (v). We fix $K \subset [\![1,N]\!]$ with cardinal $|K| \in [\![3, k_2 - 1]\!]$, so that $d_{\theta,N}(|K|) \ge 2$. We want to prove that quasi-everywhere, $\mathbb{P}_x^X$-a.s., for all $t \in [0,\zeta)$, there is no K-collision in the configuration X_t. We introduce $\sigma_K = \inf\{t > 0 : \text{there is a } K\text{-collision in the configuration } X_t\}$, with the convention that $\inf \emptyset = \zeta$, and we have to verify that quasi-everywhere, $\mathbb{P}_x^X(\sigma_K < \zeta) = 0$.

On the event $\{\sigma_K < \zeta\}$, there exist $\varepsilon \in \mathbb{Q} \cap (0,1]$ and $r \in \mathbb{Q}_+^* \cap [0, \sigma_K)$ such that $X_t \in G_{K,\varepsilon}$ for all $t \in [r, \sigma_K]$. Hence it suffices to check that for all $\varepsilon \in \mathbb{Q} \cap (0,1]$, all $r \in \mathbb{Q}_+^*$ and all $T \in \mathbb{Q}_+^* \cap (r,\infty)$, which we now fix, quasi-everywhere, $\mathbb{P}_x^X(F_{K,r,T,\varepsilon}) = 0$, where

$$F_{K,r,T,\varepsilon} = \{\sigma_K \in (r, \zeta \wedge T) \text{ and } X_t \in G_{K,\varepsilon} \text{ for all } t \in [r, \sigma_K]\}.$$

By the Markov property, it suffices that $\mathbb{P}_x^X(F_{K,0,T,\varepsilon}) = 0$ quasi-everywhere in $G_{K,\varepsilon}$ and, by equivalence, that $\mathbb{Q}_x^{T,\varepsilon,K}(F_{K,0,T,\varepsilon}) = 0$. Recalling the preliminaries, we write

$$\begin{aligned}\mathbb{Q}_x^{T,\varepsilon,K}(F_{K,0,T,\varepsilon}) =& \mathbb{Q}_x^{T,\varepsilon,K}\Big(\sigma_K \in (0, \zeta \wedge T) \text{ and } \tau_{K,\varepsilon} \ge \sigma_K\Big) \\ =& \mathbb{P}\Big(\tilde\sigma_K \in (0, \zeta' \wedge T) \text{ and } \tilde\tau_{K,\varepsilon} \ge \tilde\sigma_K\Big) \\ \le& \mathbb{P}\Big(\inf_{t\in[0,T]} R_K(Y_t) = 0\Big),\end{aligned}$$

where we have set $\tilde{\sigma}_K = \inf\{t > 0 : \text{there is a } K\text{-collision in the configuration } (Y_t, Z_t)\}$. Since $(Y_t)_{t\geq 0}$ is a $QKS(|K|\theta/N, |K|)$-process, we know from Lemma 2.11 that $(R_K(Y_t))_{t\geq 0}$ is a squared Bessel process with dimension $d_{|K|\theta/N,|K|}(|K|) = d_{\theta,N}(|K|) \geq 2$. Such a process does a.s. never reach 0.

Point (vi). The proof is exactly the same as that of (iv), replacing everywhere k_1 by k_2 and k_2 by 2, and using Proposition 2.19-(iii) instead of Proposition 2.19-(ii), which is licit because $0 < d_{k_2\theta/N,k_2}(k_2) < 2 \leq d_{k_2\theta/N,k_2}(k_2 - 1)$, since $d_{k_2\theta/N,k_2}(k_2) = d_{\theta,N}(k_2)$ and $d_{k_2\theta/N,k_2}(k_2 - 1) = d_{\theta,N}(k_2 - 1)$ and by Lemma 2.1. □

2.11 Extension to all initial conditions in E_2

We first prove Proposition 2.2 : we can build a $KS(\theta, N)$-process, i.e. a $QKS(\theta, N)$-process such that $\mathbb{P}^X_x \circ X_t^{-1}$ is absolutely continuous for all $x \in E_2$ and all $t > 0$. We next conclude the proofs of Proposition 2.3 and of Theorem 2.5.

2.11.1 Construction of a $KS(\theta, N)$-process

We fix $\theta > 0$ and $N \geq 2$ such that $N > \theta$ during the whole subsection. For each $n \in \mathbb{N}^*$, we introduce $\phi_n \in C^\infty(\mathbb{R}_+, \mathbb{R}_+^*)$ such that $\phi_n(r) = r$ for all $r \geq 1/n$ and we set, for $x \in (\mathbb{R}^2)^N$,

$$\mathbf{m}_n(x) = \prod_{1\leq i\neq j\leq N} [\phi_n(\|x^i - x^j\|^2)]^{-\theta/N} \qquad \text{and} \qquad \mu_n(\mathrm{d}x) = \mathbf{m}_n(x)\mathrm{d}x.$$

We then consider the $(\mathbb{R}^2)^N$-valued S.D.E

$$X_t^n = x + B_t + \int_0^t \frac{\nabla \mathbf{m}_n(X_s^n)}{2\mathbf{m}_n(X_s^n)}\mathrm{d}s, \tag{2.67}$$

which is strongly well-posed, for every initial condition, since the drift coefficient is smooth and bounded. We denote by $\mathbb{X}^n = (\Omega^n, \mathcal{M}^n, (X_t^n)_{t\geq 0}, (\mathbb{P}_x^n)_{x\in(\mathbb{R}^2)^N})$ the corresponding Markov process.

Lemma 2.24. *For all $n \geq 1$, $\mathbb{X}^n$ is a μ_n-symmetric $(\mathbb{R}^2)^N$-valued diffusion with regular Dirichlet space $(\mathcal{E}^n, \mathcal{F}^n)$ with core $C_c^\infty((\mathbb{R}^2)^N)$ such that for all $\varphi \in C_c^\infty((\mathbb{R}^2)^N)$,*

$$\mathcal{E}^n(\varphi, \varphi) = \frac{1}{2}\int_{(\mathbb{R}^2)^N} \|\nabla\varphi\|^2 \mathrm{d}\mu_n.$$

Moreover $\mathbb{P}_x^n \circ (X_t^n)^{-1}$ has a density with respect to the Lebesgue measure on $(\mathbb{R}^2)^N$ for all $t > 0$ and all $x \in (\mathbb{R}^2)^N$.

Démonstration. Classically, $\mathbb{X}^n$ is a μ_n-symmetric diffusion and its (strong) generator $\mathcal{L}^n$ satisfies that for all $\varphi \in C_c^\infty((\mathbb{R}^2)^N)$, all $x \in (\mathbb{R}^2)^N$, $\mathcal{L}^n\varphi(x) = \frac{1}{2}\Delta\varphi(x) + \frac{\nabla \mathbf{m}_n(x)}{2\mathbf{m}_n(x)} \cdot \nabla\varphi(x)$. Hence, see Subsection 2.13.1, one easily shows that for $(\mathcal{E}^n, \mathcal{F}^n)$ the Dirichlet space of $\mathbb{X}^n$, we have $C_c^\infty((\mathbb{R}^2)^N) \subset \mathcal{F}^n$ and, for $\varphi \in C_c^\infty((\mathbb{R}^2)^N)$, $\mathcal{E}^n(\varphi, \varphi) = \frac{1}{2}\int_{(\mathbb{R}^2)^N} \|\nabla\varphi\|^2 \mathrm{d}\mu_n$. Since $(\mathcal{E}^n, \mathcal{F}^n)$ is closed, we deduce that

$$\overline{C_c^\infty((\mathbb{R}^2)^N)}^{\mathcal{E}_1^n} \subset \mathcal{F}^n,$$

where $\mathcal{E}_1^n(\cdot,\cdot) = \mathcal{E}^n(\cdot,\cdot) + \|\cdot\|^2_{L^2((\mathbb{R}^2)^N,\mu_n)}$. But thanks to [24, Lemma 3.3.5 page 136],

$$\mathcal{F}^n \subset \{\varphi \in L^2((\mathbb{R}^2)^N,\mu_n) : \nabla\varphi \in L^2((\mathbb{R}^2)^N,\mu_n)\},$$

where ∇ is understood in the sense of distributions. Since finally

$$\overline{C_c^\infty((\mathbb{R}^2)^N)}^{\mathcal{E}_1^n} = \{\varphi \in L^2((\mathbb{R}^2)^N,\mu_n) : \nabla\varphi \in L^2((\mathbb{R}^2)^N,\mu_n)\},$$

$\mathbb{X}^n$ has the announced Dirichlet space. Finally, the absolute continuity of $\mathbb{P}_x^n \circ (X_t^n)^{-1}$, for $t > 0$ and $x \in (\mathbb{R}^2)^N$, immediately follows from the (standard) Girsanov theorem, since the drift coefficient is bounded. $\square$

For all $x \in E_2$ we set $d_x = \min_{i\neq j} \|x^i - x^j\|^2$. For $n \geq 1$, we introduce the open set

$$E_2^n = \Big\{x \in (\mathbb{R}^2)^N : d_x > \frac{1}{n} \text{ and } ||x|| < n\Big\}. \tag{2.68}$$

We also fix a $QKS(\theta, N)$-process $\mathbb{X} = (\Omega^X, \mathcal{M}^X, (X_t)_{t\geq 0}, (\mathbb{P}_x^X)_{x\in\mathcal{X}_\triangle})$ for the whole subsection.

Lemma 2.25. *There exists an exceptional set $\mathcal{N}_0 \subset E_2$ with respect to $\mathbb{X}$ such that for all $n \geq 1$, for all $x \in E_2^n \setminus \mathcal{N}_0$, the law of $(X^n_{t\wedge\tau_n})_{t\geq 0}$ under $\mathbb{P}_x^n$ equals the law of $(X_{t\wedge\sigma_n})_{t\geq 0}$ under $\mathbb{P}_x^X$, where*

$$\tau_n = \inf\{t > 0 : X_t^n \notin E_2^n\} \qquad \textit{and} \qquad \sigma_n = \inf\{t > 0 : X_t \notin E_2^n\}.$$

Démonstration. We fix $n \geq 1$. Applying Lemma 2.38 to $\mathbb{X}^n$ and $\mathbb{X}$ with the open set E_2^n, using that $\mathbf{m}_n = \mathbf{m}$ on E_2^n and Lemma 2.24, we find that the processes $\mathbb{X}^n$ and $\mathbb{X}$ killed when leaving E_2^n have the same Dirichlet space. By uniqueness, see [24, Theorem 4.2.8 page 167], there exists an exceptional set $\mathcal{N}_n$ such that for all $x \in E_2^n \setminus \mathcal{N}_n$, the law of $(X_t^n)_{t\geq 0}$ killed when leaving E_2^n under $\mathbb{P}_x^n$ equals the law of $(X_t)_{t\geq 0}$ killed when leaving E_2^n under $\mathbb{P}_x^X$. We conclude setting $\mathcal{N}_0 = \cup_{n\geq 1}\mathcal{N}_n$. $\square$

Lemma 2.26. *For all exceptional set $\mathcal{N}$ with respect to $\mathbb{X}$, all $n \geq 1$ and all $x \in E_2^n$, we have $\mathbb{P}_x^n(X^n_{\tau_n} \notin \mathcal{N}) = 1$.*

Démonstration. We fix $\mathcal{N}$ an exceptional set with respect to $\mathbb{X}$, $n \geq 1$ and $x \in E_2^n$. For $\varepsilon \in (0, 1]$, we write

$$\mathbb{P}_x^n(X^n_{\tau_n} \in \mathcal{N}) \leq \mathbb{P}_x^n(\tau_n \leq \varepsilon) + \mathbb{P}_x^n(\tau_n > \varepsilon, X^n_{\tau_n} \in \mathcal{N}) = \mathbb{P}_x^n(\tau_n \leq \varepsilon) + \mathbb{E}_x^n[\mathbb{1}_{\{\tau_n>\varepsilon\}}\mathbb{P}^n_{X^n_\varepsilon}(X^n_{\tau_n} \in \mathcal{N})]$$

by the Markov property. But by Lemma 2.25, for all $y \in E_2^n \setminus \mathcal{N}_0$, the law of $(X^n_{t\wedge\tau_n})_{t\geq 0}$ under $\mathbb{P}_y^n$ is equal to the law of $(X_{t\wedge\sigma_n})_{t\geq 0}$ under $\mathbb{P}_y^X$. Since $\mathcal{N}_0 \cup \mathcal{N}$ is exceptional for $\mathbb{X}$, we can find $\mathcal{N}' \supset \mathcal{N}_0 \cup \mathcal{N}$ properly exceptional for $\mathbb{X}$ (see Subsection 2.13.1). Hence for all $y \in E_2^n \setminus \mathcal{N}'$,

$$\mathbb{P}_y^n(X^n_{\tau_n} \in \mathcal{N}) \leq \mathbb{P}_y^n(X^n_{\tau_n} \in \mathcal{N}') = \mathbb{P}_y^X(X_{\sigma_n} \in \mathcal{N}') = 0.$$

Since $\mathbb{P}_x^n \circ (X_\varepsilon^n)^{-1}$ has a density by Lemma 2.25, we conclude that $\mathbb{P}_x^n(X_\varepsilon^n \in \mathcal{N}') = 0$ and thus that $\mathbb{P}_x^n$-a.s., we have $\mathbb{P}^n_{X^n_\varepsilon}(X^n_{\tau_n} \in \mathcal{N}) = 0$. All in all, we have proved that $\mathbb{P}_x^n(X^n_{\tau_n} \in \mathcal{N}) \leq \mathbb{P}_x^n(\tau_n \leq \varepsilon)$, and it suffices to let $\varepsilon \to 0$, since $\mathbb{P}_x^n(\tau_n > 0) = 1$ by continuity and since $x \in E_2^n$. $\square$

Using Lemmas 2.25 and 2.26, it is slightly technical but not difficult to build from $\mathbb{X}$ and the family $(\mathbb{X}^n)_{n\geq 1}$ a $\mathcal{X}_\triangle$-valued diffusion $\tilde{\mathbb{X}} = (\tilde{\Omega}^X, \tilde{\mathcal{M}}^X, (\tilde{X}_t)_{t\geq 0}, (\tilde{\mathbb{P}}^X_x)_{x\in\mathcal{X}_\triangle})$ such that

- for all $x \in \mathcal{X}_\triangle \setminus \mathcal{N}_0$, the law of $(\tilde{X}_t)_{t\geq 0}$ under $\tilde{\mathbb{P}}^X_x$ equals the law of $(X_t)_{t\geq 0}$ under $\mathbb{P}^X_x$,
- for all $x \in \mathcal{N}_0$, setting $n = 1+\lfloor \max(1/d_x, ||x||)\rfloor$ (so that $x \in E_2^n$), the law of $(\tilde{X}_{t\wedge\tilde{\sigma}_n})_{t\geq 0}$ under $\tilde{\mathbb{P}}^X_x$ is the same as that of $(X^n_{t\wedge\tau_n})_{t\geq 0}$ under $\mathbb{P}^n_x$ and the law of $(\tilde{X}_{\tilde{\sigma}_n+t})_{t\geq 0}$ under $\tilde{\mathbb{P}}^X_x$ conditionally on $\tilde{\mathcal{M}}^X_{\tilde{\sigma}_n}$ equals the law of $(X_t)_{t\geq 0}$ under $\mathbb{P}^X_{\tilde{X}_{\sigma_n}}$. We have used the notation $\tilde{\sigma}_n = \inf\{t > 0 : \tilde{X}_t \notin E_2^n\}$ and $\tilde{\mathcal{M}}^X_t = \sigma(\tilde{X}_s : s \in [0, t])$.

Remark 2.27. *For all $x \in E_2$, setting $n = 1 + \lfloor \max(1/d_x, ||x||)\rfloor$, the law of $(\tilde{X}_{t\wedge\tilde{\sigma}_n})_{t\geq 0}$ under $\tilde{\mathbb{P}}^X_x$ is the same as that of $(X^n_{t\wedge\tau_n})_{t\geq 0}$ under $\mathbb{P}^n_x$.*

Démonstration. This follows from Lemma 2.25 when $x \in E_2 \setminus \mathcal{N}_0$ and from the definition of $\tilde{\mathbb{X}}$ otherwise. □

We can finally give the

Proof of Proposition 2.2. We fix $N \geq 2$ and $\theta > 0$ such that $N > \theta$ and we prove that $\tilde{\mathbb{X}}$ defined above is a $KS(\theta, N)$-process. First, it is clear that $\tilde{\mathbb{X}}$ is a $QKS(\theta, N)$-process because $\tilde{\mathbb{X}}$ is a $\mathcal{X}_\triangle$-valued diffusion and since for all $x \in \mathcal{X}_\triangle \setminus \mathcal{N}_0$, the law of $(\tilde{X}_t)_{t\geq 0}$ under $\tilde{\mathbb{P}}^X_x$ equals the law of $(X_t)_{t\geq 0}$ under $\mathbb{P}^X_x$, with $\mathcal{N}_0$ exceptional for $\mathbb{X}$. It remains to prove that for all $x \in E_2$, all $t > 0$ and all Lebesgue-null $A \subset (\mathbb{R}^2)^N$, we have $\tilde{\mathbb{P}}^X_x(\tilde{X}_t \in A) = 0$. We set $n = 1+\lfloor \max(1/d_x, ||x||)\rfloor$ and write, for any $\varepsilon \in (0, t)$,

$$\tilde{\mathbb{P}}^X_x(\tilde{X}_t \in A) \leq \tilde{\mathbb{P}}^X_x(\tilde{\sigma}_n > \varepsilon, \tilde{X}_t \in A) + \tilde{\mathbb{P}}^X_x(\tilde{\sigma}_n \leq \varepsilon) = \tilde{\mathbb{E}}^X_x[\mathbb{1}_{\{\tilde{\sigma}_n>\varepsilon\}}\tilde{\mathbb{P}}^X_{\tilde{X}_\varepsilon}(\tilde{X}_{t-\varepsilon} \in A)] + \tilde{\mathbb{P}}^X_x(\tilde{\sigma}_n \leq \varepsilon).$$

Since $\tilde{\mathbb{X}}$ is μ-symmetric (because it is a $QKS(\theta, N)$-process), since $\tilde{P}_{t-\varepsilon}1 \leq 1$, where $\tilde{P}_t$ is the semi-group of $\tilde{\mathbb{X}}$ and since A is Lebesgue-null,

$$\int_{(\mathbb{R}^2)^N} \tilde{\mathbb{P}}_y(\tilde{X}_{t-\varepsilon} \in A)\mu(\mathrm{d}y) \leq \mu(A) = 0.$$

Hence there is a Lebesgue-null subset B of $(\mathbb{R}^2)^N$ (depending on $t-\varepsilon$) such that $\tilde{\mathbb{P}}_y(\tilde{X}_{t-\varepsilon} \in A) = 0$ for every $y \in (\mathbb{R}^2)^N \setminus B$. We conclude that

$$\tilde{\mathbb{P}}^X_x(\tilde{X}_t \in A) \leq \tilde{\mathbb{P}}^X_x(\tilde{\sigma}_n > \varepsilon, \tilde{X}_\varepsilon \in B) + \tilde{\mathbb{P}}^X_x(\tilde{\sigma}_n \leq \varepsilon) = \mathbb{P}^n_x(\tau_n > \varepsilon, X^n_\varepsilon \in B) + \tilde{\mathbb{P}}^X_x(\tilde{\sigma}_n \leq \varepsilon),$$

where we finally used Remark 2.27. Since B is Lebesgue-null, we deduce from Lemma 2.24 that $\mathbb{P}^n_x(\tau_n > \varepsilon, X^n_\varepsilon \in B) = 0$. Thus $\tilde{\mathbb{P}}^X_x(\tilde{X}_t \in A) \leq \tilde{\mathbb{P}}^X_x(\tilde{\sigma}_n \leq \varepsilon)$, which tends to 0 as $\varepsilon \to 0$ because $\tilde{\mathbb{P}}^X_x(\tilde{\sigma}_n > 0) = 1$ by continuity. □

2.11.2 Final proofs

We fix $\theta > 0$, $N \geq 2$ such that $N > \theta$ and a $KS(\theta, N)$-process $\mathbb{X}$, which exists thanks to Subsection 2.11.1. We recall that E_2^n was introduced in (2.68) and define, for all $n \geq 1$, $\sigma_n = \inf\{t \geq 0 : X_t \notin E_2^n\}$, as well as the σ-field

$$\mathcal{G} = \cap_{n\geq 1}\sigma(X_{\sigma_n+t}, t \geq 0).$$

Lemma 2.28. *Fix $A \in \mathcal{G}$. If $\mathbb{P}_x^X(A) = 0$ quasi-everywhere, then $\mathbb{P}_x^X(A) = 0$ for all $x \in E_2$.*

Démonstration. We fix $A \in \mathcal{G}$ such that $\mathbb{P}_x^X(A) = 0$ quasi-everywhere. There is an exceptional set $\mathcal{N}$ such that for all $x \in E_2 \setminus \mathcal{N}$, $\mathbb{P}_x^X(A) = 0$. We now fix $x \in E_2$ and set $n = 1 + \lfloor \max(1/d_x, ||x||) \rfloor$. For any $\varepsilon \in (0, 1]$,

$$\mathbb{P}_x^X(A) \le \mathbb{P}_x^X(\sigma_n \le \varepsilon) + \mathbb{P}_x^X[\sigma_n > \varepsilon, A].$$

By the Markov property and since $A \in \mathcal{G} \subset \sigma(X_{\sigma_n+t}, t \ge 0)$, we get

$$\mathbb{P}_x^X[\sigma_n > \varepsilon, A] = \mathbb{E}_x^X[\mathbb{1}_{\{\sigma_n > \varepsilon\}} \mathbb{P}_{X_\varepsilon}^X(A)].$$

But the law of X_ε under $\mathbb{P}_x^X$ has a density, so that $\mathbb{P}_x^X(X_\varepsilon \in \mathcal{N}) = 0$, whence $\mathbb{P}_x^X(\mathbb{P}_{X_\varepsilon}^X(A) = 0) = 1$. Hence $\mathbb{P}_x^X[\sigma_n > \varepsilon, A] = 0$ and we end with $\mathbb{P}_x^X(A) \le \mathbb{P}_x^X(\tau_n \le \varepsilon)$. As usual, we conclude that $\mathbb{P}_x^X(A) = 0$ by letting $\varepsilon \to 0$. □

We are now ready to give the

Proof of Proposition 2.3. Let $\theta \in (0, 2)$ and $N \ge 2$. Since our $KS(\theta, N)$-process $\mathbb{X}$ is a $QKS(\theta, N)$-process, we know from Proposition 2.16-(i) that $\mathbb{P}_x^X(\zeta = \infty) = 1$ quasi-everywhere. We want to prove that $\mathbb{P}_x^X(\zeta = \infty) = 1$ for all $x \in E_2$. By Lemma 2.28, it thus suffices to check that $\{\zeta = \infty\}$ belongs to $\mathcal{G}$, which is not hard since for each $n \ge 1$,

$$\{\zeta = \infty\} = \{X_t \in \mathcal{X} \text{ for all } t \ge 0\} = \{X_t \in \mathcal{X} \text{ for all } t \ge \sigma_n\} \in \sigma(X_{\sigma_n+t}, t \ge 0).$$

For the second equality, we used that $X_t \in \bar{E}_2^n \subset \mathcal{X}$ for all $t \in [0, \sigma_n]$ by definition. □

Proof of Theorem 2.5. Let $\theta \ge 2$ and $N > 3\theta$. Since our $KS(\theta, N)$-process $\mathbb{X}$ is a $QKS(\theta, N)$-process, we know from Section 2.10 that all the conclusions of Theorem 2.5 hold quasi-everywhere. In other words, $\mathbb{P}_x^X(A) = 1$ quasi-everywhere, where A is the event on which we have $\zeta < \infty$, $X_{\zeta-} = \lim_{t \to \zeta-} X_t \in (\mathbb{R}^2)^N$, there is $K_0 \in [\![1, N]\!]$ with cardinal $|K_0| = k_0$ such that there is a K_0-collision in the configuration $X_{\zeta-}$, etc. We want to prove that $\mathbb{P}_x^X(A) = 1$ for all $x \in E_2$. By Lemma 2.28, it thus suffices to check that A belongs to $\mathcal{G}$. But for each $n \ge 1$, A indeed belongs to $\sigma(X_{\sigma_n+t}, t \ge 0)$, because no collision (nor explosion) may happen before getting out of E_2^n. □

We end this section with the following remark (that we will not use anywhere).

Remark 2.29. *Fix $\theta \ge 0$ and $N \ge 2$ such that $N > \theta$. Consider a $KS(\theta, N)$ process $\mathbb{X}$ and define $\sigma = \inf\{t \ge 0 : X_t \notin E_2\}$. For all $x \in E_2$, there is some $(\mathcal{M}_t^X)_{t \ge 0}$-Brownian motion $((B_t^i)_{t \ge 0})_{i \in [\![1,N]\!]}$ (of dimension $2N$) under $\mathbb{P}_x^X$ such that for all $t \in [0, \sigma)$, all $i \in [\![1, N]\!]$,*

$$X_t^i = x^i + B_t^i - \frac{\theta}{N} \sum_{j \ne i} \int_0^t \frac{X_s^i - X_s^j}{||X_s^i - X_s^j||^2} \mathrm{d}s. \tag{2.69}$$

Démonstration. It of course suffices to prove the result during $[0, \sigma_n)$, where $\sigma_n = \inf\{t \ge 0 : X_t \notin E_2^n\}$. For any $x \in E_2^n$ and for a given Brownian motion, the solutions to (2.69) and (2.67) classically coincide while they remain E_2^n, because their drift coefficients coincide and are smooth inside E_2^n. Hence, recalling the notation of Subsection 2.11.1, it suffices to prove that the semi-groups $P_t(x, \cdot)$ and $P_t^n(x, \cdot)$ of the Markov processes $\mathbb{X}$ and $\mathbb{X}^n$ killed when getting out of E_2^n coincide for all $x \in E_2^n$.

By Lemma 2.25, there is an exceptional set $\mathcal{N}_0$ such that $P_t(x,\cdot) = P_t^n(x,\cdot)$ for all $x \in E_2^n \setminus \mathcal{N}_0$. We next fix $x \in E_2^n$. For any $\varepsilon \in (0,t)$, using that $P_\varepsilon(x,\cdot)$ has a density and that $\mathcal{N}_0$ is Lebesgue-null, we easily deduce that $P_t(x,\cdot) = (P_\varepsilon P_{t-\varepsilon})(x,\cdot) = (P_\varepsilon P^n_{t-\varepsilon})(x,\cdot)$. It is then not difficult, using that P_t^n is Feller, to let $\varepsilon \to 0$ and conclude that indeed, $P_t(x,\cdot) = P_t^n(x,\cdot)$. □

2.12 Appendix : A few elementary computations

We recall that $d_{\theta,N}(k) = (k-1)(2-\theta k/N)$ for $k \geq 2$ and give the

Proof of Lemma 2.1. First, (2.3), which says that $d_{\theta,N}(k) > 0$ if and only if $k < k_0 = \lceil 2N/\theta \rceil$, is clear. We next fix $N > 3\theta \geq 6$, so that $k_0 \in [\![7, N]\!]$ and $d_{\theta,N}(2) = 2 - 2\theta/N \in (4/3, 2)$. By concavity of $x \to (x-1)(2-\theta x/N)$, it only remains to check that (i) $d_{\theta,N}(3) \geq 2$, (ii) $d_{\theta,N}(k_0-3) \geq 2$, and (iii) $d_{\theta,N}(k_0-1) < 2$. We introduce $a = 2N/\theta > 6$ and observe that $d_{\theta,N}(k) = 2a^{-1}(k-1)(a-k)$ and that $k_0 = \lceil a \rceil$.

For (i), we write $d_{\theta,N}(3) = 4a^{-1}(a-3) = 4 - 12a^{-1} > 2$ since $a > 6$.

For (ii), we have $d_{\theta,N}(k_0-3) = 2a^{-1}(\lceil a \rceil - 4)(a - \lceil a \rceil + 3)$ and we need $(\lceil a \rceil - 4)(a - \lceil a \rceil + 3) \geq a$. Writing $a = n + \alpha$ with an integer $n \geq 6$ and $\alpha \in (0,1]$, we need that $(n-3)(2+\alpha) \geq n + \alpha$, and this holds true because $2(n-3) \geq n$ and $(n-3)\alpha \geq \alpha$.

For (iii), we write $d_{\theta,N}(k_0-1) = 2a^{-1}(\lceil a \rceil - 2)(a - \lceil a \rceil + 1) \leq 2a^{-1}(\lceil a \rceil - 2) < 2$. □

We next study the reference measure of the Keller-Segel particle system.

Proposition 2.30. *Let $N \geq 2$ and $\theta > 0$ be such that $N > \theta$. Recall that $k_0 = \lceil 2N/\theta \rceil$ and the definition* (2.4) *of $\mu(\mathrm{d}x) = \mathbf{m}(x)\mathrm{d}x$.*

(i) The measure μ is Radon on E_{k_0}.

(ii) If $k_0 \leq N$, then μ is not Radon on E_{k_0+1}.

Démonstration. (i) To show that μ is radon on E_{k_0}, we have to check that for all $x = (x^1, \dots, x^N) \in E_{k_0}$, which we now fix, there is an open set $O_x \subset E_{k_0}$ such that $x \in O_x$ and $\mu(O_x) < \infty$. We choose $O_x = \prod_{i=1}^N B(x^i, d_x)$, where the balls are subsets of $\mathbb{R}^2$ and where

$$d_x = 1 \wedge \min\Big\{ \frac{\|x^i - x^j\|}{3} : i,j \in [\![1,N]\!] \text{ such that } x^i \neq x^j \Big\} > 0.$$

We consider the partition $K_1, \dots, K_\ell$ of $[\![1,N]\!]$ such that for all $p \neq q$ in $[\![1,\ell]\!]$, for all $i,j \in K_p$ and all $k \in K_q$, $x^i = x^j$ and $x^i \neq x^k$. Since $x \in E_{k_0}$, it holds that $\max_{p \in [\![1,\ell]\!]} |K_p| \leq k_0 - 1$. By definition of O_x and d_x, we see that for all $y \in O_x$, for all $p \neq q$ in $[\![1,\ell]\!]$, for all $i \in K_p$, all $j \in K_q$,

$$\|y^i - y^j\| \geq \|x^i - x^j\| - \|x^i - y^i\| - \|x^j - y^j\| \geq \|x^i - x^j\| - 2d_x \geq d_x.$$

This implies that for some finite constant C depending on x, for all $y \in O_x$,

$$\mathbf{m}(y) = \prod_{1 \leq i \neq j \leq N} ||y^i - y^j||^{-\theta/N} \leq C \prod_{p=1}^{\ell} \Big(\prod_{i,j \in K_p, i \neq j} ||y^i - y^j||^{-\theta/N} \Big).$$

Recall now that $\mu(\mathrm{d}y) = \mathbf{m}(y)\mathrm{d}y$ and that we want to show that $\mu(O_x) < \infty$. Since $x^i = x^j$ for all $i, j \in K_p$ and all $p \in [\![1, \ell]\!]$, since $|K_p| \le k_0 - 1$, $d_x \le 1$ and by a translation argument, we are reduced to show that for any $n \in [\![2, k_0 - 1]\!]$, (when $k_0 > N$, one could study only $n \in [\![2, N]\!]$)

$$I_n = \int_{(B(0,1))^n} \Big(\prod_{1 \le i \ne j \le n} \|y^i - y^j\|^{-\theta/N} \Big) \mathrm{d}y^1 \dots \mathrm{d}y^n < \infty.$$

We fix $n \in [\![2, k_0 - 1]\!]$ and show that $I_n < \infty$. Since $\|u\|^2 \ge |u_1 u_2|$ for all $u = (u_1, u_2) \in \mathbb{R}^2$, we have $I_n \le J_n^2$, where

$$J_n = \int_{[-1,1]^n} \Big(\prod_{1 \le i \ne j \le n} |t^i - t^j|^{-\theta/(2N)} \Big) \mathrm{d}t^1 \dots \mathrm{d}t^n.$$

But for all $t^1, \dots, t^n \in \mathbb{R}$,

$$\prod_{1 \le i \ne j \le n} |t^i - t^j|^{-\theta/(2N)} = \prod_{i=1}^n \Big(\prod_{j=1, j \ne i}^n |t^i - t^j|^{-\theta/(2N)} \Big) \le \frac{1}{n} \sum_{i=1}^n \prod_{j=1, j \ne i}^n |t^i - t^j|^{-\theta n/(2N)}$$

by the inequality of arithmetic and geometric means. Thus by symmetry,

$$J_n \le \int_{[-1,1]^n} \Big(\prod_{j=2}^n |t^1 - t^j|^{-\theta n/(2N)} \Big) \mathrm{d}t^1 \dots \mathrm{d}t^n = \int_{-1}^1 \Big(\int_{-1}^1 |t^1 - t^2|^{-\theta n/(2N)} \mathrm{d}t^2 \Big)^{n-1} \mathrm{d}t^1.$$

Consequently,

$$J_n \le \int_{-1}^1 \Big(\int_{-2}^2 |s|^{-\theta n/(2N)} \mathrm{d}s \Big)^{n-1} \mathrm{d}t^1.$$

Since $n \le k_0 - 1 = \lceil 2N/\theta \rceil - 1 < 2N/\theta$, we have $\theta n/(2N) < 1$, so that $J_n < \infty$, whence $I_n < \infty$.

(ii) We next assume that $k_0 \in [\![2, N]\!]$. To prove that μ is not radon on E_{k_0+1}, we show that $\mu(K) = \infty$ for the compact subset

$$K = \prod_{i=1}^{k_0} \overline{B}(0,1) \times \prod_{k=k_0+1}^{N} \overline{B}((2k,0), 1/2)$$

of E_{k_0+1}. All the balls in the previous formula are balls of $\mathbb{R}^2$. For $x = (x^1, \dots, x^N) \in K$, it holds that $x^{k_0+1}, \dots, x^N$ are far from each other and far from $x^1, \dots, x^{k_0}$, which explains that K is indeed compact in E_{k_0+1}. There is a positive constant $c > 0$ such that for all $x \in K$,

$$\mathbf{m}(x) = \prod_{1 \le i \ne j \le N} ||x^i - x^j||^{-\theta/N} \ge c \prod_{1 \le i \ne j \le k_0} ||x^i - x^j||^{-\theta/N},$$

whence, the value of $c > 0$ being allowed to vary,

$$\mu(K) \ge c \int_{(B(0,1))^{k_0}} \Big(\prod_{1 \le i \ne j \le k_0} \|x^i - x^j\|^{-\theta/N} \Big) \mathrm{d}x^1 \dots \mathrm{d}x^{k_0}.$$

We now observe that

$$A = \{x = (x^1, \dots, x^{k_0}) : x^1, x^2 \in B(0, 1/3),\ \forall i \notin \{1,2\},\ x^i \in B(x^1, \|x^1 - x^2\|)\} \subset (B(0,1))^{k_0}$$

and that for $x \in A$, we have $||x^i - x^j|| \leq ||x^i - x^1|| + ||x^j - x^1|| \leq 2||x^1 - x^2||$ for all $i, j = 1, \dots, k_0$, from which

$$\prod_{1 \leq i \neq j \leq k_0} \|x^i - x^j\|^{-\theta/N} \geq c\|x^1 - x^2\|^{-k_0(k_0-1)\theta/N}.$$

As a conclusion,

$$\begin{aligned}
\mu(K) \geq & c \int_{(B(0,1/3))^2} \|x^1 - x^2\|^{-k_0(k_0-1)\theta/N} \mathrm{d}x^1 \mathrm{d}x^2 \int_{(B(x_1, \|x^1-x^2\|))^{k_0-2}} \mathrm{d}x^3 \dots \mathrm{d}x^{k_0} \\
\geq & c \int_{(B(0,1/3))^2} \|x^1 - x^2\|^{-k_0(k_0-1)\theta/N + 2(k_0-2)} \mathrm{d}x^1 \mathrm{d}x^2 \\
\geq & c \int_{B(0,1/3)} \|u\|^{-k_0(k_0-1)\theta/N + 2(k_0-2)} \mathrm{d}u,
\end{aligned}$$

where we finally used the change of variables $u = x^1 - x^2$ and $v = x^1 + x^2$. This last integral diverges, because $-k_0(k_0 - 1)\theta/N + 2(k_0 - 2) = d_{\theta,N}(k_0) - 2 \leq -2$, recall that $d_{\theta,N}(k_0) = (k_0 - 1)(2 - k_0\theta/N) \leq 0$ by definition of k_0. □

We need a similar result on the sphere $\mathbb{S}$ defined in Section 2.2, where $\gamma : \mathbb{R}^2 \to (\mathbb{R}^2)^N$ and $\Psi : \mathbb{R}^2 \times \mathbb{R}_+^* \times \mathbb{S} \to E_N \subset (\mathbb{R}^2)^N$ were also introduced. First, we show an explicit link between $\mu(\mathrm{d}x) = \mathbf{m}(x)\mathrm{d}x$ and $\beta(\mathrm{d}u) = \mathbf{m}(u)\sigma(\mathrm{d}u)$ defined in (2.4) and (2.8), that we use several times.

Lemma 2.31. *We fix $N \geq 2$, $\theta > 0$ and set $\nu = d_{\theta,N}(N)/2 - 1$. For all Borel $\varphi : (\mathbb{R}^2)^N \to \mathbb{R}_+$,*

$$\int_{(\mathbb{R}^2)^N} \varphi(x)\mu(\mathrm{d}x) = \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}_+^* \times \mathbb{S}} \varphi(\Psi(z, r, u)) r^\nu \mathrm{d}z \mathrm{d}r \beta(\mathrm{d}u).$$

Démonstration. Since $H = \{y = (y^1, \dots, y^N) \in (\mathbb{R}^2)^N : \sum_1^N y^i = 0\}$ and since $\mathbf{m}$ is translation invariant,

$$\int_{(\mathbb{R}^2)^N} \varphi(x)\mu(\mathrm{d}x) = \int_{(\mathbb{R}^2)^N} \varphi(x)\mathbf{m}(x)\mathrm{d}x = \int_{\mathbb{R}^2 \times H} \varphi(\gamma(z) + y)\mathbf{m}(y)\mathrm{d}z\mathrm{d}y.$$

We next note that $\mathbb{S}$ is the (true) unit sphere of the $(2N-2)$-dimensional Euclidean space H and proceed to the substitution $(\ell, u) = (\|y\|, y/||y||)$:

$$\int_{(\mathbb{R}^2)^N} \varphi(x)\mu(\mathrm{d}x) = \int_{\mathbb{R}^2 \times \mathbb{R}_+ \times \mathbb{S}} \varphi(\gamma(z) + \ell u)\mathbf{m}(\ell u)\ell^{2N-3} \mathrm{d}z\mathrm{d}\ell\sigma(\mathrm{d}u).$$

We finally substitute $\ell = \sqrt{r}$ and obtain

$$\int_{(\mathbb{R}^2)^N} \varphi(x)\mu(\mathrm{d}x) = \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}_+ \times \mathbb{S}} \varphi(\gamma(z) + \sqrt{r}u)\mathbf{m}(\sqrt{r}u)r^{N-2} \mathrm{d}z\mathrm{d}r\sigma(\mathrm{d}u).$$

But $\mathbf{m}(\sqrt{r}u)r^{N-2} = r^{N-2-\theta(N-1)/2}\mathbf{m}(u)$ by (2.4) and $\beta(\mathrm{d}u) = \mathbf{m}(u)\sigma(\mathrm{d}u)$, whence

$$\int_{(\mathbb{R}^2)^N} \varphi(x)\mu(\mathrm{d}x) = \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}_+ \times \mathbb{S}} \varphi(\Psi(z, r, u)) r^{N-2-\theta(N-1)/2} \mathrm{d}z\mathrm{d}r\beta(\mathrm{d}u).$$

Since finally $\nu = d_{\theta,N}(N)/2 - 1 = N - 2 - \theta(N-1)/2$, the conclusion follows. □

We can now study the measure β on $\mathbb{S}$.

Proposition 2.32. *Let $N \geq 2$ and $\theta > 0$ such that $N > \theta$. Recall that $k_0 = \lceil 2N/\theta \rceil$.*

(i) The measure β is Radon on $\mathbb{S} \cap E_{k_0}$.

(ii) If $k_0 \geq N$, then $\beta(\mathbb{S}) < \infty$.

Démonstration. We start with (i). For $\varepsilon \in (0,1]$, we introduce

$$\mathcal{K}_\varepsilon = \{x \in (\mathbb{R}^2)^N : \forall K \subset [\![1,N]\!] \text{ such that } |K| \geq k_0, \text{ we have } R_K(x) \geq \varepsilon\} \quad \text{and} \quad \mathcal{L}_\varepsilon = \mathcal{K}_\varepsilon \cap \mathbb{S}.$$

Since $\mathcal{K}_\varepsilon \cap \overline{B}(0,1)$ is compact in E_{k_0}, with here $B(0,1)$ the unit ball of $(\mathbb{R}^2)^N$, we know from Proposition 2.30-(i) that $\mu(\mathcal{K}_\varepsilon \cap B(0,1)) < \infty$. Now by Lemma 2.31,

$$\mu(\mathcal{K}_\varepsilon \cap B(0,1)) = \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}_+ \times \mathbb{S}} \mathbb{I}_{\{\gamma(z) + \sqrt{r}u \in \mathcal{K}_\varepsilon \cap B(0,1)\}} r^\nu \mathrm{d}z \mathrm{d}r \beta(\mathrm{d}u).$$

But for $(z,r,u) \in \mathbb{R}^2 \times \mathbb{R}_+ \times \mathbb{S}$,

$$\gamma(z) + \sqrt{r}u \in \mathcal{K}_\varepsilon \cap B(0,1) \qquad \text{if and only if} \qquad u \in \mathcal{L}_{\varepsilon/r} \quad \text{and} \quad N||z||^2 + r < 1.$$

Indeed, $R_K(\gamma(z) + \sqrt{r}u) = rR_K(u)$ for all $K \subset [\![1,N]\!]$ and $||\gamma(z) + \sqrt{r}u||^2 = \sum_1^N ||z + \sqrt{r}u^i||^2 = N||z||^2 + r$ because $\sum_1^N u^i = 0$ and $\sum_1^N ||u^i||^2 = 1$. Thus

$$\mu(\mathcal{K}_\varepsilon \cap B(0,1)) = \int_{\mathbb{R}^2 \times \mathbb{R}_+} \mathbb{I}_{\{N||z||^2 + r < 1\}} r^\nu \beta(\mathcal{L}_{\varepsilon/r}) \mathrm{d}z \mathrm{d}r.$$

All this implies that for all $\varepsilon \in (0,1]$, for almost all $r \in (0,1)$, $\beta(\mathcal{L}_{\varepsilon/r}) < \infty$. Since $\varepsilon \to \mathcal{L}_\varepsilon$ is monotone, we conclude that $\beta(\mathcal{L}_\varepsilon) < \infty$ for all $\varepsilon \in (0,1]$. Since finally $\cup_{\varepsilon \in (0,1]} \mathcal{L}_\varepsilon = \mathbb{S} \cap E_{k_0}$ and since $\mathcal{L}_\varepsilon$ is compact in $\mathbb{S} \cap E_{k_0}$ for each $\varepsilon \in (0,1]$, we conclude as desired that β is Radon on $\mathbb{S} \cap E_{k_0}$.

We next prove (ii). It holds that $\mathbb{S} \subset E_N$, because for $u \in \mathbb{S}$, we have $R_{[\![1,N]\!]}(u) = 1$. Hence if $k_0 \geq N$, then $\mathbb{S} \subset E_N \subset E_{k_0}$, whence $\mathbb{S} = \mathbb{S} \cap E_{k_0}$ and thus β is Radon on $\mathbb{S}$ by point (i). Since finally $\mathbb{S}$ is compact, we conclude that $\beta(\mathbb{S}) < \infty$. □

2.13 Appendix : Markov processes and Dirichlet spaces

In a first subsection, we recall some classical definitions and results about Hunt processes, diffusions and Dirichlet spaces found in Fukushima-Oshima-Takeda [24]. In a second subsection, we mention a few results about martingales, times-changes, concatenation, killing and Girsanov transformation of Hunt processes found in [24] and elsewhere.

2.13.1 Main definitions and properties

Let E be a locally compact separable metrizable space endowed with a Radon measure α such that Supp $\alpha = E$. We set $E_\triangle = E \cup \{\triangle\}$, where $\triangle$ is a cemetery point. See [24, Section A2] for the definition of a Hunt process $\mathbb{Y} = (\Omega, \mathcal{M}, (Y_t)_{t \geq 0}, (\mathbb{P}_y)_{y \in E_\triangle})$: it is a strong Markov process in its canonical filtration, $\mathbb{P}_y(Y_0 = y) = 1$ for all $y \in E_\triangle$, $\triangle$ is an absorbing state, i.e. $Y_t = \triangle$ for all $t \geq 0$ under $\mathbb{P}_\triangle$, and a few more technical properties are satisfied. The life-time of $\mathbb{Y}$ is defined by $\zeta = \inf\{t \geq 0 : Y_t = \triangle\}$.

Let us denote by $P_t(y, \mathrm{d}z)$ its transition kernel. Our Hunt process is said to be α-symmetric if $\int_E \varphi P_t \psi \mathrm{d}\alpha = \int_E \psi P_t \varphi \mathrm{d}\alpha$ for all measurable $\varphi, \psi : E \to \mathbb{R}_+$ and all $t \geq 0$, see [24, page 30]. The Dirichlet space $(\mathcal{E}, \mathcal{F})$ of our Hunt process on $L^2(E, \alpha)$ is then defined, see [24, page 23], by

$$\mathcal{F} = \Big\{\varphi \in L^2(E, \alpha) : \lim_{t\to 0} \frac{1}{t} \int_E \varphi(P_t\varphi - \varphi)\mathrm{d}\alpha \text{ exists}\Big\},$$
$$\mathcal{E}(\varphi, \psi) = -\lim_{t\to 0} \frac{1}{t} \int_E \varphi(P_t\psi - \psi)\mathrm{d}\alpha \qquad \text{for all } \varphi, \psi \in \mathcal{F}.$$

The generator $(\mathcal{A}, \mathcal{D}_A)$ of $\mathbb{Y}$ is defined as follows :

$$\mathcal{D}_{\mathcal{A}} = \Big\{\varphi \in L^2(E, \alpha) : \lim_{t\to 0} \frac{1}{t}(P_t\varphi - \varphi) \text{ exists in } L^2(E, \alpha)\Big\},$$

and for $\varphi \in \mathcal{D}_{\mathcal{A}}$, we denote by $\mathcal{A}\varphi \in L^2(E, \alpha)$ this limit. By [24, Pages 20-21], it holds that

$$\mathcal{D}_{\mathcal{A}} = \Big\{\varphi \in \mathcal{F} : \exists\, h \in L^2(E, \alpha) \text{ such that } \forall\, \psi \in \mathcal{F}, \;\text{ we have }\; \mathcal{E}(\varphi, \psi) = -\int_E h\psi \mathrm{d}\alpha\Big\} \tag{2.70}$$

and in such a case $\mathcal{A}\varphi = h$.

The one-point compactification $E_\triangle = E \cup \{\triangle\}$ of E is endowed with the topology consisting of all the open sets of E and of all the sets of the form $K^c \cup \{\triangle\}$ with K compact in E, see page [24, page 69]. Observe that for a $E_\triangle$-valued sequence $(x_n)_{n\geq 0}$, we have $\lim_n x_n = x$ if and only if

- either $x \in E$, $x_n \in E$ for all n large enough, and $\lim_n x_n = x \in E$ in the usual sense;
- or $x = \triangle$ and for all compact subset K of E, there is $n_K \in \mathbb{N}$ such that for all $n \geq n_K$, $x_n \notin K$.

We say that our Hunt process is continuous if $t \to Y_t$ is continuous from $\mathbb{R}_+$ into $E_\triangle$, where $E_\triangle$ is endowed with the one-point compactification topology. A continuous Hunt process is called a *diffusion*.

A Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(E, \alpha)$ is said to be regular if it has a core, see [24, page 6], i.e. a subset $\mathcal{C} \subset C_c(E) \cap \mathcal{F}$ which is dense in $\mathcal{F}$ for the norm $||\varphi|| = [\int_E \varphi^2 \mathrm{d}\alpha + \mathcal{E}(\varphi, \varphi)]^{1/2}$ and dense in $C_c(E)$ for the uniform norm.

Observe two regular Dirichlet spaces $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}', \mathcal{F}')$ such that $\mathcal{E}(\varphi, \varphi) = \mathcal{E}'(\varphi, \varphi)$ for all φ in a common core $\mathcal{C}$ are necessarily equal, i.e. $\mathcal{F} = \mathcal{F}'$ and $\mathcal{E} = \mathcal{E}'$. This follows from the fact that by definition, see [24, page 5], a Dirichlet space is closed.

We say that a Borel set A of E is $(P_t)_{t\geq 0}$-invariant if for all $\varphi \in L^2(E, \alpha)$, all $t > 0$ we have $P_t(\mathbb{1}_A\varphi) = \mathbb{1}_A P_t\varphi$ α-a.e, see [24, page 53]. According to [24, page 55], we say that $(\mathcal{E}, \mathcal{F})$ is irreducible if for all $(P_t)_{t\geq 0}$-invariant set A, we have either $\alpha(A) = 0$ or $\alpha(E \setminus A) = 0$.

We say that $(\mathcal{E}, \mathcal{F})$ is recurrent if for all nonnegative $\varphi \in L^1(E, \alpha)$, for α-a.e. $y \in E$, we have $\mathbb{E}_y[\int_0^\infty \varphi(Y_s)\mathrm{d}s] \in \{0, \infty\}$, see [24, page 55].

We finally say that $(\mathcal{E}, \mathcal{F})$ is transient if for all nonnegative $\varphi \in L^1(E, \alpha)$, for α-a.e. $y \in E$, we have $\mathbb{E}_y[\int_0^\infty \varphi(Y_s)\mathrm{d}s] < \infty$, with the convention that $\varphi(\triangle) = 0$, see [24, page 55].

By [24, Lemma 1.6.4 page 55], if $(\mathcal{E}, \mathcal{F})$ is irreducible, then it is either recurrent or transient.

A Borel set $\mathcal{N} \subset E$ is properly exceptional if $\alpha(\mathcal{N}) = 0$ and $\mathbb{P}_y(\exists t \geq 0 : Y_t \in \mathcal{N}) = 0$ for all $y \in E \setminus \mathcal{N}$, see [24, page 153]. A property is said to hold true quasi-everywhere if it holds true outside a properly exceptional set.

Remark 2.33. *Two Hunt processes with the same Dirichlet space share the same quasi-everywhere notion, up to the restriction that the capacity of every compact set is finite, which is always the case in the present work.*

Démonstration. We fix a Hunt process $\mathbb{Y}$ and explain why its quasi-everywhere notion depends only on its Dirichlet space. A set $\mathcal{N} \subset E$ is exceptional, see [24, page 152], if there exists a Borel set $\tilde{\mathcal{N}}$ such that $\mathcal{N} \subset \tilde{\mathcal{N}}$ and $\mathbb{P}_y(\exists t \geq 0 : Y_t \in \tilde{\mathcal{N}}) = 0$ for α-a.e. $y \in E$. A properly exceptional set is clearly exceptional and [24, Theorem 4.1.1 page 155] tells us that any exceptional set is included in a properly exceptional set. Thus, a property is true quasi-everywhere if and only if it holds true outside an exceptional set. Next, [24, Theorem 4.2.1-(ii) page 161] tells us that a set $\mathcal{N}$ is exceptional if and only if its capacity is 0, where the capacity of $\mathcal{N} \subset E$ is entirely defined from the Dirichlet space. And for [24, Theorem 4.2.1-(ii) page 161] to apply, one needs that the capacity of all compact sets is finite. □

2.13.2 Toolbox

We start with martingales.

Lemma 2.34. *Let E be a locally compact separable metrizable space endowed with a Radon measure α such that Supp $\alpha = E$, and $(\Omega, \mathcal{M}, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in E_\triangle})$ a α-symmetric $E_\triangle$-valued diffusion with regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(E, \alpha)$ and generator $(\mathcal{A}, \mathcal{D}_\mathcal{A})$. Assume that $\varphi : E \mapsto \mathbb{R}$ belongs to $\mathcal{D}_\mathcal{A}$ and that both φ and $\mathcal{A}\varphi$ are bounded. Define*

$$M_t^\varphi = \varphi(Z_t) - \varphi(Z_0) - \int_0^t \mathcal{A}\varphi(Z_s)\mathrm{d}s,$$

with the convention that $\varphi(\triangle) = \mathcal{A}\varphi(\triangle) = 0$. Quasi-everywhere, $(M_t^\varphi)_{t\geq 0}$ is a $\mathbb{P}_z$-martingale in the canonical filtration of $(Z_t)_{t\geq 0}$.

This can be found in [24, page 332]. There the assumption on φ is that there is f bounded and measurable such that $\varphi = R_1 f$, i.e. $\varphi = (I - \mathcal{A})^{-1} f$, which simply means that $\varphi - \mathcal{A}\varphi$ is bounded. Also, the conclusion is that $(M_t^\varphi)_{t\geq 0}$ is a MAF, which indeed implies that $(M_t^\varphi)_{t\geq 0}$ is a martingale, see [24, page 243].

Next, we deal with time-changes.

Lemma 2.35. *Let E be a C^∞-manifold, α a Radon measure on E such that* $\mathrm{Supp}(\alpha) = E$, *and $(\Omega, \mathcal{M}, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in E_\triangle})$ a α-symmetric $E_\triangle$-valued diffusion with regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(E, \alpha)$ with core $C_c^\infty(E)$. We also fix $g : E \to (0, \infty)$ continuous and take the convention that $g(\triangle) = 0$. We consider the time-change $A_t = \int_0^t g(Z_s)\mathrm{d}s$ and its generalized inverse $\rho_t = \inf\{s > 0 : A_s > t\}$. We introduce $Y_t = Z_{\rho_t} \mathbb{1}_{\{\rho_t < \infty\}} + \triangle \mathbb{1}_{\{\rho_t = \infty\}}$. Then $(\Omega, \mathcal{M}, (Y_t)_{t\geq 0}, (\mathbb{P}_y)_{y\in E_\triangle})$ is a $g\alpha$-symmetric $E_\triangle$-valued diffusion with regular Dirichlet space $(\mathcal{E}, \mathcal{F}')$ on $L^2(E, g\alpha)$ with core $C_c^\infty(E)$, i.e. $\mathcal{F}'$ is the closure of $C_c^\infty(E)$ with respect to the norm $[\int_E \varphi^2 g \mathrm{d}\alpha + \mathcal{E}(\varphi, \varphi)]^{1/2}$.*

Remark 2.36. *If we apply the preceding result to the simple case where E is an open subset of $\mathbb{R}^d$ and where $\mathcal{E}(\varphi, \varphi) = \int_{\mathbb{R}^d} \|\nabla\varphi\|^2 \mathrm{d}\alpha$ for all $\varphi \in C_c^\infty(E)$, then when $\mathcal{E}$ is seen as the Dirichlet form of a $g\alpha$-symmetric process, it may be better understood as $\mathcal{E}(\varphi, \varphi) = \int_{\mathbb{R}^d} \|g^{-1/2}\nabla\varphi\|^2 g\mathrm{d}\alpha$.*

This lemma is nothing but a particular case of [24, Theorem 6.2.1 page 316], see also the few pages before. We only have to check that the Revuz measure in our case is $g\alpha$, i.e., see [24, (5.1.13) page 229], that for all bounded nonnegative measurable functions φ, ψ on E, for all $t > 0$,

$$\int_E \mathbb{E}_x\Big[\int_0^t \varphi(Z_s)g(Z_s)\mathrm{d}s\Big]\psi(x)\alpha(\mathrm{d}x) = \int_0^t \int_E (P_s^Z\psi)\varphi g\mathrm{d}\alpha,$$

where P_t^Z is the semi-group of Z. The left hand side equals $\int_0^t \int_E P_s^Z(\varphi g)\psi \mathrm{d}\alpha$, so that the claim is obvious since Z is α-symmetric.

The following concatenation result can be found in Li-Ying [37, Proposition 3.2].

Lemma 2.37. *Let E_V, E_W be two C^∞-manifolds, α_V, α_W be some Radon measures on E_V and E_W such that* $\mathrm{Supp}(\alpha_V) = E_V$ *and* $\mathrm{Supp}(\alpha_W) = E_W$. *Let $(\Omega^V, \mathcal{M}^V, (V_t)_{t\geq 0}, (\mathbb{P}_v^V)_{v\in E_V\cup\{\triangle\}})$ be a α_V-symmetric $(E_V \cup \{\triangle\})$-valued diffusion with regular Dirichlet space $\big(\mathcal{E}^V, \mathcal{F}^V\big)$ on $L^2(E_V, \alpha_V)$ with core $C_c^\infty(E_V)$. Consider $(\Omega^W, \mathcal{M}^W, (W_t)_{t\geq 0}, (\mathbb{P}_w^W)_{w\in E_W\cup\{\triangle\}})$, a α_W-symmetric $(E_W \cup \{\triangle\})$-valued diffusion with regular Dirichlet space $\big(\mathcal{E}^W, \mathcal{F}^W\big)$ on $L^2(E_W, \alpha_W)$ with core $C_c^\infty(E_W)$. Introduce the measure $\alpha = \alpha_V \otimes \alpha_W$ on $E = E_V \times E_W$. We take the convention that $(v, \triangle) = (\triangle, w) = (\triangle, \triangle) = \triangle$ for all $v \in E_V$, all $w \in E_W$. Moreover, we set $\mathcal{M}^{(V,W)} = \sigma(\{(V_t, W_t) : t \geq 0\})$ and we define $\mathbb{P}_{(v,w)}^{(V,W)} = \mathbb{P}_v^V \otimes \mathbb{P}_w^W$ if $(v, w) \in E_V \times E_W$ and $\mathbb{P}_\triangle^{(V,W)} = \mathbb{P}_\triangle^V \otimes \mathbb{P}_\triangle^W$. The process*

$$\Big(\Omega^V \times \Omega^W, \mathcal{M}^{(V,W)}, (V_t, W_t)_{t\geq 0}, (\mathbb{P}_{(v,w)}^{(V,W)})_{(v,w)\in(E_V\times E_W)\cup\{\triangle\}}\Big)$$

is a $E_\triangle$-valued α-symmetric diffusion , with regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(E, \alpha)$ with core $C_c^\infty(E)$ and, for $\varphi \in C_c^\infty(E)$,

$$\mathcal{E}(\varphi, \varphi) = \int_{E_V} \mathcal{E}^W(\varphi(v, \cdot), \varphi(v, \cdot))\alpha_V(\mathrm{d}v) + \int_{E_W} \mathcal{E}^V(\varphi(\cdot, w), \varphi(\cdot, w))\alpha_W(\mathrm{d}w).$$

Observe that $\mathcal{M}^{(V,W)}$ may be strictly smaller than $\mathcal{M}^V \otimes \mathcal{M}^W$ due to the identification of all the cemetery points. Also, it actually holds true that $\mathbb{P}_\triangle^V \otimes \mathbb{P}_w^W = \mathbb{P}_v^V \otimes \mathbb{P}_\triangle^W = \mathbb{P}_\triangle^V \otimes \mathbb{P}_\triangle^W$ on $\mathcal{M}^{(V,W)}$ so that the choice $\mathbb{P}_\triangle^{(V,W)} = \mathbb{P}_\triangle^V \otimes \mathbb{P}_\triangle^W$ is arbitrary but legitimate.

The following killing result is a summary, adapted to our context, of Theorems 4.4.2 page 173 and 4.4.3-(i) page 174 in [24, Section 4.4].

Lemma 2.38. *Let E be a C^∞-manifold, let α be a Radon measure on E such that* $\mathrm{Supp}(\alpha) = E$, *and let $(\Omega, \mathcal{M}, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in E_\triangle})$ be a α-symmetric $E_\triangle$-valued diffusion with regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(E, \alpha)$ with core $C_c^\infty(E)$. Let O be an open subset of E and consider $\tau_O = \inf\{t \geq 0 : X_t \notin O\}$, with the convention that* $\inf \emptyset = \infty$. *Then, setting*

$$Z_t^O = Z_t \mathbb{1}_{\{t<\tau_O\}} + \triangle \mathbb{1}_{\{t\geq\tau_O\}},$$

$(\Omega, \mathcal{M}, (Z_t^O)_{t\geq 0}, (\mathbb{P}_z)_{z\in O\cup\{\triangle\}})$ is a $\alpha|_O$-symmetric $O \cup \{\triangle\}$-valued diffusion with regular Dirichlet space $(\mathcal{E}_O, \mathcal{F}_O)$ on $L^2(O, \alpha|_O)$ with core $C_c^\infty(O)$ and for $\varphi \in \mathcal{F}_O$,

$$\mathcal{E}_O(\varphi, \varphi) = \mathcal{E}(\varphi, \varphi).$$

Note that since O is an open subset of the manifold E and since the Hunt process is continuous, the regularity condition (4.4.6) of [24, Theorem 4.4.2 page 173] is obviously satisfied.

We finally give an adaptation of the Girsanov theorem in the context of Dirichlet spaces, which is a particular case of Chen-Zhang [15, Theorem 3.4].

Lemma 2.39. *Let E be an open subset of $\mathbb{R}^d$, with $d \geq 1$, α be a Radon measure on E such that* $\mathrm{Supp}(\alpha) = E$ *and* $(\Omega, \mathcal{M}, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in E_\triangle})$ *be a α-symmetric $E_\triangle$-valued diffusion with regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(E, \alpha)$ with core $C_c^\infty(E)$ such that for all $\varphi \in C_c^\infty(E)$,*

$$\mathcal{E}(\varphi, \varphi) = \int_E \|\nabla\varphi\|^2 \mathrm{d}\alpha.$$

Let $(\mathcal{A}, \mathcal{D}_\mathcal{A})$ stand for its generator. Let $u \in \mathcal{F}$ be bounded, such that for $\varrho = e^u$, we have $\varrho - 1 \in \mathcal{D}_\mathcal{A}$ with $\mathcal{A}[\varrho - 1]$ is bounded. Set

$$L_t^\varrho = \frac{\varrho(Z_t)}{\varrho(Z_0)} \exp\Big(- \int_0^t \frac{\mathcal{A}[\varrho - 1](Z_s)}{\varrho(Z_s)} \mathrm{d}s \Big),$$

with the conventions that $\varrho(\triangle) = 1$ and $\mathcal{A}[\varrho - 1](\triangle) = 0$.

Assume that ϱ is continuous on $E_\triangle$. Then quasi-everywhere, $(L_t^\varrho)_{t\geq 0}$ is a bounded $(\mathcal{M}_t)_{t\geq 0}$-martingale under $\mathbb{P}_z$, where we have set $\mathcal{M}_t = \sigma(\{Z_s : s \in [0, t]\})$, and there exists a probability measure $\tilde{\mathbb{P}}_z$ on $(\Omega, \mathcal{M})$, such that for all $t > 0$, $\tilde{\mathbb{P}}_z = L_t^\varrho \cdot \mathbb{P}_z$ on $\mathcal{M}_t$.

Moreover $(\Omega,\mathcal{M},(Z_t)_{t\geq 0},(\tilde{\mathbb{P}}_z)_{z\in E_\triangle})$ is a $\varrho^2\alpha$-symmetric $E_\triangle$-valued diffusion with regular Dirichlet space $(\tilde{\mathcal{E}}, \mathcal{F})$ on $L^2(E, \varrho^2\alpha)$ such that for all $\varphi \in \mathcal{F}$,

$$\tilde{\mathcal{E}}(\varphi, \varphi) = \frac{1}{2} \int_E \|\nabla\varphi\|^2 \varrho^2 \mathrm{d}\alpha.$$

Actually, they speak of *right processes* in [15], but this is not an issue since we only consider continuous Hunt processes. Also, they assume that L^ϱ is bounded from above and from below by some deterministic constants, on each compact time interval, but this is obvious under our assumptions on u and $\mathcal{A}\varrho$. Finally, their expression of L^ϱ is different, see [15, pages 485-486] : first, they define M_t^ϱ as the martingale part of $\varrho(X_t)$. By Lemma 2.34 (applied to $\varrho - 1$), we see that

$$M_t^\varrho = \varrho(Z_t) - \varrho(Z_0) - \int_0^t \mathcal{A}[\varrho - 1](Z_s)\mathrm{d}s.$$

Then they put $M_t = \int_0^t [\varrho(Z_s)]^{-1} \mathrm{d}M_s^\varrho$ and define L^ϱ as

$$L_t^\varrho = \exp\Big(M_t - \frac{1}{2}\langle M\rangle_t \Big).$$

But by Itô's formula, $\log\varrho(Z_t) = \log\varrho(Z_0) + \int_0^t [\varrho(Z_s)]^{-1}\mathrm{d}M_s^\varrho + \int_0^t [\varrho(Z_s)]^{-1}\mathcal{A}[\varrho - 1](Z_s)\mathrm{d}s - \frac{1}{2}\int_0^t [\varrho(Z_s)]^{-2}\mathrm{d}\langle M^\varrho\rangle_s$, whence $\log\varrho(Z_t) = \log\varrho(Z_0) + M_t + \int_0^t [\varrho(Z_s)]^{-1}\mathcal{A}[\varrho - 1](Z_s)\mathrm{d}s - \frac{1}{2}\langle M\rangle_t$, so that $L_t^\varrho = \exp(M_t - \frac{1}{2}\langle M\rangle_t) = [\varrho(Z_0)]^{-1}\varrho(Z_t)\exp(-\int_0^t \varrho(Z_s)^{-1}\mathcal{A}[\varrho - 1](Z_s)\mathrm{d}s)$ as desired.

Chapitre 3

A simple proof of non-explosion for measure solutions of the Keller-Segel equation

Abstract. We give a simple proof, relying on a *two-particles* moment computation, that there exists a global weak solution to the 2-dimensional parabolic-elliptic Keller-Segel equation when starting from any initial measure f_0 such that $f_0(\mathbb{R}^2) < 8\pi$.

3.1 Introduction

3.1.1 The model

We consider the classical parabolic-elliptic Keller-Segel model, also called Patlak-Keller-Segel, of chemotaxis in $\mathbb{R}^2$, which writes

$$\partial_t f + \nabla \cdot (f \nabla c) = \Delta f \qquad \text{and} \qquad \Delta c + f = 0. \tag{3.1}$$

The unknown (f, c) is composed of two nonnegative functions $f_t(x)$ and $c_t(x)$ of $t \geq 0$ and $x \in \mathbb{R}^2$, and the initial condition f_0 is given.

This equation models the collective motion of a population of bacteria which emit a chemical substance that attracts them. The quantity $f_t(x)$ represents the density of bacteria at position $x \in \mathbb{R}^2$ at time $t \geq 0$, while $c_t(x)$ represents the concentration of chemical substance at position $x \in \mathbb{R}^2$ at time $t \geq 0$. Note that in this model, the speed of diffusion of the chemo-attractant is supposed to be infinite. This equation has been introduced by Keller and Segel [35], see also Patlak [43]. We refer to the recent book of Biler [3] and to the review paper of Arumugam and Tyagi [1] for some complete descriptions of what is known about this model.

We classically observe, see e.g. Blanchet-Dolbeault-Perthame [9, Page 4], that necessarily $\nabla c_t = K * f_t$ for each $t \geq 0$, where

$$K(x) = -\frac{x}{2\pi \|x\|^2} \quad \text{for } x \in \mathbb{R}^2 \setminus \{0\} \quad \text{and (arbitrarily)} \quad K(0) = 0.$$

Hence (3.1) may be rewritten as

$$\partial_t f + \nabla \cdot [f\ (K * f)] = \Delta f. \tag{3.2}$$

3.1.2 Weak solutions

We will deal with weak measure solutions. For each $M > 0$, we set

$$\mathcal{M}_M(\mathbb{R}^2) = \Big\{\mu \text{ nonnegative measure on } \mathbb{R}^2 \text{ such that } \mu(\mathbb{R}^2) = M\Big\}$$

and we endow $\mathcal{M}_M(\mathbb{R}^2)$ with the weak convergence topology, i.e. taking $C_b(\mathbb{R}^d)$, the set of continuous and bounded functions, as set of test functions. We also denote by $C_b^2(\mathbb{R}^d)$ the set of C^2-functions, bounded together with all their derivatives. The following notion of weak solutions is classical, see e.g. Blanchet-Dolbeault-Perthame [9, Page 5].

Definition 3.1. *Fix $M > 0$. We say that $f \in C([0,\infty), \mathcal{M}_M(\mathbb{R}^2))$ is a weak solution of (3.2) if for all $\varphi \in C_b^2(\mathbb{R}^2)$, all $t \geq 0$,*

$$\int_{\mathbb{R}^2} \varphi(x) f_t(\mathrm{d}x) = \int_{\mathbb{R}^2} \varphi(x) f_0(\mathrm{d}x) + \int_0^t \int_{\mathbb{R}^2} \Delta\varphi(x) f_s(\mathrm{d}x)\mathrm{d}s$$
$$+ \frac{1}{2}\int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} K(x-y)\cdot(\nabla\varphi(x) - \nabla\varphi(y)) f_s(\mathrm{d}x) f_s(\mathrm{d}y)\mathrm{d}s.$$

All the terms in this equality are well-defined. In particular concerning the last term, it holds that $|K(x-y)\cdot(\nabla\varphi(x)-\nabla\varphi(y))| \leq \|\nabla^2\varphi\|_\infty/2\pi$. However, $K(x-y)\cdot(\nabla\varphi(x)-\nabla\varphi(y))$, which equals 0 when $x = y$ because we (arbitrarily) imposed that $K(0) = 0$, is not continuous near $x = y$. Hence a *good* weak solution has to verify that $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathbb{1}_{\{x=y\}} f_s(\mathrm{d}x) f_s(\mathrm{d}y) = 0$ for a.e. $s \geq 0$.

3.1.3 Main result

Our goal is to give a simple proof of the following global existence result.

Theorem 3.2. *Fix $M \in (0, 8\pi)$ and assume that $f_0 \in \mathcal{M}_M(\mathbb{R}^2)$. There exists a global weak solution f to (3.2) with initial condition f_0. Moreover, for all $\gamma \in (M/(4\pi), 2)$, there is a constant $A_{M,\gamma} > 0$ depending only on M and γ such that for all $T > 0$,*

$$\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \|x-y\|^{\gamma-2} f_s(\mathrm{d}x) f_s(\mathrm{d}y)\mathrm{d}s \leq A_{M,\gamma}(1+T). \tag{3.3}$$

These solutions indeed satisfy that $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathbb{1}_{\{x=y\}} f_s(\mathrm{d}x) f_s(\mathrm{d}y) = 0$ for a.e. $s \geq 0$.

3.1.4 References

Actually, a stronger result is already known : gathering the results of Bedrossian-Masmoudi [2] and Wei [51], there exists a global *mild* solution for any $f_0 \in \mathcal{M}_M(\mathbb{R}^2)$ with $M < 8\pi$. But the proof in [2, 51] is long and complicate, and the goal of the present paper is to provide a simple and robust non explosion proof, even if the solution we build is weaker. Actually, a global solution is also built in [2, 51] when $f_0 \in \mathcal{M}_{8\pi}(\mathbb{R}^2)$ satisfies $\max_{x\in\mathbb{R}^2} f_0(\{x\}) < 8\pi$, a case we could also treat with a little more work.

This model was first introduced by Patlak [43] and Keller-Segel [35], as a model for chemotaxis. For an exhaustive summary of the knowledge about this equation and related models, we refer the reader to the review paper Arumugam-Tyagi [1] and to the book of Biler [3]. The main difficulty of this model lies in the tight competition between diffusion and attraction. Therefore it is not clear that a solution exists because a blow-up could occur due to the emergence of a cluster, i.e. a Dirac mass. Thus, the whole problem is about determining if the solution ends-up by being concentrated in finite time or not.

As shown in Jäger-Luckhaus [31], this depends on the initial mass of the solution $M = \int_{\mathbb{R}^2} f_0(\mathrm{d}x)$, the solution globally exists if M is small enough and explodes in the other case. The fact that solutions must explode in finite time if $M > 8\pi$ is rather easy to show. But the fact that 8π is indeed the correct threshold was much more difficult.

Biler-Karch-Laurençot-Nadzieja [4, 5] proved the global existence of a weak solution in the subcritical case for every initial data which is a radially symmetric measure such that $f_0(\{0\}) = 0$ and $f_0(\mathbb{R}^2) = M \leq 8\pi$, with a few other anodyne technical conditions.

At the same time, Blanchet-Dolbeault-Perthame [9] proved the existence of a global weak *free energy* solution for initial data $f_0 \in L^1_+(\mathbb{R}^2)$ with mass $M < 8\pi$, a finite moment of order 2 and a finite entropy. The core of the argument lies in the use of the logarithmic Hardy-Littlewood-Sobolev inequality applied on a well chosen free-energy quantity. Something noticeable is that the authors use this inequality with its optimal constant to get the correct threshold 8π.

In Bedrossian-Masmoudi [2], it is proven that one can build mild solutions even in the supercritical case under the condition that $\max_{x\in\mathbb{R}^2} f_0(\{x\}) < 8\pi$, which are stronger solutions than weak solutions, but these solutions are local in time. Wei [51] built global mild solutions in the subcritical and critical cases and local mild solutions in the supercritical case for every initial data $f_0 \in L^1(\mathbb{R}^2)$, without any other condition. And these two last papers can be put together to build global mild solutions as soon as $f_0(\mathbb{R}^2) \leq 8\pi$ and $\max_{x\in\mathbb{R}^2} f_0(\{x\}) < 8\pi$.

Let us finally mention [20], where global weak solutions were built for any measure initial condition f_0 such that $f_0(\mathbb{R}^2) < 2\pi$, with a light additional moment condition. This work was inspired by the work of Osada [42] on vortices. The present paper consists in refining this approach, and surprisingly, this allows us to treat the whole subcritical case.

3.1.5 Motivation

Our main goal is to present a simple proof of non explosion. This proof relies on a two-particles moment computation : roughly, we show that for $\gamma \in (0, 2)$ and for $(f_t)_{t\geq 0}$ a solution to (3.2), it *a priori* holds that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Big(||x-y||^\gamma \wedge 1\Big) f_t(\mathrm{d}x) f_t(\mathrm{d}y) \geq c_{\gamma,M} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} ||x-y||^{\gamma-2} \mathbb{1}_{\{||x-y||\leq 1\}} f_t(\mathrm{d}x) f_t(\mathrm{d}y),$$

with $c_{\gamma,M} > 0$ as soon as $\gamma \in (4M/\pi, 2)$. By integration, this implies (3.3) and such an *a priori* estimate is sufficient to build a global solution.

This computation seems simple and robust. Although they build a more regular weak solution, Blanchet-Dolbeault-Perthame [9] use some optimal Hardy-Littlewood-Sobolev inequality. Moreover, they have some little restrictions on the initial conditions (finite entropy and moment of order

2). The proof of Bedrossian-Masmoudi [2] and Wei [51] is much longer and relies on a fine study of what happens near each possible atom of the initial condition. Let us say again that they build a much stronger solution.

In particular, due to its robusteness, we hope to be able to apply such a method to study the convergence of the empirical measure of some stochastic particle system, as the number of particles tends to infinity, to the solution of (3.2). To establish such a convergence, one needs to show the non explosion of the particle system, uniformly in N in some sense. It seems that the present method works very well and we hope to be able to treat the whole subcritical case $M \in (0, 8\pi)$ and even the critical case $M = 8\pi$. To our knowledge, only the case where $M \in (0, 4\pi)$ has been studied, by an entropy method, see Bresch-Jabin-Wang [11, 12].

We do not treat the criticial case $M = 8\pi$ in the present paper for the sake of conciseness.

3.2 Proof

We fix $M > 0$ and $f_0 \in \mathcal{M}_M(\mathbb{R}^2)$. For $\varepsilon \in (0, 1]$, we introduce the following regularized versions

$$K_\varepsilon(x) = -\frac{x}{2\pi(\|x\|^2 + \varepsilon)} \quad \text{and} \quad f_0^\varepsilon(x) = \frac{1}{2\pi\varepsilon}\int_{\mathbb{R}^2} e^{-\|x-y\|^2/(2\varepsilon)} f_0(\mathrm{d}y).$$

of K and f_0. Since K_ε and f_0^ε are smooth, the equation

$$\partial_t f^\varepsilon + \nabla \cdot [f^\varepsilon(K_\varepsilon * f^\varepsilon)] = \Delta f^\varepsilon \tag{3.4}$$

starting from f_0^ε has a unique classical solution $(f_t^\varepsilon(x))_{t\geq 0, x\in\mathbb{R}^2}$. This solution preserves mass, i.e

$$\int_{\mathbb{R}^2} f_t^\varepsilon(\mathrm{d}x) = \int_{\mathbb{R}^2} f_0^\varepsilon(\mathrm{d}x) = \int_{\mathbb{R}^2} f_0(\mathrm{d}x) = M \qquad \text{for all } t \geq 0, \tag{3.5}$$

where we write $f_t^\varepsilon(\mathrm{d}x) = f_t^\varepsilon(x)\mathrm{d}x$. Multiplying (3.4) by $\varphi \in C_b^2(\mathbb{R}^2)$, integrating on $[0, t] \times \mathbb{R}^2$, proceeding to some integrations by parts and using a symmetry argument, we classically find that

$$\begin{aligned}\int_{\mathbb{R}^2} \varphi(x) f_t^\varepsilon(\mathrm{d}x) = &\int_{\mathbb{R}^2} \varphi(x) f_0^\varepsilon(\mathrm{d}x) + \int_0^t \int_{\mathbb{R}^2} \Delta\varphi(x) f_s^\varepsilon(\mathrm{d}x)\mathrm{d}s \\ &+ \frac{1}{2}\int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} K_\varepsilon(x-y)\cdot[\nabla\varphi(x) - \nabla\varphi(y)] f_s^\varepsilon(\mathrm{d}x) f_s^\varepsilon(\mathrm{d}y)\mathrm{d}s.\end{aligned} \tag{3.6}$$

We now prove some compactness result.

Proposition 3.3. *Fix $M > 0$, $f_0 \in \mathcal{M}_M(\mathbb{R}^2)$ and consider the corresponding family $(f^\varepsilon)_{\varepsilon\in(0,1]}$. The family $(f^\varepsilon)_{\varepsilon\in(0,1]}$ is relatively compact in $C([0, \infty), \mathcal{M}_M(\mathbb{R}^2))$, endowed with the uniform convergence on compact time intervals, $\mathcal{M}_M(\mathbb{R}^2)$ being endowed with the weak convergence topology.*

Démonstration. We first prove that for each $t \geq 0$, the family $(f_t^\varepsilon)_{\varepsilon\in(0,1]}$ is tight in $\mathcal{P}(\mathbb{R}^2)$. Since the family $(f_0^\varepsilon)_{\varepsilon\in(0,1]}$ is clearly tight, by the de la Vallée Poussin theorem, there exists $\psi : \mathbb{R}^2 \to \mathbb{R}_+$ such that $A = \sup_{\varepsilon\in(0,1]} \int_{\mathbb{R}^2} \psi(x) f_0^\varepsilon(\mathrm{d}x) < \infty$ and $\lim_{|x|\to\infty} \psi(x) = \infty$. Moreover, we can choose ψ smooth and such that $||\nabla^2\psi||$ is bounded by some constant C. It then immediately follows from (3.6), since $\|z\|\,\|K_\varepsilon(z)\| \leq 1/(2\pi)$, that for all $\varepsilon \in (0, 1]$, all $t \geq 0$,

$$\int_{\mathbb{R}^2} \psi(x) f_t^\varepsilon(\mathrm{d}x) \leq \int_{\mathbb{R}^2} \psi(x) f_0^\varepsilon(\mathrm{d}x) + C\Big(M + \frac{M^2}{4\pi}\Big)t \leq A + C\Big(M + \frac{M^2}{4\pi}\Big)t.$$

As $\lim_{|x|\to\infty}\psi(x)=\infty$, we conclude that indeed, $(f_t^\varepsilon)_{\varepsilon\in(0,1]}$ is tight for each $t\geq 0$.

By the Arzela-Ascoli theorem, it is enough to prove that f^ε is uniformly Lipschitz continuous in time, in that there exists a constant $C>0$ such that for all $\varepsilon\in(0,1]$, all $t\geq s\geq 0$, $\delta(f_t^\varepsilon,f_s^\varepsilon)\leq C|t-s|$, where δ metrizes the weak convergence topology on $\mathcal{M}_M(\mathbb{R}^2)$. As is well-known, we may find a family $(\varphi_n)_{n\geq 0}$ of elements of $C_b^2(\mathbb{R}^2)$ satisfying

$$\|\varphi_n\|_\infty+\|\nabla\varphi_n\|_\infty+\|\nabla^2\varphi_n\|_\infty\leq 1 \qquad \text{for all } n\geq 0$$

and such that the distance δ on $\mathcal{M}_M(\mathbb{R}^2)$ defined through

$$\delta(f,g)=\sum_{n\geq 0}2^{-n}\Big|\int_{\mathbb{R}^2}\varphi_n(x)f(\mathrm{d}x)-\int_{\mathbb{R}^2}\varphi_n(x)g(\mathrm{d}x)\Big|$$

is suitable. But using (3.6), for all $n\geq 0$,

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^2}\varphi_n(x)f_t^\varepsilon(\mathrm{d}x)-\int_{\mathbb{R}^2}\varphi_n(x)f_s^\varepsilon(\mathrm{d}x)\Big|\\
=&\Big|\int_s^t\int_{\mathbb{R}^2}\Delta\varphi_n(x)f_u^\varepsilon(\mathrm{d}x)\mathrm{d}u\\
&+\frac{1}{2}\int_s^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}K_\varepsilon(x-y)\cdot[\nabla\varphi_n(x)-\nabla\varphi_n(y)]f_u^\varepsilon(\mathrm{d}x)f_u^\varepsilon(\mathrm{d}y)\mathrm{d}u\Big|\\
\leq&(M+M^2/(4\pi))(t-s),
\end{aligned}$$

by (3.5), since $\|\nabla^2\varphi_n\|_\infty\leq 1$ and since $\|z\|\,\|K_\varepsilon(z)\|\leq 1/(2\pi)$. We conclude that

$$\delta(f_t^\varepsilon,f_s^\varepsilon)\leq\sum_{n\geq 0}2^{-n}(M+M^2/(4\pi))(t-s)=2(M+M^2/(4\pi))(t-s)$$

as desired. □

The following simple geometrical observation is crucial for our purpose.

Lemma 3.4. *For all pair of nonincreasing functions $\varphi,\psi:(0,\infty)\to(0,\infty)$, for all $X,Y,Z\in\mathbb{R}^2$ such that $X+Y+Z=0$, we have*

$$\Delta=[\varphi(\|X\|)X+\varphi(\|Y\|)Y+\varphi(\|Z\|)Z]\cdot[\psi(\|X\|)X+\psi(\|Y\|)Y+\psi(\|Z\|)Z]\geq 0.$$

Démonstration. We may study only the case where $\|X\|\leq\|Y\|\leq\|Z\|$. Since $Y=-X-Z$,

$$\begin{aligned}
\varphi(\|X\|)X+\varphi(\|Y\|)Y+\varphi(\|Z\|)Z&=\lambda X-\mu Z,\\
\psi(\|X\|)X+\psi(\|Y\|)Y+\psi(\|Z\|)Z&=\lambda' X-\mu' Z,
\end{aligned}$$

where $\lambda=\varphi(\|X\|)-\varphi(\|Y\|)\geq 0$, $\mu=\varphi(\|Y\|)-\varphi(\|Z\|)\geq 0$, $\lambda'=\psi(\|X\|)-\psi(\|Y\|)\geq 0$ and $\mu'=\psi(\|Y\|)-\psi(\|Z\|)\geq 0$. Therefore,

$$\Delta=\lambda\lambda'\|X\|^2+\mu\mu'\|Z\|^2-(\lambda\mu'+\lambda'\mu)X\cdot Z\geq 0$$

as desired, because $X\cdot Z\leq 0$. Indeed, if $X\cdot Z>0$, then $\|Y\|^2=\|Z+X\|^2=\|Z\|^2+\|X\|^2+2X\cdot Z>\|Z\|^2\geq\|Y\|^2$, which is absurd. □

The following computation is the core of the paper.

Proposition 3.5. *Recall that $M \in (0, 8\pi)$ and $f_0 \in \mathcal{M}_M(\mathbb{R}^2)$. For all $\gamma \in (M/(4\pi), 2)$, there is a constant $A_{M,\gamma} > 0$ depending only on M and γ such that for all $\varepsilon \in (0,1]$, all $T > 0$,*

$$\int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \|x-y\|^{\gamma-2} f_s^\varepsilon(\mathrm{d}x) f_s^\varepsilon(\mathrm{d}y)\mathrm{d}s \le A_{M,\gamma}(1+T).$$

Démonstration. For any smooth $\psi : (\mathbb{R}^2)^2 \to \mathbb{R}$ such that $\psi(x,y) = \psi(y,x)$, it holds that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \psi(x,y) f_t^\varepsilon(\mathrm{d}x) f_t^\varepsilon(\mathrm{d}y) =& 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \psi(x,y)[\Delta f_t^\varepsilon(x) \\ & - \nabla \cdot (f_t^\varepsilon(x)(K_\varepsilon * f_t^\varepsilon))(x)] f_t^\varepsilon(y)\mathrm{d}x\mathrm{d}y \\ =& 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} [\Delta_x \psi(x,y) \\ & + (K_\varepsilon * f_t^\varepsilon)(x) \cdot \nabla_x \psi(x,y)] f_t^\varepsilon(x) f_t^\varepsilon(y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

We fix $\gamma \in (M/(4\pi), 2)$, introduce $\varphi(r) = r^{\gamma/2}/(1+r^{\gamma/2})$, and set $\psi(x,y) = \varphi(||x-y||^2)$. We have

$$\varphi'(r) = \frac{\gamma}{2} \frac{r^{\gamma/2-1}}{(1+r^{\gamma/2})^2} \qquad \text{and} \qquad \varphi''(r) = \frac{\gamma}{2} \frac{r^{\gamma/2-2}}{(1+r^{\gamma/2})^2} \Big(\frac{\gamma}{2} - 1 - \frac{\gamma r^{\gamma/2}}{1+r^{\gamma/2}} \Big)$$

and

$$\begin{aligned}\nabla_x \psi(x,y) =& 2\varphi'(||x-y||^2)(x-y) = \gamma \frac{||x-y||^{\gamma-2}}{(1+||x-y||^\gamma)^2}(x-y), \\ \Delta_x \psi(x,y) =& 4\varphi'(||x-y||^2) + 4||x-y||^2 \varphi''(||x-y||^2) \\ =& \gamma^2 \frac{||x-y||^{\gamma-2}}{(1+||x-y||^\gamma)^2} \Big(1 - 2\frac{||x-y||^\gamma}{1+||x-y||^\gamma}\Big).\end{aligned}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \varphi(||x-y||^2) f_t^\varepsilon(\mathrm{d}x) f_t^\varepsilon(\mathrm{d}y) = J_t^\varepsilon + S_t^\varepsilon, \tag{3.7}$$

where

$$\begin{aligned}J_t^\varepsilon =& 2\gamma^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{||x-y||^{\gamma-2}}{(1+||x-y||^\gamma)^2} \Big(1 - 2\frac{||x-y||^\gamma}{1+||x-y||^\gamma}\Big) f_t^\varepsilon(x) f_t^\varepsilon(y)\mathrm{d}x\mathrm{d}y, \\ S_t^\varepsilon =& 2\gamma \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{||x-y||^{\gamma-2}}{(1+||x-y||^\gamma)^2}(x-y) \cdot K_\varepsilon(x-z) f_t^\varepsilon(x) f_t^\varepsilon(y) f_t^\varepsilon(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z.\end{aligned}$$

First, we have

$$J_t^\varepsilon \ge \gamma(\gamma + M/(4\pi)) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{||x-y||^{\gamma-2}}{(1+||x-y||^\gamma)^2} f_t^\varepsilon(x) f_t^\varepsilon(y)\mathrm{d}x\mathrm{d}y - M^2 C_{M,\gamma}, \tag{3.8}$$

where $C_{M,\gamma} > 0$ is a constant such that for all $a > 0$, recall that $\gamma > M/(4\pi)$,

$$2\gamma^2 \frac{a^{\gamma-2}}{(1+a^\gamma)^2} \Big(1 - 2\frac{a^\gamma}{1+a^\gamma}\Big) \ge 2\gamma \Big(\frac{\gamma + M/(4\pi)}{2}\Big) \frac{a^{\gamma-2}}{(1+a^\gamma)^2} - C_{M,\gamma}.$$

Next, by symmetrization, we have

$$\begin{aligned} S_t^\varepsilon =&\gamma \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{||x-y||^{\gamma-2}}{(1+||x-y||^\gamma)^2} \\ &(x-y)\cdot(K_\varepsilon(x-z)-K_\varepsilon(y-z))f_t^\varepsilon(x)f_t^\varepsilon(y)f_t^\varepsilon(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ =&\frac{\gamma}{3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} F_\varepsilon(x,y,z)f_t^\varepsilon(x)f_t^\varepsilon(y)f_t^\varepsilon(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} F_\varepsilon(x,y,z) =&[K_\varepsilon(x-z)-K_\varepsilon(y-z)]\cdot(x-y)\frac{||x-y||^{\gamma-2}}{(1+||x-y||^\gamma)^2} \\ &+[K_\varepsilon(y-x)-K_\varepsilon(z-x)]\cdot(y-z)\frac{||y-z||^{\gamma-2}}{(1+||y-z||^\gamma)^2} \\ &+[K_\varepsilon(z-y)-K_\varepsilon(x-y)]\cdot(z-x)\frac{||z-x||^{\gamma-2}}{(1+||z-x||^\gamma)^2}. \end{aligned}$$

Introducing $X=x-y$, $Y=y-z$ and $Z=z-x$ and recalling that $K_\varepsilon(X)=\frac{-X}{2\pi(\|X\|^2+\varepsilon)}$, we find

$$\begin{aligned} 2\pi F_\varepsilon(x,y,z) =&\Big[\frac{Z}{\|Z\|^2+\varepsilon}+\frac{Y}{\|Y\|^2+\varepsilon}\Big]\cdot X\frac{||X||^{\gamma-2}}{(1+||X||^\gamma)^2} \\ &+\Big[\frac{X}{\|X\|^2+\varepsilon}+\frac{Z}{\|Z\|^2+\varepsilon}\Big]\cdot Y\frac{||Y||^{\gamma-2}}{(1+||Y||^\gamma)^2} \\ &+\Big[\frac{Y}{\|Y\|^2+\varepsilon}+\frac{X}{\|X\|^2+\varepsilon}\Big]\cdot Z\frac{||Z||^{\gamma-2}}{(1+||Z||^\gamma)^2}. \end{aligned}$$

We now introduce

$$\begin{aligned} G(x,y,z) =&\frac{\|X\|^{\gamma-2}}{(1+\|X\|^\gamma)^2}+\frac{\|Y\|^{\gamma-2}}{(1+\|Y\|^\gamma)^2}+\frac{\|Z\|^{\gamma-2}}{(1+\|Z\|^\gamma)^2} \\ \geq& X\cdot\frac{X}{\|X\|^2+\varepsilon}\frac{\|X\|^{\gamma-2}}{(1+\|X\|^\gamma)^2}+Y\cdot\frac{Y}{\|Y\|^2+\varepsilon}\frac{\|Y\|^{\gamma-2}}{(1+\|Y\|^\gamma)^2} \\ &+Z\cdot\frac{Z}{\|Z\|^2+\varepsilon}\frac{\|Z\|^{\gamma-2}}{(1+\|Z\|^\gamma)^2}. \end{aligned}$$

Hence $G(x,y,z)+2\pi F_\varepsilon(x,y,z)$ is larger than

$$\Big(\frac{X}{\|X\|^2+\varepsilon}+\frac{Y}{\|Y\|^2+\varepsilon}+\frac{Z}{\|Z\|^2+\varepsilon}\Big)\cdot\Big(X\frac{||X||^{\gamma-2}}{(1+||X||^\gamma)^2}+Y\frac{||Y||^{\gamma-2}}{(1+||Y||^\gamma)^2}+Z\frac{||Z||^{\gamma-2}}{(1+||Z||^\gamma)^2}\Big),$$

which is nonnegative according to Lemma 3.4, since $r\to 1/(r^2+\varepsilon)$ and $r\to r^{\gamma-2}/(1+r^\gamma)^2$ are both nonincreasing on $(0,\infty)$ and since $X+Y+Z=0$. Thus $F_\varepsilon(x,y,z)\geq -G(x,y,z)/(2\pi)$. Recalling (3.9), we conclude that

$$\begin{aligned} S_t^\varepsilon \geq& -\frac{\gamma}{6\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G(x,y,z)f_t^\varepsilon(\mathrm{d}x)f_t^\varepsilon(\mathrm{d}y)f_t^\varepsilon(\mathrm{d}z) \\ =&-\frac{\gamma M}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{\|x-y\|^{\gamma-2}}{(1+\|x-y\|^\gamma)^2}f_t^\varepsilon(\mathrm{d}x)f_t^\varepsilon(\mathrm{d}y) \end{aligned} \tag{3.10}$$

by symmetry again.

Gathering (3.7)-(3.8)-(3.10), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varphi(||x-y||^2)f_t^\varepsilon(\mathrm{d}x)f_t^\varepsilon(\mathrm{d}y)\geq\gamma\Big(\gamma-\frac{M}{4\pi}\Big)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{\|x-y\|^{\gamma-2}}{(1+\|x-y\|^\gamma)^2}f_t^\varepsilon(\mathrm{d}x)f_t^\varepsilon(\mathrm{d}y) - M^2C_{M,\gamma}.$$

Integrating on $[0,T]$, using that $\gamma>M/(4\pi)$ and and that φ is $[0,1]$-valued, we end with

$$\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{\|x-y\|^{\gamma-2}}{(1+\|x-y\|^\gamma)^2}f_t^\varepsilon(\mathrm{d}x)f_t^\varepsilon(\mathrm{d}y)\mathrm{d}t\leq\frac{M^2+M^2C_{M,\gamma}T}{\gamma(\gamma-M/(4\pi))}.$$

One easily completes the proof, using that there is $D_\gamma>0$ such that $a^{\gamma-2}\leq 2a^{\gamma-2}/(1+a^\gamma)^2+D_\gamma$ for all $a>0$ □

We finally give the

Proof of Theorem 3.2. Recall that $M\in(0,8\pi)$, that $f_0\in\mathcal{M}_M(\mathbb{R}^2)$, and that $(f^\varepsilon)_{\varepsilon\in(0,1]}$ is the corresponding family of regularized solutions. By Proposition 3.3, we can find $(\varepsilon_k)_{k\geq0}$ and $f\in C([0,\infty),\mathcal{M}_M(\mathbb{R}^2))$ such that $\lim_k\varepsilon_k=0$ and $\lim_k f^{\varepsilon_k}=f$ in $C([0,\infty),\mathcal{M}_M(\mathbb{R}^2))$, endowed with the uniform convergence on compact time intervals, $\mathcal{M}_M(\mathbb{R}^2)$ being endowed with the weak convergence topology. By definition of f_0^ε, we obviously have $f|_{t=0}=f_0$. By Proposition 3.5 and the Fatou lemma, for all $\gamma\in(M/(4\pi),2)$, we have

$$\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\|x-y\|^{\gamma-2}f_s(\mathrm{d}x)f_s(\mathrm{d}y)\mathrm{d}s\leq A_{M,\gamma}(1+T). \tag{3.11}$$

It only remains to check that f is a weak solution to (3.2). We fix $\varphi\in C_b^2(\mathbb{R}^2)$ and use (3.6) to write $\int_{\mathbb{R}^2}\varphi(x)f_t^{\varepsilon_k}(\mathrm{d}x)=I_k(t)+J_k(t)$, where

$$I_k(t)=\int_{\mathbb{R}^2}\varphi(x)f_0^{\varepsilon_k}(\mathrm{d}x)+\int_0^t\int_{\mathbb{R}^2}\Delta\varphi(x)f_s^{\varepsilon_k}(\mathrm{d}x)\mathrm{d}s,$$
$$J_k(t)=\frac12\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}K_{\varepsilon_k}(x-y)\cdot[\nabla\varphi(x)-\nabla\varphi(y)]f_s^{\varepsilon_k}(\mathrm{d}x)f_s^{\varepsilon_k}(\mathrm{d}y)\mathrm{d}s.$$

Since φ and $\Delta\varphi$ are continuous and bounded, we immediately conclude that

$$\lim_k\int_{\mathbb{R}^2}\varphi(x)f_t^{\varepsilon_k}(\mathrm{d}x)=\int_{\mathbb{R}^2}\varphi(x)f_t(\mathrm{d}x),$$
$$\lim_k I_k(t)=\int_{\mathbb{R}^2}\varphi(x)f_0(\mathrm{d}x)+\int_0^t\int_{\mathbb{R}^2}\Delta\varphi(x)f_s(\mathrm{d}x)\mathrm{d}s,$$

and it only remains to check that for $J(t)=\frac12\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}K(x-y)\cdot[\nabla\varphi(x)-\nabla\varphi(y)]f_s(\mathrm{d}x)f_s(\mathrm{d}y)\mathrm{d}s$, we have $\lim_k J_k(t)=J(t)$. To this end, we write $J_k(t)=J_k^1(t)+J_k^2(t)$, where

$$J_k^1(t)=\frac12\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}K(x-y)\cdot[\nabla\varphi(x)-\nabla\varphi(y)]f_s^{\varepsilon_k}(\mathrm{d}x)f_s^{\varepsilon_k}(\mathrm{d}y)\mathrm{d}s,$$
$$J_k^2(t)=\frac12\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}[K_{\varepsilon_k}(x-y)-K(x-y)]\cdot[\nabla\varphi(x)-\nabla\varphi(y)]f_s^{\varepsilon_k}(\mathrm{d}x)f_s^{\varepsilon_k}(\mathrm{d}y)\mathrm{d}s.$$

Recalling the expression of K and that $\varphi \in C_b^2(\mathbb{R}^2)$, we see that $g(x,y) = K(x-y)\cdot[\nabla\varphi(x)-\nabla\varphi(y)]$ is bounded and continuous on the set $\mathbb{R}^2 \setminus D$, where $D = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : x = y\}$. Since $f_s^{\varepsilon_k} \otimes f_s^{\varepsilon_k}$ goes weakly to $f_s \otimes f_s$ for each $s \geq 0$ and since $(f_s \otimes f_s)(D) = 0$ for a.e. $s \geq 0$ by (3.11), 1, we conclude that $\lim_k \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} g(x,y) f_s^{\varepsilon_k}(\mathrm{d}x) f_s^{\varepsilon_k}(\mathrm{d}y) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} g(x,y) f_s(\mathrm{d}x) f_s(\mathrm{d}y)$ for a.e. $s \geq 0$, whence $\lim_k J_k^1(t) = J_k(t)$ by dominated convergence.

We finally have to verify that $\lim_k J_k^2(t) = 0$. We fix $\gamma \in (M/(4\pi), 2)$ and write

$$\begin{aligned}\|z\|\,\|K(z) - K_\varepsilon(z)\| = &\frac{\varepsilon}{2\pi(\varepsilon + \|z\|^2)} \\ &\leq \min(1, \varepsilon\|z\|^{-2}) \leq (\varepsilon\|z\|^{-2})^{1-\gamma/2} = \varepsilon^{1-\gamma/2}\|z\|^{\gamma-2}.\end{aligned}$$

Thus

$$|J_k^2(t)| \leq \|\nabla^2\varphi\|_\infty \varepsilon_k^{1-\gamma/2} \int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \|x-y\|^{\gamma-2} f_s^{\varepsilon_k}(\mathrm{d}x) f_s^{\varepsilon_k}(\mathrm{d}y)\mathrm{d}s,$$

which tends to 0 as desired since $\sup_{\varepsilon \in (0,1]} \int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \|x-y\|^{\gamma-2} f_s^{\varepsilon_k}(\mathrm{d}x) f_s^{\varepsilon_k}(\mathrm{d}y)\mathrm{d}s < \infty$ by Proposition 3.5. $\square$

Chapitre 4

Convergence of the empirical measure for the Keller-Segel model in both subcritical and critical cases

Abstract. We show the weak convergence, up to extraction of a subsequence, of the empirical measure for the Keller-Segel system of particles in both subcritical and critical cases. We use a simple *two particles* moment argument in order to show that the particle system does not explodes in finite time, uniformly in the number of particles in some sense.

4.1 Introduction and results

4.1.1 The model

We consider the classical Keller-Segel model of chemotaxis (or Patlak-Keller-Segel model) in the parabolic-elliptic case, which writes

$$\partial_t f = \frac{1}{2}\Delta f - 2\pi\theta\nabla\cdot(f\nabla c), \tag{4.1}$$

$$0 = \Delta c + f, \tag{4.2}$$

with initial condition $f_0 \in \mathcal{P}(\mathbb{R}^2)$, where $\mathcal{P}(\mathbb{R}^2)$ denotes the set of probability measures on $\mathbb{R}^2$, where (f, c) is the unknown of the equation and $f_t(x)$ and $c_t(x)$ respectively stand for the density of particles and chemoattractant at position $x \in \mathbb{R}^2$ and time $t \geq 0$. The scalar $\theta > 0$ describes the intensity of the attraction of the chemoattractant.

Remark 4.1. *In the litterature,* (4.1) *is written*

$$\partial_t f = \Delta f - \chi\nabla\cdot(f\nabla c),$$
$$0 = \Delta f + c,$$

and f_0 is assumed to have a finite mass $M > 0$. One can easily convince oneself that setting $\tilde{f}_t(x) = f_{t/2}(x)/M$ and $c_{t/2}(x)/M = \tilde{c}_t(x)$, $(\tilde{f}, \tilde{c})$ solves (4.1)-(4.2) *with $\theta = \chi M/(4\pi)$. Observe that if $\chi = 1$, one recovers that the classical threshold $M = 8\pi$ of* (4.1)-(4.2) *corresponds to $\theta = 2$ in our case.*

Since c is the solution of the Poisson equation on $\mathbb{R}^2$ with f as source term, the equation (4.1) can we rewritten

$$\partial_t f = \frac{1}{2}\Delta f - \theta \nabla \cdot (f(K \star f)), \tag{4.3}$$

with

$$K(x) = -x/\|x\|^2 \quad \text{if } x \neq 0 \quad \text{and} \quad K(0) = 0$$

and $\star$ stands for the convolution product. One can understand this equation as the modeling of an infinite number of particles moving as Brownian motions which attract each other with a Coulombian force.

The main interest of this equation is the tight competition between the diffusion and the concentration of particles. As it is presented in Blanchet-Dolbeaut-Perthame [9], setting the variance of the problem $V_t = \int_{\mathbb{R}^2} \|x - \int_{\mathbb{R}^2} x f_t(\mathrm{d}x)\|^2 \mathrm{d}f_t(x)$, we informally have, using multiple integration by parts, that $\frac{\mathrm{d}}{\mathrm{d}t} V_t = (2-\theta)$. This highlights that there are informally three cases :

- the subcritical case which correspond to the case where $\theta < 2$: V_t tends to infinty as time goes to infinity which suggests that the diffusion is dominant over the concentration,
- the critical case where $\theta = 2$,
- the supercritical case where $\theta > 2$: V_t ends up by being non positive, which suggests that a blow up occurs.

4.1.2 Weak solutions

In this paper, we will deal with a very weak type of solutions that we define here, see e.g. Blanchet-Dolbeault-Perthame [9].

Definition 4.2. *We say that $f \in C([0,\infty), \mathcal{P}(\mathbb{R}^2))$ is a weak solution of (4.1)-(4.2) with initial condition $f_0 \in \mathcal{P}(\mathbb{R}^2)$ if for all $\varphi \in C_b^2(\mathbb{R}^2)$,*

$$\begin{aligned}\int_{\mathbb{R}^2} \varphi(x) f_t(\mathrm{d}x) = &\int_{\mathbb{R}^2} \varphi(x) f_0(\mathrm{d}x) + \frac{1}{2}\int_0^t \int_{\mathbb{R}^2} \Delta\varphi(x) f_s(\mathrm{d}x)\mathrm{d}s \\ &+ \frac{\theta}{2}\int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} K(x-y)\cdot[\nabla\varphi(x) - \nabla\varphi(y)] f_s(\mathrm{d}x) f_s(\mathrm{d}y)\mathrm{d}s.\end{aligned}$$

Observe that since $K(0) = 0$ and $\varphi \in C_b^2$, the last integral is well defined. Since $(x,y) \mapsto K(x-y)\cdot[\nabla\varphi(x) - \nabla\varphi(y)]$ equals 0 (arbitrilarly) on the set $\{(x,y) \in \mathbb{R}^2 : x = y\}$, we expect that a satisfactory solution must verify $\int_0^t \mathbb{1}_{x\neq y} f_s(\mathrm{d}x) f_s(\mathrm{d}y)\mathrm{d}s = 0$.

4.1.3 The associated trajectories

The equation (4.3) comes from an Eulerian point of view. The following SDE models the same problem with a Lagrangian point of view : we consider the behaviour of one typical particle among the other.

Definition 4.3. *We say that $(X_t)_{t\geq 0}$ is a solution of the nonlinear SDE (4.4) with initial condition X_0 if*

$$\textit{for all } t \geq 0, \quad X_t = X_0 + B_t + \theta \int_0^t K \star f_s(X_s)\mathrm{d}s, \tag{4.4}$$

where $(B_t)_{t\geq 0}$ is a 2-dimensional Brownian motion and f_s is the law of X_s.

The idea is to follow the motion of one specific particle instead of considering the proportion of particles at a precise place and a precise time. One easily checks from the Itô formula that if $(X_t)_{t\geq 0}$ is a solution of the nonlinear SDE (4.4), then $(f_t)_{t\geq 0}$ is a weak solution to (4.1)-(4.2).

We consider for $N \geq 2$ and $\theta > 0$ a natural discretization introduced by Keller-Segel [36] of the nonlinear SDE (4.4) : we consider $N \geq 2$ particles with positions $(X_t^{i,N})_{i\in[\![1,N]\!]}$, solving (recall that $K(0) = 0$)

$$X_t^{i,N} = X_0^{i,N} + B_t^i + \frac{\theta}{N}\int_0^t \sum_{j=1}^N K(X_s^{i,N} - X_s^{j,N})\mathrm{d}s. \tag{4.5}$$

where $((B_t^i)_{i\in[\![1,N]\!]})_{t\geq 0}$ is a family of N independant 2-dimensional Brownian motion independant from $(X_0^{i,N})_{i\in[\![1,N]\!]}$. We call such a process a $KS(\theta,N)$-process, $\theta \leq 2(N-2)/(N-1)$ and its existence is guaranteed by [20, Theorem 7] for any initial datum $(X_0^{i,N})_{i\in[\![1,N]\!]}$ with its law F_0^N in $\mathcal{P}^*_{sym,1}((\mathbb{R}^2)^N)$, where $\mathcal{P}^*_{sym,1}((\mathbb{R}^2)^N)$ denotes the set of exchangeable probability measures on $(\mathbb{R}^2)^N$ with a finite 1-order moment and such that

$$F_0^N(\{\text{There exists } i \neq j \in [\![1,N]\!] \text{ such that } x^i = x^j\}) = 0.$$

Moreover, $(X_t^{i,N})_{i\in[\![1,N]\!]}$ is exchangeable and we have, for all $t > 0$,

$$\int_0^t \sum_{j=1}^N ||K(X_s^{1,N} - X_s^{2,N})||\mathrm{d}s < \infty.$$

4.1.4 The subcritical case

Our goal is to prove the following result. We endow the space $C([0,\infty),\mathcal{P}(\mathbb{R}^2))$ with the topology of the uniform convergence on the compact sets, where $\mathcal{P}(\mathbb{R}^2)$ is endowed with the topology of the weak convergence.

Theorem 4.4. *Let $\theta \in (0,2)$ and $f_0 \in \mathcal{P}(\mathbb{R}^2)$. For each $N \geq N_0 := (1 + \lceil 2/(2-\theta)\rceil) \vee 5$, consider $F_0^N \in \mathcal{P}^*_{sym,1}((\mathbb{R}^2)^N)$ and a $KS(\theta,N)$-process $(X_t^{i,N})_{t\geq 0, i\in[\![1,N]\!]}$ with initial law F_0^N, as well as the empirical measure for all $t \geq 0$, $\mu_t^N := N^{-1}\sum_{i=1}^N \delta_{X_t^{i,N}}$, which a.s. belongs to $\mathcal{P}(\mathbb{R}^2)$. We assume that μ_0^N goes weakly to f_0 in probability as $N \to \infty$.*

(i) The sequence $((\mu_t^N)_{t\geq 0})_{N\geq N_0}$ is tight in $C([0,\infty),\mathcal{P}(\mathbb{R}^2))$.

(ii) For any sequence $(N_k)_{k\geq 0}$ such that $(\mu_t^{N_k})_{t\geq 0}$ goes in law in $C([0,\infty),\mathcal{P}(\mathbb{R}^2))$ as $k \to \infty$ to some $(\mu_t)_{t\geq 0}$, this limit $(\mu_t)_{t\geq 0}$ is a.s a weak solution to (4.1)-(4.2) starting from $\mu_0 = f_0$. Moreover, for all $T > 0$, all $\gamma \in (\theta, 2)$.

$$\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \|x-y\|^{\gamma-2}\mu_t(\mathrm{d}x)\mu_t(\mathrm{d}y)\mathrm{d}t\Big] < \infty.$$

Observe that the condition $N \geq N_0$ implies that the particle system exists thanks to [20, Theorem 7]. There is no condition at all on f_0 : it can be a Dirac mass, it does not need to have finite moments.

However, we impose some conditions on F_0^N. These conditions are not necessary for our proof, but only to apply the existence result of [20].

For any given f_0, we can easily build F_0^N satisfying all the assumptions : take for example F_0^N the law of $(\chi_N(X_0^i)+(1/N)G^i)_{i\in[\![1,N]\!]}$ where $(X_0^i)_{i\geq 1}$ are i.i.d f_0-distributed random variables, where $(G^i)_{i\geq 1}$ are i.i.d standard bidimensional Gaussian random variables and where $\chi_N(x_1,x_2) = (x_1\wedge N \vee (-N), x_2 \wedge N \vee (-N))$.

We highlight that this result has to be linked with Fournier-Tardy [21], where the critical case was not treated for the sake of conciseness (although it was treated in an unpublished preprint version).

4.1.5 The critical and supercritical particle system

In the case where $\theta > 2(N-2)/(N-1)$, the existence of a process solving (4.5) for all time is not guaranteed, so that we need to precise our definition of a solution. If τ is a stopping time for a filtration $(\mathcal{F}_t)_{t\geq 0}$, a continuous $(\mathbb{R}^2)^N$-valued process $(X_t^{i,N})_{t\in[0,\tau), i\in[\![1,N]\!]}$ is a $KS(\theta,N)$-process on $[0,\tau)$ if for all sequences of increasing stopping times $(\tau_n)_{n\in\mathbb{N}}$ with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\lim_{n\to\infty}\tau_n = \tau$ a.s. and for all $n\geq 1$, $(X_{t\wedge\tau_n}^{i,N})_{t\geq 0, i\in[\![1,N]\!]}$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted and (4.5) holds true on $[0,\tau_n]$.

For the whole paper, we define for all $N\geq 2$, all $l\geq 1$, all $k\geq 2$,

$$\tau_k^{N,l} := \inf\Big\{t>0 : \exists K\subset [\![1,N]\!] : |K| = k \text{ and } R_K(X_t^N) \leq \frac{1}{l}\Big\} \quad \text{and} \quad \tau_k^N = \sup_{l\geq 1}\tau_k^{N,l},$$

where for all $K\subset[\![1,N]\!]$, all $x=(x^1,\cdots,x^N)\in(\mathbb{R}^2)^N$,

$$S_K(x) = \frac{1}{|K|}\sum_{i\in K} x^i \qquad \text{and} \qquad R_K(x) = \sum_{i\in K}\|x^i - S_K(x)\|^2.$$

One can find the following existence theorem in Fournier-Jourdain [20, Theorem 7 and Lemma 14], recall that our θ corresponds to $\chi/(4\pi)$ in [20].

Theorem 4.5. *If $N\geq 5$, $\theta>0$ and $F_0^N\in\mathcal{P}^*_{sym,1}(\mathbb{R}^2)$, then there exists a $KS(\theta,N)$-process $(X_t^{i,N})_{t\in[0,\tau_3^N), i\in[\![1,N]\!]}$ to (4.5) with initial law F_0^N. The family $((X_t^{i,N})_{t\in[0,\tau_3^N)}, i\in[\![1,N]\!])$ is exchangeable and for any $\ell\geq 1$ we have*

$$\mathbb{E}\Big(\int_0^{t\wedge\tau_3^{N,\ell}} \|X_s^1 - X_s^2\|^{-1}\mathrm{d}s\Big) < \infty.$$

Moreover $\tau_3^N<\infty$ a.s. if $\theta > 2(N-2)/(N-1)$, and $\tau_3^N = \infty$ a.s. if $\theta\in(0,2(N-2)/(N-1)]$.

The idea is that collisions between 2 particles do not prevent the well definition of the KS-process for a selection of generic parameters θ and N, but the existence in the classical sense of such a process beyond a collision between 3 or more particles is more tricky if not impossible depending of the size of the cluster involved in the collision.

4.1.6 The critical case

Our second main result is the following.

Theorem 4.6. *Assume $\theta = 2$. Let $f_0 \in \mathcal{P}(\mathbb{R}^2)$ checking $\max_{x\in\mathbb{R}^2} f_0(\{x\}) < 1$. For each $N \geq N_0 := 5$, consider $F_0^N \in \mathcal{P}^*_{sym,1}((\mathbb{R}^2)^N)$ and $(X_t^{i,N})_{t\in[0,\tau_3^N), i\in[\![1,N]\!]}$ a $KS(2,N)$-process on $[0,\tau_3^N)$ with initial law F_0^N. We set $\mu_t^N := N^{-1}\sum_{i=1}^N \delta_{X_t^{i,N}}$ and we assume that μ_0^N goes weakly to f_0 in probability as $N \to \infty$.*

There exists $(\ell_N)_{N\geq N_0}$ an increasing sequence of integers such that, setting $\beta_N = \tau_3^{N,\ell_N}$, it holds that $\lim_{N\to\infty} \beta_N = \infty$ in probability and

(i) the sequence $((\mu^N_{t\wedge\beta_N})_{t\geq 0})_{N\geq 5}$ is tight in $C([0,\infty),\mathcal{P}(\mathbb{R}^2))$;

(ii) for any sequence $(N_k)_{k\geq 0}$ such that $(\mu^{N_k}_{t\wedge\beta_{N_k}})_{t\geq 0}$ goes in law, as $k \to \infty$, in $C([0,\infty),\mathcal{P}(\mathbb{R}^2))$, to some $(\mu_t)_{t\geq 0}$, it holds that μ_t is a.s a weak solution to (4.1) starting from $\mu_0 = f_0$. Moreover,

$$\mathbb{E}\Big[\int_0^\infty \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathbb{1}_{\{x=y\}}\mu_t(\mathrm{d}x)\mu_t(\mathrm{d}y)\mathrm{d}t\Big] = 0. \tag{4.6}$$

The only condition on f_0 is that it is not a full Dirac mass, it can have any atoms as it wants provided none of these atoms have mass 1.

Actually, it can be shown that with $\theta = 2$, the particle system explodes in finite time for each fixed N, see [21] in a slightly different context. However, we will show the explosion time tends to infinity as $N \to \infty$, which explains that at the limit, the solution is global.

Roughly speaking, since $\beta_N \to \infty$, the above theorem implies that if we fix $T > 0$, $(\mu_t^N)_{t\in[0,T]}$ is well-defined with high probability (as $N \to \infty$), is tight in some sense, and the accumulation points of this sequence are almost surely weak solutions of (4.1)-(4.2) on $[0,T]$.

4.1.7 References

The Keller-Segel equation (4.1)-(4.2) was first introduced by Patlak [43] and Keller and Segel [35] as a model for chemotaxis. One can find an exhaustive content on what is understood about this model in the recent book of Biler [3] and a review of Arumugan-Tyagi [1].

The notion of propagation of chaos was introduced by Kac [33], the idea was to make a step to the rigourous justification of the Boltzmann equation. The idea is to approximate the solution of a mesoscopic PDE by solutions of the associated microscopic problem, which involve interacting particles. The case of Lipschitz continuous interaction kernels is now well understood, we mention for example McKean [38], Sznitman [45], Méléard [39] and Mischler-Mouhot [40] for significant contributions to the theory. The case of singular kernels is much more complicated and has been studied for different kind of kernels as the kernel of the viscous Burgers equation by Bossy-Talay [10] and Jourdain [32], the one of the Dyson model by Cépa-Lépingle [14], etc...

A model relatively close to our subject is the $2d$-Navier-Stokes particle system studied by Osada [42] where the kernel K is replaced by $x^\perp/\|x\|^2$ which is as singular as the kernel of Keller-Segel but with rotation instead of attraction. This implies that collisions do not occur so that the singularity is not visited. Osada [42] obtained the convergence of the corresponding stochastic particle system under a large viscosity assumption. This condition has been removed by Fournier-Hauray-Mischler [19] thanks to a compactness-uniqueness method, and then, more recently, Jabin-Wang [29] obtained a quantitative convergence result, using a *modulated free entropy* method. We mention Serfaty [46] who introduced the modulated energy method in the deterministic case.

Let us mention Stevens [47], who studied the case where the chemoattractant is not in the stationnary regime (parabolic-parabolic Keller-Segel equation), by considering a particle system with two types of particles, representing bacteria and chemoattractant.

This paper has been written in the same spirit as Fournier-Tardy [21] where we gave a simple proof of nonexplosion of the weak solution to (4.1)-(4.2) in the subcritical case. We used a *two-particles* moment method that allowed us to extend to the general case the simple non-explosion of Biler-Karch-Laurençot-Nadzieja [4, 5] concerning the radial case. The method in [21] is simple enough so that we are able to adapt it to show that the particle system does not explode, uniformly in N in some sense, and even to include the critical case. Let us mention however that the existence result included in Theorems 4.4 and 4.6, under the weak conditions that $f_0 \in \mathcal{P}(\mathbb{R}^2)$ if $\theta \in (0,2)$ and $f_0 \in \mathcal{P}(\mathbb{R}^2)$ and $\max\{f_0(\{x\}) : x \in \mathbb{R}^2\} < 1$ if $\theta = 2$, are actually already known : some stronger (mild) global solutions do exist under the same conditions, combining the results of Bedrossian-Masmoudi [2] and Wei [51].

Concerning the Keller-Segel particle system, there are four main available results :

(a) Godinho-Quininao [25] proved the convergence of the particle system without using a subsequence, following the ideas of Fournier-Hauray-Mischler [19] with K replaced by $-x/(2\pi\|x\|^{1+\alpha})$ with $\alpha \in (0,1)$. This kernel is strictly less singular, this prevents from any explosion issue.

(b) Olivera-Richard-Tomasevic [41] have shown the convergence of a smoothed particle system where K is replaced, roughly, by $-x/(\varepsilon_N + ||x||^2)$ with ε_N very large in front of $N^{-1/d}$. The method is based on a semigroup approach developped by Flandoli [18].

(c) In Fournier-Jourdain [20], the convergence of the empirical measure is shown up to an extraction and for $\theta < 1/2$. We highlight that this is exactly the same result as Theorem 4.4 but with $\theta < 1/2$, with a slight additional moment condition. Cattiaux-Pedèches [13] give another proof of the existence without the assumption of exchangeability on the initial conditions and establish the (weak) uniqueness of the Keller-Segel particle system.

(d) Finally, Bresch-Jabin-Wang [11, 12] studied the convergence of the empirical measure with quantitative estimates in the case where $\theta < 2$ by using a *modulated free entropy* method in the same spirit as [29]. They do not really deal with (4.1)-(4.2) but rather with a weak solution to the associated Fokker-Planck P.D.E. They obtain a full convergence result (not only along a subsequence), when replacing $\mathbb{R}^2$ by a torus, and for regular initial conditions, namely at least f_0 of class $W^{2,\infty}$ on the torus.

In the subcritical case, our method seems much simpler than that of [11, 12], we work in the whole space instead of the torus and we do not have any requirement on the initial condition f_0, we e.g. allow for Dirac masses. But of course, one drawback is of that we prove convergence only up to extraction of a subsequence. At least, its seems that nothing was known about the critical case.

4.1.8 Main a priori estimate in the subcritical case

The subcritical case (which means the case where $\theta \in (0,2)$) is the easiest to handle and here we briefly present the main computation. The idea is to find an a priori estimate which testifies to the fact that particles don't have the tendency to aggregate. We fix $\gamma \in (\theta, 2)$. Informally using the Itô formula and taking the expectation we find

$$S_t = S_0 + \int_0^t \mathbb{E}\Big[\mathbb{1}_{\|X_s^{1,N}-X_s^{2,N}\|\le 1}\Big(I_s^1 + \frac{\theta}{N} I_s^2\Big)\Big]\mathrm{d}s, \tag{4.7}$$

where,

$$S_t = \mathbb{E}[\|X_t^{1,N} - X_t^{2,N}\|^\gamma \wedge 1],$$
$$I_s^1 = \Big(\gamma^2 - \frac{2\gamma\theta}{N}\Big)\|X_s^{1,N} - X_s^{2,N}\|^{\gamma-2},$$
$$I_s^2 = \gamma \sum_{k=3}^N (K(X_s^{1,N} - X_s^{k,N}) + K(X_s^{k,N} - X_s^{2,N})) \cdot (X_s^{1,N} - X_s^{2,N})\|X_s^{1,N} - X_s^{2,N}\|^{\gamma-2}.$$

Using exchangeability we get that $\mathbb{E}[\mathbb{1}_{\|X_s^{1,N}-X_s^{2,N}\|\le 1} I_s^2]$ equals

$$\gamma(N-2)\mathbb{E}[\mathbb{1}_{\|X_s^{1,N}-X_s^{2,N}\|\le 1}(K(X_s^{1,N} - X_s^{3,N}) + K(X_s^{3,N} - X_s^{2,N})) \cdot (X_s^{1,N} - X_s^{2,N})\|X_s^{1,N} - X_s^{2,N}\|^{\gamma-2}],$$

so that using exchangeability again,

$$\frac{\theta}{N}\int_0^t \mathbb{E}\Big[\mathbb{1}_{\|X_s^{1,N}-X_s^{2,N}\|\le 1} I_s^2\Big]\mathrm{d}s = \frac{\gamma\theta(N-2)}{3N}\int_0^t \mathbb{E}[F(X_s^{1,N}, X_s^{2,N}, X_s^{3,N})]\mathrm{d}s, \tag{4.8}$$

where

$$\begin{aligned} F(x,y,z) =& \Big(\frac{Y}{\|Y\|^2} + \frac{Z}{\|Z\|^2}\Big) \cdot X\|X\|^{\gamma-2}\mathbb{1}_{\|X\|\le 1} \\ &+ \Big(\frac{Z}{\|Z\|^2} + \frac{X}{\|X\|^2}\Big) \cdot Y\|Y\|^{\gamma-2}\mathbb{1}_{\|Y\|\le 1} \\ &+ \Big(\frac{X}{\|X\|^2} + \frac{Y}{\|Y\|^2}\Big) \cdot Z\|Z\|^{\gamma-2}\mathbb{1}_{\|Z\|\le 1}, \end{aligned}$$

where we have set $X = x - y$, $Y = y - z$ and $Z = z - x$. By furthermore setting $G(x,y,z) = \|X\|^{\gamma-2}\mathbb{1}_{\|X\|\le 1} + \|Y\|^{\gamma-2}\mathbb{1}_{\|Y\|\le 1} + \|Z\|^{\gamma-2}\mathbb{1}_{\|Z\|\le 1}$, we get that $G(x,y,z) + F(x,y,z)$ equals

$$\Big(\frac{X}{\|X\|^2} + \frac{Y}{\|Y\|^2} + \frac{Z}{\|Z\|^2}\Big) \cdot \Big(X\|X\|^{\gamma-2}\mathbb{1}_{\|X\|\le 1} + Y\|Y\|^{\gamma-2}\mathbb{1}_{\|Y\|\le 1} + Z\|Z\|^{\gamma-2}\mathbb{1}_{\|Z\|\le 1}\Big),$$

which is positive according to Lemma 4.9. Injecting in (4.8), we get

$$\begin{aligned} \frac{\theta}{N}\int_0^t \mathbb{E}\Big[\mathbb{1}_{\|X_s^{1,N}-X_s^{2,N}\|\le 1} I_s^2\Big]\mathrm{d}s &\ge -\frac{\gamma\theta(N-2)}{3N}\int_0^t \mathbb{E}[G(X_s^{1,N}, X_s^{2,N}, X_s^{3,N})]\mathrm{d}s \\ &= -\frac{\gamma\theta(N-2)}{N}\int_0^t \mathbb{E}[\mathbb{1}_{\|X_s^{1,N}-X_s^{2,N}\|\le 1}\|X_s^{1,N} - X_s^{2,N}\|^{\gamma-2}]\mathrm{d}s. \end{aligned}$$

Gathering this and (4.7) we get

$$S_t \ge S_0 + \gamma(\gamma - \theta)\int_0^t \mathbb{E}[\mathbb{1}_{\|X_s^{1,N}-X_s^{2,N}\|\le 1}\|X_s^{1,N} - X_s^{2,N}\|^{\gamma-2}]\mathrm{d}s.$$

Since $\gamma \in (\theta, 2)$, S_t is bounded by 1 and S_0 is nonnegative, we get that

$$\mathbb{E}\Big[\int_0^t \mathbb{1}_{\|X_s^{1,N}-X_s^{2,N}\|\leq 1}\|X_s^{1,N}-X_s^{2,N}\|^{\gamma-2}\mathrm{d}s\Big] \leq \frac{1}{\gamma(\gamma-\theta)}.$$

This provides some bound saying that uniformly in N, the particles do not spend too much time close to each other. As we will see, this is sufficient to conclude in the subcritical case. The critical case is of course more complicated.

4.1.9 Plan of the paper

In Section 4.2 we show some basic formulas verified by a KS-process. In Section 4.3 we will prove Theorem 4.4-(i) and Theorem 4.6-(i) and we give the complete proof of Theorem 4.4-(ii) in Section 4.4. Finally, in Section 4.5 we show some probabilistic results on the behaviour of collisions in the critical case which is crucial for the proof of Theorem 4.6-(ii) in the critical case that is presented in Section 4.6.

4.2 The particle system and some basic properties

This section is devoted to prove the following preliminary results about KS-processes.

Lemma 4.7. *Let $N \geq 2$, $\theta > 0$ and $(\mathcal{F}_t)_{t\geq 0}$ be a filtration. If τ is a stopping time for the filtration $(\mathcal{F}_t)_{t\geq 0}$, then for all $KS(\theta, N)$-process $(X_t^{i,N})_{t\in[0,\tau), i\in[\![1,N]\!]}$ on $[0,\tau)$, for all sequence of increasing $(\mathcal{F}_t)_{t\geq 0}$-stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\tau_n = \tau$ a.s., all $t \geq 0$, all $n \in \mathbb{N}$, we have the following identities.*

(i) For all $i \in [\![1,N]\!]$, all $\varphi \in C^2(\mathbb{R}_+)$,

$$\begin{aligned}\varphi(\|X_{t\wedge\tau_n}^{i,N}\|^2) =& \varphi(\|X_0^{i,N}\|^2) + 2\int_0^{t\wedge\tau_n} \varphi'(\|X_s^{i,N}\|^2)X_s^{i,N}\cdot \mathrm{d}B_s^i \\ &+ 2\int_0^{t\wedge\tau_n}[\varphi'(\|X_s^{i,N}\|^2) + \|X_s^{i,N}\|^2\varphi''(\|X_s^{i,N}\|^2)]\mathrm{d}s \\ &+ \frac{2\theta}{N}\int_0^{t\wedge\tau_n}\sum_{j=1}^N K(X_s^{i,N}-X_s^{j,N})\cdot X_s^{j,N}\varphi'(\|X_s^{j,N}\|^2)\mathrm{d}s.\end{aligned}$$

(ii) For all $i,j \in [\![1,N]\!]$ such that $i \neq j$, all $\varphi \in C^2(\mathbb{R}_+)$,

$$\varphi(\|X_{t\wedge\tau_n}^{i,N} - X_{t\wedge\tau_n}^{j,N}\|^2) = \varphi(\|X_0^{i,N} - X_0^{j,N}\|^2) + M_t^{N,n} + I_t^{N,n} + \frac{2\theta}{N}J_t^{N,n},$$

where

$$M_t^{N,n} = 2\int_0^{t\wedge\tau_n}\varphi'(\|X_s^{i,N}-X_s^{j,N}\|^2)(X_s^{i,N}-X_s^{j,N})\cdot\mathrm{d}(B_s^i - B_s^j),$$

$$I_t^{N,n} = 4\int_0^{t\wedge\tau_n}\Big[\varphi'(\|X_s^{i,N}-X_s^{j,N}\|^2) + \|X_s^{i,N}-X_s^{j,N}\|^2\varphi''(\|X_s^{i,N}-X_s^{j,N}\|^2)\Big]\mathrm{d}s,$$

and

$$J_t^{N,n}=\int_0^{t\wedge\tau_n}\sum_{k=1}^{N}(K(X_s^{i,N}-X_s^{k,N})+K(X_s^{k,N}-X_s^{j,N}))\cdot(X_s^{i,N}-X_s^{j,N})\varphi'(\|X_s^{i,N}-X_s^{j,N}\|^2)\mathrm{d}s.$$

(iii) Setting $\mu_t^N=N^{-1}\sum_{i=1}^N\delta_{X_t^{i,N}}$ *for all* $t\in[0,T)$, *we have all* $\varphi\in C^2(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2}\varphi(x)\mu_{t\wedge\tau_n}^N(\mathrm{d}x)=\int_{\mathbb{R}^2}\varphi(x)\mu_0^N(\mathrm{d}x)+\frac{1}{N}\int_0^{t\wedge\tau_n}\sum_{i=1}^N\nabla\varphi(X_s^{i,N})\cdot\mathrm{d}B_s^i+\frac{1}{2}\int_0^{t\wedge\tau_n}\int_{\mathbb{R}^2}\Delta\varphi(x)\mu_s^N(\mathrm{d}x)\mathrm{d}s+\frac{\theta}{2}\int_0^{t\wedge\tau_n}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}K(x-y)\cdot[\nabla\varphi(x)-\nabla\varphi(y)]\mu_s^N(\mathrm{d}x)\mu_s^N(\mathrm{d}y)\mathrm{d}s.$$

Démonstration. First, (i) follows directly from the Itô formula, knowing that setting $\psi:x\mapsto\varphi(\|x\|^2)$, we have $\nabla\psi(x)=2x\varphi'(\|x\|^2)$ and $\Delta\psi(x)=4\Big(\varphi'(\|x\|^2)+\|x\|^2\varphi''(\|x\|^2)\Big)$. Moreover (ii) is true for the same reason so that it only remains to show (iii).

We fix $\varphi\in C^2(\mathbb{R}^2)$. Applying the Itô formula to $\varphi(X_{t\wedge\tau_n}^{i,N})$ and summig over $i\in[\![1,N]\!]$, we get

$$\frac{1}{N}\sum_{i=1}^N\varphi(X_{t\wedge\tau_n}^{i,N})=\frac{1}{N}\sum_{i=1}^N\varphi(X_0^{i,N})+\frac{1}{N}\sum_{i=1}^N\int_0^{t\wedge\tau_n}\nabla\varphi(X_s^{i,N})\cdot\mathrm{d}B_s^i+\frac{1}{2N}\sum_{i=1}^N\int_0^{t\wedge\tau_n}\Delta\varphi(X_s^{i,N})\mathrm{d}s+\frac{\theta}{N^2}\int_0^{t\wedge\tau_n}\sum_{i,j=1}^N K(X_s^{i,N}-X_s^{j,N})\cdot\nabla\varphi(X_s^{i,N})\mathrm{d}s. \tag{4.9}$$

By symmetrizing, we get that for all $s\in[0,t\wedge\tau_n]$,

$$\sum_{i,j=1}^N K(X_s^{i,N}-X_s^{j,N})\cdot\nabla\varphi(X_s^{i,N})=\frac{1}{2}\sum_{i,j=1}^N K(X_s^{i,N}-X_s^{j,N})\cdot[\nabla\varphi(X_s^{i,N})-\nabla\varphi(X_s^{i,N})],$$

which together with (4.9) implies the result. □

4.3 Compactness

In this section, and only in this section, we allow for any value of $\theta>0$. We consider $f_0\in\mathcal{P}(\mathbb{R}^2)$. We consider also for all $N\geq 5$, $F_0^N\in\mathcal{P}_{sym,1}^*((\mathbb{R}^2)^N)$ with associated random variable $(X_0^{1,N},\dots,X_0^{N,N})$ such that μ_0^N goes weakly to f_0 in probability as $N\to\infty$ where $\mu_0^N=N^{-1}\sum_{i=1}^N\delta_{X_0^{1,N}}$. Applying Theorem 4.5, there exists a $KS(\theta,N)$-process on $[0,\tau_3^N)$ with initial law F_0^N which is denoted $(X_t^{i,N})_{t\in[0,\tau_3^N),i\in[\![1,N]\!]}$. We recall that $N_0 = (1+\lceil 2/(2-\theta)\rceil)\vee 5$ if $\theta \in (0,2)$ and $N_0 = 5$ if $\theta \geq 2$. We define the empirical measure for all $N \geq N_0$ and all $t \geq 0$

$$\mu_t^{N,\beta_N} = \mu_{t\wedge\beta_N}^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_{t\wedge\beta_N}^{i,N}},$$

where we set

$$\beta_N = \infty \quad \text{if} \quad \theta \in (0,2) \qquad \text{and} \qquad \beta_N = \tau_3^{N,\ell_N} \quad \text{if} \quad \theta \geq 2,$$

with $(\ell_N)_{N\geq 5}$ some increasing sequence of integers. It is clear that $(\mu_t^{N,\beta_N})_{t\geq 0} \in C([0,\infty),\mathcal{P}(\mathbb{R}^2))$ a.s. because $(X_t^{i,N})_{t\in[0,\beta_N),i\in[\![1,N]\!]}$ is a.s. continuous. This section is devoted to show the following Theorem 4.4-(i) and Theorem 4.6-(i). To this end, we need to prove some preliminary estimates.

Proposition 4.8. *(i) There exists some nondecreasing $\psi \in C(\mathbb{R}_+)$ such that $\psi(0) = 1$, $\psi(2r) \leq C\psi(r)$ for some constant $C > 0$, $\lim_{r\to\infty}\psi(r) = \infty$ and*

$$\text{for all } T \geq 0, \qquad M_T := \sup_{N\geq N_0} \mathbb{E}\Big[\sup_{t\in[0,T]} \frac{1}{N}\sum_{i=1}^N \psi(\|X_{t\wedge\beta_N}^{i,N}\|^2)\Big] < \infty.$$

(ii) For all $T > 0$, $\varepsilon > 0$, there exists $K_{\varepsilon,T}$ a compact set of $\mathcal{P}(\mathbb{R}^2)$ such that,

$$\text{for all } N \geq N_0, \quad \mathbb{P}(\forall t \in [0,T], \mu_t^{N,\beta_N} \in K_{\varepsilon,T}) \geq 1-\varepsilon.$$

Démonstration. We first prove (i).

Step 1. We first show that $(\|X_0^{1,N}\|)_{N\geq N_0}$ is tight. Since μ_0^N goes to f_0 in probability, we conclude that $\limsup_{N\to\infty}\mathbb{E}[\mu_0^N(\{x \in \mathbb{R}^2 : ||x|| \geq K\}] \leq f_0(\{x \in \mathbb{R}^2 : ||x|| \geq K\})$ for all $K \geq 1$. Since it holds that $\mathbb{E}[\mu_0^N(\{x \in \mathbb{R}^2 : ||x|| \geq K\}] = \mathbb{P}(||X_0^{1,N}|| \geq K)$, we conclude that $\limsup_{K\to\infty}\limsup_{N\to\infty}\mathbb{P}(||X_0^{1,N}|| \geq K) = 0$.

Step 2. We conclude the proof of (i). Since $(X_0^{1,N})_{N\geq N_0}$ is tight, by the de la Vallée Poussin theorem there exists some non decreasing $\psi \in C(\mathbb{R}_+)$ such that $\lim_{r\to\infty}\psi(r) = \infty$ and

$$\sup_{N\geq N_0}\mathbb{E}[\psi(\|X_0^{1,N}\|^2)] < \infty. \tag{4.10}$$

Moreover, this is tedious but not so difficult to show that one can choose $\psi \in C^2(\mathbb{R}_+)$ such that $\psi(0) = 1$, such that $\psi(2r) \leq C\psi(r)$ for some constant $C > 0$ and $(1+r)|\psi'(r)| + r|\psi''(r)| \leq 1$. We fix $T > 0$. Applying Lemma 4.7-(iii) and taking the supremum we get,

$$\sup_{t\in[0,T]}\frac{1}{N}\sum_{i=1}^N \psi(\|X_{t\wedge\beta_N}^{i,N}\|^2) \leq \frac{1}{N}\sum_{i=1}^N \psi(\|X_0^{i,N}\|^2) + I_T^1 + I_T^2 + I_T^3, \tag{4.11}$$

where

$$I_T^1 = \frac{2}{N}\sum_{i=1}^N \int_0^{T\wedge\beta_N} (|\psi'(\|X_s^{i,N}\|^2)| + \|X_s^{i,N}\|^2|\psi''(\|X_s^{i,N}\|^2))|\mathrm{d}s,$$

$$I_T^2 = 2\sup_{t\in[0,T]}\Big(\frac{1}{N}\sum_{i=1}^N \int_0^{t\wedge\beta_N} \psi'(\|X_s^{i,N}\|^2)X_s^{i,N}\cdot \mathrm{d}B_s^i\Big),$$

$$I_T^3 = \frac{\theta}{N^2}\sup_{t\in[0,T]}\int_0^{t\wedge\beta_N}\sum_{i,j=1}^N K(X_s^{i,N}-X_s^{j,N})\cdot[X_s^{i,N}\psi'(\|X_s^{i,N}\|^2) - X_s^{j,N}\psi'(\|X_s^{j,N}\|^2)]\mathrm{d}s.$$

where I_T^3 was obtained thanks to a symmetrization argument. Thanks to the hypothesis we made on ψ, we have $I_T^1 \leq 2(T \wedge \beta_N)$ a.s. Moreover, since $||K(x)|| \leq ||x||^{-1}$, and since $g : x \in \mathbb{R}^2 \mapsto x\psi'(\|x\|^2)$ is differentiable with a bounded differential according to the properties of ψ, we get that $I_T^3 \leq \theta \|\nabla g\|_\infty (T \wedge \beta_N)$ a.s. Finally, using the Cauchy-Schwarz and the Doob inequalities, we find that there exists a constant $C > 0$ such that

$$\mathbb{E}[I_T^2] \leq C\Big(\frac{1}{N^2}\sum_{i=1}^N \mathbb{E}\Big[\int_0^{T\wedge\beta_N} (\psi'(\|X_s^{i,N}\|^2))^2 \|X_s^{i,N}\|^2 \mathrm{d}s\Big]\Big)^{1/2} \leq CT^{1/2},$$

where we used the properties of ψ. This implies, by taking the expectation in (4.11) that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \frac{1}{N}\sum_{i=1}^N \psi(\|X_{t\wedge\beta_N}^{i,N}\|^2)\Big] \leq \mathbb{E}\Big[\sup_{N\geq N_0} \psi(\|X_0^{1,N}\|^2)\Big] + (2+\theta\|\nabla g\|_\infty)T + CT^{1/2}.$$

Combining with (4.10), we get the result.

We now prove (ii). We fix $T > 0$ and $\varepsilon > 0$ and, for some positive sequences $(A_k^\varepsilon)_{k\geq 0}$ and $(\eta_k^\varepsilon)_{k\geq 0}$ such that $\lim_{k\to\infty} A_k^\varepsilon = \infty$ and $\lim_{k\to\infty} \eta_k^\varepsilon = 0$, to be chosen later, we introduce the compact

$$K_{\varepsilon,T} = \bigcap_{k\geq 0} \Big\{\mu \in \mathcal{P}(\mathbb{R}^2) : \mu(B(0, A_k^\varepsilon)^c) \leq \eta_k^\varepsilon\Big\}$$

of $\mathcal{P}(\mathbb{R}^2)$. Now for $N \geq N_0$,

$$\begin{aligned}
\mathbb{P}(\exists t \in [0,T] : \mu_t^{N,\beta_N} \notin K_{\varepsilon,T}) &\leq \sum_{k\geq 0} \mathbb{P}\Big(\exists t \leq T, \mu_t^{N,\beta_N}(B(0, A_k^\varepsilon)) < 1 - \eta_k^\varepsilon\Big) \\
&= \sum_{k\geq 0} \mathbb{P}\Big(\exists t \leq T, \frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\|X_{t\wedge\beta_N}^{i,N}\| > A_k^\varepsilon} \geq \eta_k^\varepsilon\Big) \\
&\leq \sum_{k\geq 0} \mathbb{P}\Big(\exists t \leq T, \frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\psi(\|X_{t\wedge\beta_N}^{i,N}\|^2) > \psi((A_k^\varepsilon)^2)} \geq \eta_k^\varepsilon\Big) \\
&\leq \sum_{k\geq 0} \mathbb{P}\Big(\exists t \leq T, \frac{1}{N}\sum_{i=1}^N \psi(\|X_{t\wedge\beta_N}^{i,N}\|^2) \geq \eta_k^\varepsilon \psi((A_k^\varepsilon)^2)\Big) \\
&\leq \sum_{k\geq 0} \frac{\mathbb{E}\Big(\sup_{t\in[0,T]} \frac{1}{N}\sum_{i=1}^N \psi(\|X_{t\wedge\beta_N}^{i,N}\|^2)\Big)}{\eta_k^\varepsilon \psi((A_k^\varepsilon)^2)} \\
&\leq M_T \sum_{k\geq 0} \frac{1}{\eta_k^\varepsilon \psi((A_k^\varepsilon)^2)}
\end{aligned}$$

by (i). We now choose A_k^ε such that $\sum_{k\geq 0} 1/\sqrt{\psi((A_k^\varepsilon)^2)} \leq \varepsilon/M_T$ and $\eta_k^\varepsilon = 1/\sqrt{\psi((A_k^\varepsilon)^2)}$. We find that $\mathbb{P}(\exists t \in [0,T] : \mu_t^{N,\beta_N} \notin K_{\varepsilon,T}) \leq \varepsilon$ as desired.

□

Proof of Theorem 4.4-(i) and Theorem 4.6-(i). We recall that one can find a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of elements of $C^2(\mathbb{R}^2)$ such that

$$\|\varphi_n\|_\infty + \|\nabla\varphi_n\|_\infty + \|\nabla^2\varphi_n\|_\infty \leq 1 \quad \text{for all } n \geq 0,$$

and such that the distance δ defined for all $f,\, g \in \mathcal{P}(\mathbb{R}^2)$ by

$$\delta(f,g) = \sum_{n\geq 0} 2^{-n}\Big| \int \varphi_n(x) f(\mathrm{d}x) - \int \varphi_n(x) g(\mathrm{d}x)\Big|,$$

metrizes the weak convergence topology in $\mathcal{P}(\mathbb{R}^2)$. Since $C([0,\infty),\mathcal{P}(\mathbb{R}^2))$ is endowed with the topology of the uniform convergence on compact sets, it is sufficient to prove the tightness of $((\mu_t^N)_{t\geq 0})_{N\geq N_0}$ on $C([0,T],\mathcal{P}(\mathbb{R}^2))$ for every $T>0$. We fix $T>0$ and set the following compact set of $C([0,T],\mathcal{P}(\mathbb{R}^2))$,

$$\mathcal{K}_{A,K} := \Big\{ (f_t)_{t\geq 0} \in C([0,T],\mathcal{P}(\mathbb{R}^2)) : \forall t\in[0,T], f_t \in K \text{ and } \sup_{\substack{s,t\in[0,T]\\ s\neq t}} \frac{\delta(f(t),f(s))}{|t-s|^{1/4}} \leq A \Big\},$$

for all $A>0$ and for all K compact set of $\mathcal{P}(\mathbb{R}^2)$. It suffices to show

$$\sup_{N\geq N_0} \mathbb{E}\Big[\sup_{\substack{s,t\in[0,T]\\ s\neq t}} \frac{\delta(\mu_t^{N,\beta_N},\mu_s^{N,\beta_N})}{|t-s|^{1/4}} \Big] < \infty. \tag{4.12}$$

Indeed, if it is the case, thanks to the Markov inequality, for all $\varepsilon>0$, there exists $A_\varepsilon>0$ such that

$$\text{for all } N\geq N_0, \quad \mathbb{P}\Big(\sup_{\substack{s,t\in[0,T]\\ s\neq t}} \frac{\delta(\mu_t^{N,\beta_N},\mu_s^{N,\beta_N})}{|t-s|^{1/4}} \leq A_\varepsilon \Big) \geq 1-\varepsilon,$$

and using $(K_{\varepsilon,T})_{\varepsilon>0}$ defined in Proposition 4.8-(ii), we have for all $N\geq N_0$,

$$\mathbb{P}((\mu_t^{N,\beta_N})_{t\geq 0} \in \mathcal{K}_{A_\varepsilon,K_{\varepsilon,T}}) \geq 2(1-\varepsilon)-1 = 1-2\varepsilon,$$

which ends the proof. Let us show (4.12). Using Lemma 4.7-(iii) and the fact that $\|K(x)\| \leq \|x\|^{-1}$ for $x\neq 0$, we have for all $n\geq 0$, all $T>0$, all $s,t\in[0,T]$ such that $s\neq t$,

$$\frac{1}{N}\Big| \sum_{i=1}^N (\varphi_n(X_{t\wedge\beta_N}^{i,N}) - \varphi_n(X_{s\wedge\beta_N}^{i,N}))\Big| \leq S^1 + S^2,$$

where

$$S^1 = \frac{1}{N}\Big| \sum_{i=1}^N \int_{s\wedge\beta_N}^{t\wedge\beta_N} \nabla\varphi_n(X_u^{i,N})\mathrm{d}B_u^i \Big|,$$

$$S^2 = \frac{1}{N}\sum_{i=1}^N \int_{s\wedge\beta_N}^{t\wedge\beta_N} \frac{1}{2}|\Delta\varphi_n(X_u^{i,N})|\mathrm{d}u + \frac{\theta}{2N}\sum_{\substack{i,j=1\\ j\neq i}}^N \int_{s\wedge\beta_N}^{t\wedge\beta_N} \frac{\|\nabla\varphi_n(X_u^{i,N}) - \nabla\varphi_n(X_u^{j,N})\|}{\|X_u^{i,N}-X_u^{j,N}\|}\mathrm{d}u.$$

Since $\nabla^2\varphi_n$ is bounded, S^2 is bounded by $C(t\wedge\beta_N - s\wedge\beta_N) \leq C(t-s)$ with $C>0$ a constant. Moreover, by the Dubins-Schwarz theorem, even if it means to enlarge the probability space, there exists a Brownian motion $(\tilde{B}_t)_{t\geq 0}$ such that

$$\sum_{i=1}^N \int_0^{t\wedge\beta_N} \nabla\varphi_n(X_u^{i,N})\mathrm{d}B_u^i = \tilde{B}_{\int_0^{t\wedge\beta_N}\sum_{i=1}^N \|\nabla\varphi_n(X_u^{i,N})\|^2\mathrm{d}u}.$$

Recalling that $||\nabla\varphi_n||_\infty \le 1$ and setting $C_{\tilde B} := \sup_{u,v\in[0,T],u\neq v}(\tilde B_u - \tilde B_v)/|u-v|^{1/4}$, this implies that

$$\begin{aligned} S^1 &\le N^{-1}C_{\tilde B}\Big(\sum_{i=1}^N \int_{s\wedge\beta_N}^{t\wedge\beta_N} \|\nabla\varphi_n(X_u^{i,N})\|^2 \mathrm{d}u\Big)^{1/4} \\ &\le C_{\tilde B}N^{-3/4}(t\wedge\beta_N - s\wedge\beta_N)^{1/4} \\ &\le C_{\tilde B}N^{-3/4}(t-s)^{1/4}. \end{aligned}$$

But $\mathbb{E}[C_{\tilde B}]<\infty$ thanks to the Kolmogorov criterion, so that recalling the definition of δ,

$$\begin{aligned} \mathbb{E}\Big[\sup_{\substack{s,t\in[0,T]\\ s\neq t}} \frac{\delta(\mu_t^{N,\beta_N},\mu_s^{N,\beta_N})}{|t-s|^{1/4}}\Big] &\le \sum_{n\ge 0} 2^{-n}\mathbb{E}\Big[\frac{1}{N}\Big|\sum_{i=1}^N(\varphi_n(X^{i,N}_{t\wedge\beta_N}) - \varphi_n(X^{i,N}_{s\wedge\beta_N}))\Big|\Big] \\ &\le \mathbb{E}[2(CT^{3/4} + N^{-3/4}C_{\tilde B})]. \end{aligned}$$

This proves (4.12). □

4.4 The subcritical case

In this section we prove Theorem 4.4-(ii). We recall that in this case $\theta \in (0,2)$. We begin with a simple geometrical result which is crucial for our purpose, that we proved in [21] but that we recall for completeness.

Lemma 4.9. *For all pair of nonincreasing functions $\varphi,\psi:(0,\infty)\to\mathbb{R}_+$, for all $X,Y,Z\in\mathbb{R}^2\setminus\{0\}$ such that $X+Y+Z=0$, we have*

$$\Delta = [\varphi(\|X\|)X + \varphi(\|Y\|)Y + \varphi(\|Z\|)Z]\cdot[\psi(\|X\|)X + \psi(\|Y\|)Y + \psi(\|Z\|)Z] \ge 0.$$

More precisely, if e.g. $\|X\|\le\|Y\|\le\|Z\|$, we have

$$\Delta \ge [\varphi(\|X\|)-\varphi(\|Y\|)][\psi(\|X\|)-\psi(\|Y\|)]\|X\|^2.$$

Démonstration. We may study only the case where $\|X\|\le\|Y\|\le\|Z\|$. Since $Y=-X-Z$,

$$\begin{aligned} \varphi(\|X\|)X + \varphi(\|Y\|)Y + \varphi(\|Z\|)Z &= \lambda X - \mu Z, \\ \psi(\|X\|)X + \psi(\|Y\|)Y + \psi(\|Z\|)Z &= \lambda' X - \mu' Z, \end{aligned}$$

where $\lambda = \varphi(\|X\|)-\varphi(\|Y\|)\ge 0$, $\mu=\varphi(\|Y\|)-\varphi(\|Z\|)\ge 0$, $\lambda'=\psi(\|X\|)-\psi(\|Y\|)\ge 0$ and $\mu'=\psi(\|Y\|)-\psi(\|Z\|)\ge 0$. Therefore,

$$\Delta = \lambda\lambda'\|X\|^2 + \mu\mu'\|Z\|^2 - (\lambda\mu'+\lambda'\mu)X\cdot Z \ge \lambda\lambda'\|X\|^2$$

as desired, because $X\cdot Z\le 0$. Indeed, if $X\cdot Z>0$, then $\|Y\|^2=\|Z+X\|^2=\|Z\|^2+\|X\|^2+2X\cdot Z > \|Z\|^2\ge\|Y\|^2$, which is absurd. □

The next result follows the proof of [21, Proposition 5].

Proposition 4.10. *For all $\gamma \in (\theta, 2)$, there exists a constant $C_{\theta,\gamma} > 0$ such that for all $t \geq 0$,*

$$\sup_{N \geq N_0} \mathbb{E}\Big[\int_0^t \|X_s^{1,N} - X_s^{2,N}\|^{\gamma-2} \mathrm{d}s\Big] \leq C_{\theta,\gamma}(1+t).$$

Démonstration. We fix $a > 0$ and set $\varphi_a(r) = (r+a)^{\gamma/2}/(1+(r+a)^{\gamma/2})$ whence

$$\varphi_a'(r) = \frac{\gamma}{2}\frac{(r+a)^{\gamma/2-1}}{(1+(r+a)^{\gamma/2})^2} \quad \text{and} \quad \varphi_a''(r) = \frac{\gamma}{2}\frac{(r+a)^{\gamma/2-2}}{(1+(r+a)^{\gamma/2})^2}\Big(\frac{\gamma}{2} - 1 - \gamma\frac{(r+a)^{\gamma/2}}{1+(r+a)^{\gamma/2}}\Big),$$

which implies since $a \geq (\gamma/2)a$,

$$\begin{aligned}
\varphi_a'(r) + r\varphi_a''(r) &= \varphi_a'(r)\Big(\frac{\frac{\gamma}{2}r + a}{r+a} - \gamma\frac{r(r+a)^{\gamma/2-1}}{1+(r+a)^{\gamma/2}}\Big)\\
&\geq \frac{\gamma}{2}\varphi_a'(r)\Big(1 - 2\frac{r(r+a)^{\gamma/2-1}}{1+(r+a)^{\gamma/2}}\Big)\\
&\geq \frac{\gamma}{2}\varphi_a'(r)\Big(1 - 2\frac{(r+a)^{\gamma/2}}{1+(r+a)^{\gamma/2}}\Big).
\end{aligned}$$

Let $c = (1+\theta/\gamma)/2 \in (\theta/\gamma, 1)$ and $u = (r+a)^{\gamma/2}$. For $0 \leq u \leq (1-c)/(3+c)$ it holds $(1-c) - 2u/(1+u) \geq (1-c)/2$ so that in this case

$$\varphi_a'(r) + r\varphi_a''(r) \geq \frac{\gamma}{2}\varphi_a'(r)\Big(c + \frac{1-c}{2}\Big) \geq \frac{c\gamma}{2}\varphi_a'(r).$$

Otherwise if $u > (1-c)/(3+c)$ i.e. $(r+a) > v = ((1-c)/(3+c))^{2/\gamma}$,

$$1 - c - 2\frac{u}{1+u} \geq -c - 1,$$

since $u/(1+u) \leq 1$ so that

$$\varphi_a'(r) + r\varphi_a''(r) \geq \frac{c\gamma}{2}\varphi_a'(r) - (1+c)\frac{\gamma}{2}\varphi_a'(v).$$

We conclude the existence of $C_1 > 0$ only depending on θ and γ such that

$$\varphi_a'(r) + r\varphi_a''(r) \geq c\frac{\gamma}{2}\varphi_a'(r) - C_1. \tag{4.13}$$

Thus, applying Lemma 4.7-(ii) to $\varphi_a(\|X_t^{1,N} - X_t^{2,N}\|^2)$, using (4.13) and taking the expectation, we find

$$\mathbb{E}(\varphi_a(\|X_t^{1,N} - X_t^{2,N}\|^2)) \geq \mathbb{E}(\varphi_a(\|X_0^{1,N} - X_0^{2,N}\|^2)) + \Big(2c\gamma - \frac{4\theta}{N}\Big)S_t^1 - \frac{2\theta}{N}S_t^2 - 4C_1 t, \tag{4.14}$$

where

$$S_t^1 = \mathbb{E}\Big[\int_0^t \varphi_a'(\|X_s^{1,N} - X_s^{2,N}\|^2)\mathrm{d}s\Big]$$

and

$$S_t^2 = \mathbb{E}\Big[\int_0^t \varphi_a'(\|X_s^{1,N} - X_s^{2,N}\|^2)(X_s^{1,N} - X_s^{2,N}) \cdot \sum_{\substack{i=1\\ i\neq 1,2}}^N \Big(\frac{X_s^{1,N} - X_s^{i,N}}{\|X_s^{1,N} - X_s^{i,N}\|^2} + \frac{X_s^{i,N} - X_s^{2,N}}{\|X_s^{i,N} - X_s^{2,N}\|^2}\Big)\mathrm{d}s\Big].$$

By exchangeability, we get that $S_t^2/(N-2)$ equals

$$\mathbb{E}\Big[\int_0^t \varphi_a'(\|X_s^{1,N} - X_s^{2,N}\|^2)(X_s^{1,N} - X_s^{2,N}) \cdot \Big(\frac{X_s^{1,N} - X_s^{3,N}}{\|X_s^{1,N} - X_s^{3,N}\|^2} + \frac{X_s^{3,N} - X_s^{2,N}}{\|X_s^{3,N} - X_s^{2,N}\|^2}\Big)\mathrm{d}s\Big]$$

so that by exchangeability again,

$$\frac{S_t^2}{N-2} = -\frac{1}{3}\mathbb{E}\Big[\int_0^t F_a(X_s^{1,N}, X_s^{2,N}, X_s^{3,N})\mathrm{d}s\Big], \tag{4.15}$$

where

$$\begin{aligned} F_a(x,y,z) =& \varphi_a'(\|X\|^2)X \cdot \Big(\frac{Z}{\|Z\|^2} + \frac{Y}{\|Y\|^2}\Big) \\ &+ \varphi_a'(\|Y\|^2)Y \cdot \Big(\frac{X}{\|X\|^2} + \frac{Z}{\|Z\|^2}\Big) \\ &+ \varphi_a'(\|Z\|^2)Z \cdot \Big(\frac{Y}{\|Y\|^2} + \frac{X}{\|X\|^2}\Big), \end{aligned}$$

with $X = x-y$, $Y = y-z$ and $Z = z-x$. We now introduce $G_a(x,y,z) = \varphi_a'(\|X\|^2) + \varphi_a'(\|Y\|^2) + \varphi_a'(\|Z\|^2)$ and note that for all X, Y, $Z \in \mathbb{R}^2 \setminus \{0\}$,

$$G_a(x,y,z) = \varphi_a'(\|X\|^2)X \cdot \frac{X}{\|X\|^2} + \varphi_a'(\|Y\|^2)Y \cdot \frac{Y}{\|Y\|^2} + \varphi_a'(\|Z\|^2)Z \cdot \frac{Z}{\|Z\|^2}.$$

Hence for all X, Y, $Z \in \mathbb{R}^2 \setminus \{0\}$, $G_a(x,y,z) + F_a(x,y,z)$ equals

$$\Big(\varphi_a'(\|X\|^2)X + \varphi_a'(\|Y\|^2)Y + \varphi_a'(\|Z\|^2)Z\Big) \cdot \Big(\frac{X}{\|X\|^2} + \frac{Y}{\|Y\|^2} + \frac{Z}{\|Z\|^2}\Big),$$

which is nonnegative according to Lemma 4.9, since $r \to r^{-2}$ and $r \to \varphi_a'(r)$ are both nonnegative and nonincreasing on $(0,\infty)$ and $X+Y+Z=0$. Thus $F_a(x,y,z) \geq -G_a(x,y,z)$, which injected in (4.15) gives

$$\begin{aligned} S_t^2 &\leq \frac{N-2}{3}\mathbb{E}\Big[\int_0^t G_a(X_s^{1,N}, X_s^{2,N}, X_s^{3,N})\mathrm{d}s\Big] \\ &= (N-2)\mathbb{E}\Big[\int_0^t \varphi_a'(\|X_s^{1,N} - X_s^{2,N}\|^2)\mathrm{d}s\Big], \end{aligned} \tag{4.16}$$

since $\int_0^t \mathbb{1}_{\{X_s^{1,N}=X_s^{2,N}\}}\mathrm{d}s = 0$ a.s. according to Theorem 4.5. Gathering (4.14) and (4.16), the fact that φ_a is nonnegative and bounded by 1 and recalling the definition of φ'_a, we get the existence of a positive constant C such that

$$1 + Ct \geq 2(c\gamma - \theta)\mathbb{E}\Big[\int_0^t \frac{(\|X_s^{1,N} - X_s^{2,N}\|^2 + a)^{\gamma/2-1}}{(1 + \|X_s^{1,N} - X_s^{2,N}\|^2 +)^{\gamma/2})^2}\mathrm{d}s\Big].$$

Recalling that $c = (1 + \theta/\gamma)/2$ we get that $c\gamma - \theta = (\gamma - \theta)/2 > 0$, so that

$$\mathbb{E}\Big[\int_0^t \frac{(\|X_s^{1,N} - X_s^{2,N}\|^2 + a)^{\gamma/2-1}}{(1 + \|X_s^{1,N} - X_s^{2,N}\|^2 +)^{\gamma/2})^2}\mathrm{d}s\Big] \leq \frac{1 + Ct}{\gamma - \theta}.$$

Recalling that $\gamma/2 - 1 \leq 0$, we know that there exists $A_\gamma > 0$ only depending on γ such that $x^{\gamma/2-1} \leq 2x^{\gamma/2-1}/(1 + x^{\gamma/2})^2 + A_\gamma$, this implies

$$\mathbb{E}\Big[\int_0^t (\|X_s^{1,N} - X_s^{2,N}\|^2 + a)^{\gamma/2-1}\mathrm{d}s\Big] \leq \frac{2(1 + Ct)}{\gamma - \theta} + A_\gamma t,$$

so that the monotone convergence theorem completes the proof. □

Proof of Theorem 4.4-(ii). Recall that $\theta \in (0, 2)$, $f_0 \in \mathcal{P}(\mathbb{R}^2)$ and that $(\mu_t^N)_{t\geq 0}$ is the empirical process associated to a solution $(X_t^{i,N})_{t\geq 0, i\in [\![1,N]\!]}$ to (4.5), for each $N \geq N_0$. By Theorem 4.4-(i), we know that the family $((\mu_t^N)_{t\geq 0}, N \geq N_0)$ is tight in $C([0,\infty), \mathcal{P}(\mathbb{R}^2))$. We now consider a sequence $(N_k)_{k\geq 0}$ and a random variable $(\mu_t)_{t\geq 0} \in C([0,\infty), \mathcal{P}(\mathbb{R}^2))$ such that $\lim_k N_k = \infty$ and $(\mu_t^{N_k})_{t\geq 0}$ goes to $(\mu_t)_{t\geq 0}$ in law as $k \to \infty$. We have $\mu_0 = f_0$ since by hypothesis, μ_0^N goes weakly to f_0 in probability as $N \to \infty$.

Step 1. We prove that $\mathbb{E}[\int_0^t \|x - y\|^{\gamma-2}\mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y)\mathrm{d}s] < \infty$ First, by exchangeability, we get that for all $M > 0$,

$$\mathbb{E}\Big(\int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (\|x - y\|^{\gamma-2} \wedge M)\mu_s^{N_k}(\mathrm{d}x)\mu_s^{N_k}(\mathrm{d}y)\mathrm{d}s\Big) \leq \frac{M}{N_k} + \frac{N_k - 1}{N_k}\mathbb{E}\Big(\int_0^t \|X_s^{1,N_k} - X_s^{2,N_k}\|^{\gamma-2}\mathrm{d}s\Big).$$

According to Proposition 4.10, since $(\mu_t^{N_k})_{t\geq 0}$ goes to $(\mu_t)_{t\geq 0}$ in law as $k \to \infty$ there exists $C > 0$ such that

$$\mathbb{E}\Big(\int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (\|x - y\|^{\gamma-2} \wedge M)\mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y)\mathrm{d}s\Big) \leq C_{\theta,\gamma}(1 + T).$$

Since this holds true for every $M > 0$, the monotone convergence theorem gives us the result by letting $M \to \infty$.

Step 2. It only remains to check that μ is a.s. a weak solution to (4.1). We apply Lemma 4.7-(iii) to $\varphi \in C_b^2(\mathbb{R}^2)$ and get $I_1(\mu^{N_k}) - I_2(\mu^{N_k}) = M_k(t)$, where

$$M_k(t) = \frac{1}{N_k}\int_0^t \sum_{i=1}^{N_k} \nabla\varphi(X_s^{i,N_k}) \cdot \mathrm{d}B_s^i,$$

and for all $\nu \in C([0,\infty),\mathcal{P}(\mathbb{R}^2))$,

$$I_t^1(\nu) = \int_{\mathbb{R}^2} \varphi(x)\nu_t(\mathrm{d}x) - \int_{\mathbb{R}^2} \varphi(x)\nu_0(\mathrm{d}x) - \frac{1}{2}\int_0^t \int_{\mathbb{R}^2} \Delta\varphi(x)\nu_s(\mathrm{d}x)\mathrm{d}s,$$
$$I_t^2(\nu) = \frac{\theta}{2}\int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} K(x-y)\cdot[\nabla\varphi(x) - \nabla\varphi(y)]\nu_s(\mathrm{d}x)\nu_s(\mathrm{d}y)\mathrm{d}s.$$

Since φ and $\Delta\varphi$ are continuous and bounded, $I_t^1(\nu)$ is clearly continuous with respect to ν and bounded. Since $g : (x,y) \in (\mathbb{R}^2)^2 \mapsto K(x-y)\cdot[\nabla\varphi(x) - \nabla\varphi(y)]\mathbb{1}_{x\neq y}$ is continuous and bounded on $(\mathbb{R}^2)^2 \setminus D$ where $D = \{(x,y) \in (\mathbb{R}^2)^2 : x = y\}$, $I_t^2(\nu)$ is also continuous $\mathcal{L}((\mu_t)_{t\geq 0})$-a.e, where $\mathcal{L}((\mu_t)_{t\geq 0})$ is the law of $(\mu_t)_{t\geq 0}$. Indeed, Step 1 implies that for all $t \geq 0$, $\int_0^t (\mu_s \otimes \mu_s)(D)\mathrm{d}s = 0$ a.s. Thus, $I_t^1(\mu^{N_k}) - I_t^2(\mu^{N_k})$ goes in law to $I_t^1(\mu) - I_t^2(\mu)$ as $k \to \infty$.

Moreover, $\lim_{k\to\infty} \mathbb{E}[(I_t^1(\mu^{N_k}) - I_t^2(\mu^{N_k}))^2] = \lim_{k\to\infty} \mathbb{E}[M_k(t)^2] = 0$, which implies $I_t^1(\mu^{N_k}) - I_t^2(\mu^{N_k})$ goes to 0 in probability as $k \to \infty$. We conclude that for each $t \geq 0$,

$$I_t^1(\mu) = I_t^2(\mu) \quad \text{a.s.}$$

This implies the result, i.e. that a.s., for all $t \geq 0$, $I_t^1(\mu) = I_t^2(\mu)$ by continuity, since $(\mu_t)_{t\geq 0} \in C([0,\infty),\mathcal{P}(\mathbb{R}^2))$, since $\varphi \in C_b^2$ and since g is bounded and for all $s \geq 0$, $\mu_s \in \mathcal{P}(\mathbb{R}^2)$. □

4.5 Estimation of the first triple collision time

Since the existence of the particle system is not guaranteed after the time of the first triple collision, meaning the first collision between three particles (which occurs a.s. in the case where $\theta > 2(N-2)/(N-1)$ according to Theorem 4.5), we need to show that this time goes to the infinite as the number of particle increases to the infinite. This work has been more or less done in Fournier-Tardy [22] but since the interest in [22] was to study the KS-process near the blow-up instant and that there is no solution to (4.5) in the classical sense in this particular region of the time, the Dirichlet forms theory is used to provide a sense to such a process, which makes the proofs much more complicated.

We define for all $\theta > 0$, all $N \geq 2$,

$$d_{\theta,N}(k) = (k-1)\Big(2 - \frac{k\theta}{N}\Big) \text{ for all } k \in [\![2,N]\!] \quad \text{and} \quad k_2 = \min\{k \geq 3 : d_{2,N}(k) < 2\}.$$

This section is devoted to prove the following result. Recall the definitions of τ_k^N and $\tau_k^{N,\ell}$ for $N \geq 2$, $k \in [\![2,N]\!]$ and $\ell \geq 1$, see Subsection 4.1.5.

Proposition 4.11. *If $\theta = 2$, we consider for all $N \geq N_0$, $F_0^N \in \mathcal{P}^*_{sym,1}(\mathbb{R}^2)$ and a $KS(\theta,N)$-process $(X_t^{i,N})_{t\in[0,\tau_3^N),i\in[\![1,N]\!]}$ on $[0,\tau_3^N)$ with initial law F_0^N, then*

(i) $\tau_3^N = \tau_{k_2}^N$ a.s.

(ii) There exists an increasing sequence of deterministic integers $(\ell_N)_{N\geq 5}$ such that setting $\beta_N = \tau_3^{N,\ell_N}$, then for all $t \geq 0$, $\mathbb{P}(\beta_N \leq t) \to_{N\to\infty} 0$.

Point (i) explains that the first collision involving strictly more than two particles is a collision between at least k_2 particles. Moreover, point (ii) means that the instant of this collision tends to infinity in probability as the number of particles of the system tends to infinity.

We begin with some preliminary results about the behaviour of the empirical variance of subsets of the particles. The following has been essentially treated in Fournier-Jourdain [20]. Recall the definition of $R_{[\![1,N]\!]}$, see Subsection 4.1.5 and the definition of a squared Bessel process, see Revuz-Yor [44, Chapter XI].

Lemma 4.12. *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration, τ a $(\mathcal{F}_t)_{t\geq 0}$-stopping time such that $\tau \leq \tau_N^N$ a.s. and $(\tau_n)_{n\geq 0}$ a sequence of increasing $(\mathcal{F}_t)_{t\geq 0}$-stopping times such that $\lim_{n\to\infty} \tau_n = \tau$. If $\theta > 0$ and $N \geq 5$, then for all $KS(\theta, N)$-process $(X_t^{i,N})_{t\in[0,\tau), i\in[\![1,N]\!]}$ on $[0,\tau)$, $(R_{[\![1,N]\!]}(X_t^N))_{t\in[0,\tau)}$ is a $(\mathcal{F}_t)_{t\geq 0}$-mesurable squared Bessel process with dimension $d_{\theta,N}(N)$ restricted to $[0,\tau)$.*

Démonstration. We apply Lemma 4.7-(ii) with $\varphi = I_d$ and we get for all $i \neq j \in [\![1,N]\!]$, all $n \geq 0$,

$$\begin{aligned}\|X_{t\wedge\tau_n}^{i,N} - X_{t\wedge\tau_n}^{j,N}\|^2 =& \|X_0^{i,N} - X_0^{j,N}\|^2 + 2\int_0^{t\wedge\tau_n} (X_s^{i,N} - X_s^{j,N}) \cdot \mathrm{d}(B_s^i - B_s^j)\\ &+ 4\Big(1 - \frac{\theta}{N}\Big)(t\wedge\tau_n) - \frac{2\theta}{N} J_t^n,\end{aligned} \tag{4.17}$$

where

$$J_t^n = \sum_{\substack{k=1\\k\neq i,j}}^N \int_0^{t\wedge\tau_n} \Big(\frac{X_s^{i,N} - X_s^{k,N}}{\|X_s^{i,N} - X_s^{k,N}\|^2} + \frac{X_s^{k,N} - X_s^{j,N}}{\|X_s^{k,N} - X_s^{j,N}\|^2}\Big) \cdot (X_s^{i,N} - X_s^{j,N})\mathrm{d}s.$$

Using (4.17) together with the fact that $R_{[\![1,N]\!]}(x) = (2N)^{-1}\sum_{1\leq i\neq j\leq N} \|x^i - x^j\|^2$ imply by summing the last equality over $i \neq j \in [\![1,N]\!]$, and dividing by $2N$,

$$\begin{aligned}R_{[\![1,N]\!]}(X_{t\wedge\tau_n}^N) =& R_{[\![1,N]\!]}(X_0^N) + 2\int_0^{t\wedge\tau_n} \frac{1}{2N}\sum_{1\leq i\neq j\leq N} (X_s^{i,N} - X_s^{j,N}) \cdot \mathrm{d}(B_s^i - B_s^j)\\ &+ (N-1)\Big(2 - \frac{2\theta}{N}\Big)(t\wedge\tau_n) - \frac{\theta}{N^2}\int_0^{t\wedge\tau_n} S(X_s^N)\mathrm{d}s,\end{aligned}$$

where for all $x = (x^1, \ldots, x^N) \in (\mathbb{R}^2)^N$,

$$S(x) = \sum_{\substack{1\leq i,j,k\leq N\\ distincts}} \Big(\frac{x^i - x^k}{\|x^i - x^k\|^2} + \frac{x^k - x^j}{\|x^k - x^j\|^2}\Big) \cdot (x^i - x^j).$$

By oddness, we have for all $x \in (\mathbb{R}^2)^N$,

$$\begin{aligned}S(x) =& 2\sum_{\substack{1\leq i,j,k\leq N\\ distincts}} \frac{x^i - x^k}{\|x^i - x^k\|^2} \cdot (x^i - x^j)\\ =& 2\sum_{\substack{1\leq i,j,k\leq N\\ distincts}} \Big(1 + \frac{x^i - x^k}{\|x^i - x^k\|^2} \cdot (x^k - x^j)\Big) = 2N(N-1)(N-2) - S(x),\end{aligned}$$

so that $S(x) = N(N-1)(N-2)$. Since $(N-1)(2-2\theta/N) - \theta(N-1)(N-2)/N = (N-1)(2-\theta)$, we deduce that we can write for all $n \geq 0$,

$$R_{[\![1,N]\!]}(X^N_{t\wedge\tau_n}) = R_{[\![1,N]\!]}(X^N_0) + 2\int_0^{t\wedge\tau_n} R_{[\![1,N]\!]}(X^N_s)^{1/2}\mathrm{d}W^n_s + (N-1)(2-\theta)(t\wedge\tau_n), \quad (4.18)$$

with

$$(W^n_t)_{t\geq 0} = \Big(\frac{1}{2N}\sum_{1\leq i\neq j\leq N}\int_0^{t\wedge\tau_n}\frac{X^{i,N}_s - X^{j,N}_s}{R_{[\![1,N]\!]}(X^N_s)^{1/2}}\cdot \mathrm{d}(B^i_s - B^j_s)\Big)_{t\geq 0},$$

which is well defined because $R_{[\![1,N]\!]}(X^N_s) > 0$ for all $s\in[0,t\wedge\tau_n]$ since $\tau_n < \tau^N_N$ a.s.,

According to (4.18), the definition of a squared Bessel process (see Revuz-Yor [44, Chapter XI]) and since $d_{\theta,N}(N) = (N-1)(2-\theta)$, it only remains to show that for all $n\geq 0$, $(W^n_t)_{t\in[0,\tau_n)}$ is a 1-dimensional Brownian motion restricted to $[0,\tau_n)$ which we now do. For all $n\geq 0$, $\langle W^n\rangle_t$ equals

$$\frac{1}{4N^2}\sum_{\substack{1\leq i\neq j\leq N\\ 1\leq k\neq \ell\leq N}}\Big\langle\int_0^{\cdot}\frac{(X^{i,N}_s - X^{j,N}_s)}{R_{[\![1,N]\!]}(X^N_s)^{1/2}}\mathrm{d}(B^i_s - B^j_s), \int_0^{\cdot}\frac{(X^{k,N}_s - X^{\ell,N}_s)}{R_{[\![1,N]\!]}(X^N_s)^{1/2}}\cdot \mathrm{d}(B^k_s - B^\ell_s)\Big\rangle_{t\wedge\tau_n}$$

so that

$$\langle W^n\rangle_t = \frac{1}{4N^2}\sum_{\substack{1\leq i\neq j\leq N\\ 1\leq k\neq \ell\leq N}}(\delta_{i,k} - \delta_{i,\ell} - \delta_{j,k} + \delta_{j,\ell})\int_0^{t\wedge\tau_n}\frac{(X^{i,N}_s - X^{j,N}_s)\cdot(X^{k,N}_s - X^{\ell,N}_s)}{R_{[\![1,N]\!]}(X^N_s)}\mathrm{d}s.$$

By distinguishing cases according to whether one or two of the Kronecker symbols in the sum equals 1, we get

$$\langle W^n\rangle_t = \frac{1}{N^2}\sum_{1\leq i\neq j\leq N}\int_0^{t\wedge\tau_n}\frac{\|X^{i,N}_s - X^{j,N}_s\|^2}{R_{[\![1,N]\!]}(X^N_s)}\mathrm{d}s + \frac{1}{N^2}\int_0^{t\wedge\tau_n}\frac{\tilde{S}(X^N_s)}{R_{[\![1,N]\!]}(X^N_s)}\mathrm{d}s,$$

where

$$\tilde{S}(x) := \sum_{\substack{1\leq i,j,k\leq N\\ distincts}}(x^i - x^j)\cdot(x^i - x^k).$$

By oddness we get

$$\begin{aligned}\tilde{S}(x) &= \sum_{\substack{1\leq i,j,k\leq N\\ distincts}}\big(\|x^i - x^j\|^2 + (x^i - x^j)\cdot(x^j - x^k)\big)\\ &= 2N(N-2)R_{[\![1,N]\!]}(x) - \tilde{S}(x),\end{aligned}$$

so that $\tilde{S}(x) = N(N-2)R_{[\![1,N]\!]}(x)$ which implies that

$$\langle W^n\rangle_t = \frac{2}{N}(t\wedge\tau_n) + \frac{N-2}{N}(t\wedge\tau_n) = t\wedge\tau_n.$$

The conclusion easily follows thanks to the Lévy characterisation. □

In the following Lemma, we show that if particles indexed by $K \subset [\![1, N]\!]$ are far enough away from the other, then $(R_K(X_t^N))_{t\geq 0}$ behaves like a squared Bessel process with dimension $d_{\theta,N}(|K|)$. Indeed, we will explain in which sense we can neglect the interactions between particles indexed by K and particles indexed by K^c.

Lemma 4.13. *For all $T > 0$, $\alpha > 0$, $\theta > 0$, $N \geq 3$ and $K \subset [\![1, N]\!]$ with $|K| \in [\![2, N-1]\!]$, if τ is a stopping time for a filtration $(\mathcal{F}_t)_{t\geq 0}$ and $(X_t^{i,N})_{t\in[0,\tau), i\in[\![1,N]\!]}$ is a $KS(\theta, N)$-process on $[0, \tau)$, then the family $\Big((R_K(X^N_{\sigma^k_\alpha \wedge T+t}))_{t\in[0,\tilde{\sigma}^k_{\alpha,K}\wedge T - \sigma^k_{\alpha,K}\wedge T)}\Big)_{k\geq 1}$ is a family of squared Bessel process with dimension $d_{\theta,N}(|K|)$ restricted to $[0, \tilde{\sigma}^k_{\alpha,K} \wedge T - \sigma^k_{\alpha,K} \wedge T)$ driven by independant Brownian motion under $\mathbb{Q}^{\alpha,T}_K := \mathcal{E}(L^{\alpha,T}_K) \cdot \mathbb{P}$ where*

$$L^{\alpha,T}_K = \frac{\theta}{N} \sum_{i\in K} \int_0^T \mathbb{1}_{s\in\cup_{k\geq 0}[\sigma^k_{\alpha,K}, \tilde{\sigma}^k_{\alpha,K})} \sum_{j\notin K} \frac{X^{i,N}_s - X^{j,N}_s}{\|X^{i,N}_s - X^{j,N}_s\|^2} \mathrm{d}B^i_s,$$

with $(\sigma^k_{\alpha,K})_{k\geq 1}$ and $(\tilde{\sigma}^k_{\alpha,K})_{k\geq 0}$ defined by induction by setting $\tilde{\sigma}^0_{\alpha,K} = 0$ and for all $k \geq 1$,

$$\sigma^k_{\alpha,K} = \inf\{t \geq \tilde{\sigma}^{k-1}_{\alpha,K} : \min_{i\in K, j\notin K} \|X^{i,N}_t - X^{j,N}_t\| \geq \alpha\}$$
$$\tilde{\sigma}^k_{\alpha,K} = \inf\{t \geq \sigma^k_{\alpha,K} : \min_{i\in K, j\notin K} \|X^{i,N}_t - X^{j,N}_t\| \leq \alpha/2\},$$

with the convention $\inf \emptyset = \tau$.

Démonstration. We fix $T > 0$, $\alpha > 0$, $N \geq 3$ and $K \subset [\![1, N]\!]$ with $|K| \in [\![1, N-1]\!]$. We compute

$$\begin{aligned}
\langle L^{\alpha,\cdot}_K \rangle_T =& \frac{\theta^2}{N^2} \sum_{i\in K} \int_0^T \mathbb{1}_{s\in\cup_{k\geq 0}[\sigma^k_{\alpha,K}, \tilde{\sigma}^k_{\alpha,K})} \Big\| \sum_{j\notin K} \frac{X^{i,N}_s - X^{j,N}_s}{\|X^{i,N}_s - X^{j,N}_s\|^2} \Big\|^2 \mathrm{d}s \\
\leq& \frac{4\theta^2|K^c|^2}{\alpha^2 N^2} \sum_{i\in K} \int_0^T \mathbb{1}_{s\in\cup_{k\geq 0}[\sigma^k_{\alpha,K}, \tilde{\sigma}^k_{\alpha,K})} \mathrm{d}s \\
\leq& \frac{4\theta^2|K^c|^2|K|T}{\alpha^2 N^2},
\end{aligned}$$

so that the Novikov condition is satisfied and applying the Girsanov's theorem we get that under $\mathbb{Q}^{\alpha,T}_K$, there exists a $2N$-dimensional Brownian motion $(\tilde{B}^i_t)_{t\geq 0, i\in[\![1,N]\!]}$ such that for all $k \geq 1$, all $t \in [\sigma^k_{\alpha,K} \wedge T, \tilde{\sigma}^k_{\alpha,K} \wedge T)$, all $i \in [\![1, N]\!]$,

$$X^{i,N}_t = X^{i,N}_{\sigma^k_{\alpha,K}\wedge T} + \tilde{B}^i_t - \tilde{B}^i_{\sigma^k_{\alpha,K}\wedge T} - \frac{\theta|K|/N}{|K|} \sum_{j\in K\setminus\{i\}} \int_{\sigma^k_{\alpha,K}\wedge T}^t \frac{X^{i,N}_s - X^{j,N}_s}{\|X^{i,N}_s - X^{j,N}_s\|^2} \mathrm{d}s,$$

so that for any $k \geq 1$, $(X^{i,N}_{\sigma^k_{\alpha,K}\wedge T+t})_{t\in[0,\tilde{\sigma}^k_{\alpha,K}\wedge T - \sigma^k_{\alpha,K}\wedge T), i\in K}$ is a $KS(|K|, |K|\theta/N)$-process restricted to $[0, \tilde{\sigma}^k_{\alpha,K} \wedge T - \sigma^k_{\alpha,K} \wedge T)$. We used the fact that the sequence of processes $((\tilde{B}_{\sigma^k_{\alpha,K}\wedge T+t} - \tilde{B}_{\sigma^k_{\alpha,K}\wedge T})_{t\in[0,\tilde{\sigma}^k_{\alpha,K}\wedge T - \sigma^k_{\alpha,K}\wedge T)}))_{k\geq 1}$ is a family of independant restricted $2N$-dimensional Brownian motion under $\mathbb{Q}^{\alpha,T}_K$.

Lemma 4.12 concludes the proof. □

The following Proposition shows regularity in some sense of the process at τ^N_3.

Proposition 4.14. *If $\theta = 2$, $N \geq 2$, and $((X_t^{i,N})_{t\in[0,\tau_3^N)})_{i\in[\![1,N]\!]}$ is a $KS(\theta, N)$-process on $[0, \tau_3^N)$, then for all $K \subset [\![1, N]\!]$ such that $|K| \geq 2$, we have a.s. the alternative*

$$\lim_{t\to\tau_3^N} R_K(X_t^N) = 0 \quad \text{or} \quad \liminf_{t\to\tau_3^N} R_K(X_t^N) > 0$$

Démonstration. We proceed by reverse induction on $|K|$. First, if $|K| = N$, then the stochastic process $(R_{[\![1,N]\!]}(X_t^N))_{t\in[0,\tau_3^N)}$ is a squared Bessel process with dimension $d_{\theta,N}(N)$ restricted to $[0, \tau_3^N)$ according to Lemma 4.12, so that $\lim_{t\to\tau_3^N} R_{[\![1,N]\!]}(X_t^N)$ exists a.s.

We fix $k \in [\![3, N]\!]$ and suppose that we proved the result for all subset of $[\![1, N]\!]$ with cardinal k, we fix $K \subset [\![1, N]\!]$ such that $|K| = k - 1$ and we prove that the result is true for K. If there exists $i \notin K$ such that $R_{K\cup\{i\}}(x) = 0$ then it is clear that $R_K(x) = 0$. This, continuity of R_K for all $K \subset [\![1, N]\!]$ and the induction hypothesis gives us that it suffices to handle the study on the event $\{\liminf_{t\to\tau_3^N} \min_{i\notin K} R_{K\cup\{i\}}(X_t^N) > 0\}$. We set

$$\mathcal{A} = \Big\{ \liminf_{t\to\tau_3^N} R_K(X_t^N) = 0,\ \limsup_{t\to\tau_3^N} R_K(X_t^N) > 0,\ \liminf_{t\to\tau_3^N} \min_{i\notin K} R_{K\cup\{i\}}(X_t^N) > 0 \Big\}$$

and assume by contradiction that $\mathbb{P}(\mathcal{A}) > 0$. Since $\tau_3^N < \infty$ a.s., there exists $T > 0$ such that

$$\mathbb{P}(\mathcal{A} \cap \{\tau_3^N \leq T\}) > 0.$$

Step 1. There exists $\varepsilon \in \mathbb{Q}_+^*$, $\eta \in \mathbb{Q}_+^*$ small enough and $c \in \mathbb{Q}_+^*$, $\alpha \in \mathbb{Q}_+^*$, $T > 0$ such that with positive probability $\liminf_{t\to\tau_3^N} \min_{i\notin K} R_{K\cup\{i\}}(X_t^N) \geq 2c$, $R_K(X_t^N)$ crosses the interval $[\varepsilon, 2\varepsilon]$ an infinite amount of time during $[0, \tau_3^N)$, $\tau_3^N \leq T$ and if $R_K(x) \leq 2\varepsilon$ and $\min_{i\notin K} R_{K\cup\{i\}}(x) \geq c$, then $\|x^i - x^j\| \geq \alpha$ for all $i \in K$, $j \notin K$.

Step 2. We build in this step a sequence of i.i.d squared Bessel processes from the empirical dispersions $(R_K(X_t^N))_{t\geq 0}$. Take T, α, c and ε defined in Step 1 and we define the sequences of stopping times $(\sigma_{c,\varepsilon}^k)_{k\geq 1}$ and $(\tilde{\sigma}_{c,\varepsilon}^k)_{\geq 0}$ by induction by setting $\tilde{\sigma}_{c,\varepsilon}^0 = 0$ and for all $k \geq 1$,

$$\sigma_{c,\varepsilon}^k = \inf\{t \geq \tilde{\sigma}_{c,\varepsilon}^{k-1} : R_K(X_t^N) \leq \varepsilon \quad \text{and} \quad \min_{i\notin K} R_{K\cup\{i\}}(X_t^N) \geq 2c\}$$
$$\tilde{\sigma}_{c,\varepsilon}^k = \inf\{t \geq \sigma_{c,\varepsilon}^k : R_K(X_t^N) \geq 2\varepsilon \quad \text{or} \quad \min_{i\notin K} R_{K\cup\{i\}}(X_t^N) \leq c\},$$

with the convention $\inf \emptyset = \tau_3^N$.

According to Lemma 4.13, there exists a probability $\mathbb{Q}_K^{\alpha,T}$ absolutely continuous with respect to $\mathbb{P}$ such that $\Big((R_K(X^N_{\sigma_{c,\varepsilon}^k\wedge T+t}))_{t\in[0,\tilde{\sigma}_{c,\varepsilon}^k\wedge T-\sigma_{c,\varepsilon}^k\wedge T)}\Big)_{k\geq 1}$ is a familly of squared Bessel processes with dimension $d_{\theta,N}(|K|)$ driven by independant Brownian motion. Indeed, one has to observe that a.s., for all $k \geq 1$, there exists $k_0 \geq 1$ such that $[\sigma_{c,\varepsilon}^k \wedge T, \tilde{\sigma}_{c,\varepsilon}^k \wedge T] \subset [\sigma_{\alpha,K}^{k_0} \wedge T, \tilde{\sigma}_{\alpha,K}^{k_0} \wedge T]$, where $\sigma_{\alpha,K}^{k_0}$ and $\sigma_{\alpha,K}^{k_0}$ have been defined in Lemma 4.13, because if $t \in [\sigma_{c,\varepsilon}^k \wedge T, \tilde{\sigma}_{c,\varepsilon}^k \wedge T]$, then $R_K(X_t^N) \leq 2\varepsilon$ and $\min_{i\notin K} R_{K\cup\{i\}}(X_t^N) \geq c$, so that $\min_{i\in K, j\notin K} \|X_t^{i,N} - X_t^{j,N}\|^2 \geq \alpha$ according to Step 1.

We consider $((W_t^{k,i})_{t\geq 0})_{k\geq 1, i\in[\![1,2]\!]}$ a sequence of independant 1-dimensional $\mathbb{Q}_K^{\alpha,T}$-Brownian motions independant of all other variable we have considered and we define for all $k \geq 0$,

$$\begin{aligned} Z_t^k =& \mathbb{1}_{t\in[0,\eta^{k,1})} Z_t^{k,1} + \mathbb{1}_{t\in[\eta^{k,1},\eta^{k,1}+\tilde{\sigma}_{c,\varepsilon}^k\wedge T-\sigma_{c,\varepsilon}^k\wedge T)} R_K(X^N_{\sigma_{c,\varepsilon}^k\wedge T+t-\eta^{k,1}}) \\ &+ \mathbb{1}_{t\in[\eta^{k,1}+\tilde{\sigma}_{c,\varepsilon}^k\wedge T-\sigma_{c,\varepsilon}^k\wedge T,\eta^{k,2}]} Z_t^{k,2} \end{aligned}$$

if $\sigma^k_{c,\varepsilon} \wedge T < \tau^N_3$ where

- $(Z^{k,1}_t)_{t\geq 0}$ is a $\mathbb{Q}^{\alpha,T}_K$- squared Bessel process with dimension $d_{\theta,N}(|K|)$ starting at ε driven by $(W^{k,1}_t)_{t\geq 0}$,
- $\eta^{k,1} = \inf\{t \geq 0 : Z^{k,1}_t \geq R_K(X^N_{\sigma^k_{c,\varepsilon}\wedge T})\}$,
- $(Z^{k,2}_t)_{t\geq 0}$ is a $\mathbb{Q}^{\alpha,T}_K$-squared Bessel process with dimension $d_{\theta,N}(|K|)$ driven by the Brownian motion $(W^{k,2}_t)_{t\geq 0}$ starting at $R_K(X^N_{\tilde\sigma^k_{c,\varepsilon}\wedge T})$, (observe that if $\tilde\sigma^k_{c,\varepsilon} \wedge T = \tau^N_3$, this has a sense since a squared Bessel process is continuous),
- $\eta^{k,2} = \inf\{t \geq \eta^{k,1} + \tilde\sigma^k_{c,\varepsilon} \wedge T - \sigma^k_{c,\varepsilon} \wedge T : Z^{k,2}_t \geq 2\varepsilon\}$,

and $Z^k_t = \mathbb{1}_{t\in[0,\eta^{k,2}]} Z^{k,2}_t$ if $\sigma^k_{c,\varepsilon} \wedge T = \tau^N_3$.

Under $\mathbb{Q}^{\alpha,T}_K$, $((Z^k_t)_{t\geq 0})_{k\geq 0}$ is a family of i.i.d squared Bessel processes with dimension $d_{\theta,N}(|K|)$ starting at ε and killed when reaching 2ε.

Step 3. We conclude. According to Step 2, $(\eta^{k,2})_{k\geq 0}$ is a i.i.d family of random variable under $\mathbb{Q}^{\alpha,T}_K$ such that $\mathbb{E}_{\mathbb{Q}^{\alpha,T}_K}(\eta^{k,2}) > 0$, which together with the Borel-Cantelli Lemma imply

$$\sum_{k\geq 0} \eta^{k,2} = \infty \text{ a.s.} \tag{4.19}$$

However, we deduce from Step 1 that with positive probability, $(R_K(X^N_t))_{t\in[0,\tau^N_3)}$ upcrosses $[\varepsilon, 2\varepsilon]$ an infinite number of time and there exists $t_0 \in [0, \tau^N_3)$ such that for all $t \geq t_0$, $\min_{i\notin K} R_K(X^N_t) \geq 2c$, so that for all $n \geq 0$, $\sigma^n_{c,\varepsilon} \wedge T < \tau^N_3$ and there exists $n_0 \geq 1$ such that for all $n \geq n_0$,

$$R_K(X^N_{\sigma^n_{c,\varepsilon}}) = \varepsilon \qquad \text{and} \qquad R_K(X^N_{\tilde\sigma^n_{c,\varepsilon}}) = 2\varepsilon.$$

Thus, with positive probability, there exists $n_0 \geq 0$ such that for all $n \geq n_0$,

$$(Z^n_t)_{t\in[0,\eta^{n,2})} = (R_K(X^N_t))_{t\in[\sigma^n_{c,\varepsilon},\tilde\sigma^n_{c,\varepsilon})},$$

so that $\sum_{k\geq 1} \eta^{k,2} \leq (\sum_{k=1}^{k_0-1} \eta^{k,2}) + \tau^N_3 < \infty$. Indeed, $\tau^N_3 < \infty$ according to Theorem 4.5 and $\sum_{k=1}^{k_0-1} \eta^{k,2} < \infty$ a.s. because $d_{\theta,N}(|K|) > 0$ (see Revuz-Yor [44, Chapter XI]), since $|K| < N$ and $\theta = 2$. This is contradictory with (4.19).

□

The following Lemma will be helpful to determined which kind of collision occurs. In particular this gives us information on the dimension of the squared Bessel processes linked with the empirical variances.

Lemma 4.15. *If $N \geq 5$, then $k_2 \in \{N-2, N-1\}$.*

Démonstration. We first have $d_{2,N}(N-1) = 2(N-2)/N < 2$, so that $k_2 \leq N-1$. Thus it suffices to show that $k_2 \geq N-2$.

We can restrict our study to the case where $N \geq 6$ since the result is clear if $N = 5$. It suffices to show that $2 \leq \min(d_{2,N}(3), d_{2,N}(N-3))$ since $d_{2,N}(k)$ is increasing on $(-\infty, (N+1)/2]$ and decreasing on $[(N+1)/2, \infty)$. Indeed, $d_{2,N}(k) = 2(k-1)(N-k)/N$ is a polynomial expression with roots 1 and N.

Moreover, since $d_{2,N}(3) \leq d_{2,N}(N-3)$ because $|(N+1)/2-3| \geq |(N+1)/2-(N-3)|$, $d_{2,N}(k)$ is symmetric with respect to the line of equation $y=(N+1)/2$ and $d_{2,N}$ is increasing on $(-\infty,(N+1)/2)$, it suffices to show that $d_{2,N}(3) \geq 2$, but this is clear since $d_{2,N}(3) = 2(2-6/N) \geq 2$. □

We are now ready to give the

Proof of Proposition 4.11-(i). Let $(X_t^{i,N})_{t\in[0,\tau_3^N),i\in[\![1,N]\!]}$ be the $KS(\theta,N)$-process on $[0,\tau_3^N)$ built in Theorem 4.5. It suffices to show that $\tau_3^N = \tau_{k_2}^N$ a.s. We reason by the absurd and suppose that with positive probability, $\tau_{k_2}^N > \tau_3^N$, which implies that with positive probability there exists $\ell \geq 1$ such that $\tau_{k_2}^{N,\ell} > \tau_3^N$. Indeed, if for all $\ell \geq 1$, $\tau_{k_2}^{N,\ell} \leq \tau_3^N$, then $\tau_{k_2}^N = \lim_{\ell\to\infty} \tau_3^{N,\ell} \leq \tau_3^N$. We conclude that under this assumption, there exists $T>0$ such that

$$\text{with positive probability,}\quad \tau_3^N \leq T \quad \text{and there exists } \ell \geq 1 \text{ such that} \quad \tau_{k_2}^{N,\ell} \geq \tau_3^N. \tag{4.20}$$

Observe that on the event $\{\tau_{k_2}^{N,\ell} \geq \tau_3^N\}$, for all $K \subset [\![1,N]\!]$ with $|K| = k_2$, we have that $\min_{t\in[0,\tau_3^N)} R_K(X_t^N) \geq 1/\ell$.

Step 1. We show that there exists $\alpha>0$ and $K \subset [\![1,N]\!]$ with $|K| \in [\![3,k_2-1]\!]$ such that with positive probability,

$$\lim_{t\to\tau_3^N} R_K(X_t^N) = 0 \quad \text{and} \quad \liminf_{t\to\tau_3^N} \min_{i\in K, j\notin K} \|X_t^{i,N} - X_t^{j,N}\| \geq \alpha.$$

According to Proposition 4.14, the definition of τ_3^N and (4.20), we have with positive probability,

$$\mathcal{S} := \Big\{K \subset [\![1,N]\!] : |K| \in [\![3,k_2-1]\!] \quad \text{and} \quad \lim_{t\to\tau_3^N} R_K(X_t^N) = 0\Big\} \neq \emptyset,$$

so that according to Proposition 4.14 and (4.20) again, with positive probability there exists $K \subset [\![1,N]\!]$ with $|K| \in [\![3,k_2-1]\!]$ (which is for example a set of $\mathcal{S}$ which is maximal for the inclusion) and $c>0$ such that

$$\lim_{t\to\tau_3^N} R_K(X_t^N) = 0 \quad \text{and} \quad \liminf_{t\to\tau_3^N} \min_{i\notin K} R_{K\cup\{i\}}(X_t^N) \geq c.$$

We conclude observing that for any fixed $c'>0$, there exists $\varepsilon>0$ and $\alpha>0$ such that if $R_K(x) \leq \varepsilon$ and $\min_{i\notin K} R_{K\cup\{i\}}(x) \geq c'$, then $\min_{i\in K, j\notin K} \|x^i - x^j\| \geq \alpha$.

Step 2. We conclude. We apply Lemma 4.13 with α, T and K defined in Step 1, so that under $\mathbb{Q}_K^{\alpha,T}$ with positive probability, a squared Bessel process with dimension $d_{\theta,N}(|K|) \geq 2$ (because $|K| \in [\![3,k_2-1]\!]$ and by definition of k_2) tends to 0 in finite time, which is absurd.

□

The issue now is to prove Proposition 4.11-(ii). According to Proposition 4.11-(i), for all $N \geq 5$ we have $\tau_3^{N,\ell} - \tau_{k_2}^{N,\ell} \to 0$ in probability as $\ell \to \infty$, so that we can find a deterministic increasing sequence $(\ell_N)_{N\geq 5}$ such that

$$\lim_{N\to\infty} \mathbb{P}(|\tau_3^{N,\ell_N} - \tau_{k_2}^{N,\ell_N}| \geq 1) = 0. \tag{4.21}$$

Until now we set for all $N \geq 5$, $\kappa_N := \tau_{k_2}^{N,\ell_N}$ and $\beta_N = \tau_3^{N,\ell_N}$. We need to prove first a technical lemma.

Lemma 4.16. *If* $\sup_{N\geq 5}\mathbb{E}[\|X_0^{1,N}\|^6]<\infty$*, then*

$$\text{for all } N\geq 5,\ \text{all } t\geq 0 \qquad \mathbb{E}[\|X_{t\wedge\kappa_N}^{1,N}\|^6]\leq \Big(\sup_{N\geq 5}\mathbb{E}[\|X_0^{1,N}\|^6]+18t\Big)e^{18t}$$

Let us point out that the power 6 in this result is arbitrary and does not play any specific role.

Démonstration. We fix $M>0$ and set the stopping time $\sigma_M=\inf\{t\geq 0:\sum_{i=1}^N\|X_{t\wedge\kappa_N}^{i,N}\|^6\leq M\}$ with the convention $\inf\emptyset=\kappa_N$. We apply Lemma 4.7-(i) with $\varphi(x)=x^3$ and taking the expectation so that we get

$$\begin{aligned}\mathbb{E}[\|X_{u\wedge\sigma_M}^{1,N}\|^6]=&\|X_0^{1,N}\|^6+18\mathbb{E}\Big[\int_0^{t\wedge\sigma_M}\|X_s^{1,N}\|^4\mathrm{d}s\Big]\\&+\frac{6\theta}{N}\mathbb{E}\Big[\sum_{j=2}^N\int_0^{t\wedge\sigma_M}K(X_s^{1,N}-X_s^{j,N})\cdot\|X_s^{1,N}\|^4X_s^{1,N}\mathrm{d}s\Big].\end{aligned}$$

Using exchangeability of the family $\{(X_{t\wedge\sigma_M}^{i,N})_{t\geq 0}, i\in[\![1,N]\!]\}$ which is given by Theorem 4.5 and the fact that σ_M is invariant by permutation of the family $((X_t^{i,N})_{t\in[0,\tau_3^N)})_{i\in[\![1,N]\!]}$, we get

$$\begin{aligned}\mathbb{E}[\|X_{t\wedge\sigma_M}^{1,N}\|^6]=&\mathbb{E}[\|X_0^{1,N}\|^6]+18\mathbb{E}\Big[\int_0^{t\wedge\sigma_M}\|X_s^{1,N}\|^4\mathrm{d}s\Big]\\&+\frac{6\theta}{N^2}\mathbb{E}\Big[\sum_{1\leq i\neq j\leq N}\int_0^{t\wedge\sigma_M}K(X_s^{i,N}-X_s^{j,N})\cdot\|X_s^{i,N}\|^4X_s^{i,N}\mathrm{d}s\Big]\\\leq&\mathbb{E}[\|X_0^{1,N}\|^6]+18\mathbb{E}\Big[\int_0^{t\wedge\sigma_M}\|X_s^{1,N}\|^4\mathrm{d}s\Big],\end{aligned}$$

since by oddness of K and convexity of $x\mapsto\|x\|^6$, and recalling the definition of K,

$$\sum_{1\leq i\neq j\leq N}K(x^i-x^j)\cdot\|x^i\|^4x^i=\frac{1}{2}\sum_{1\leq i\neq j\leq N}K(x^i-x^j)\cdot[\|x^i\|^4x^i-\|x^j\|^4x^j]\leq 0.$$

Since $u\leq 1+u^{3/2}$ for all $u\geq 0$ and $t\wedge\sigma_M\leq t$, we get

$$\mathbb{E}[\|X_{t\wedge\sigma_M}^{1,N}\|^6]=\sup_{N\geq 5}\mathbb{E}[\|X_0^{1,N}\|^6]+18t+18\int_0^t\mathbb{E}[\|X_{s\wedge\sigma_M}^{1,N}\|^6\mathrm{d}s,$$

which implies the result thanks to the Gronwall's Lemma and the Fatou's Lemma because σ_M goes to κ_N a.s. as $M\to\infty$.

□

We can finally give the

Proof of Proposition 4.11-(ii). Step 1. We first show that it is sufficient to consider F_0^N such that $\sup_{N\geq 5}\mathbb{E}[\|X_0^{1,N}\|^6]<\infty$. Indeed, suppose that we have shown the result for these initial conditions and take some $F_0^N\in\mathcal{P}_{sym,1}^*$ and the associated random variable $(X_0^{i,N})_{i\in[\![1,N]\!]}$. We set the $\sigma(X_0^{i,N},N\geq 5,i\in[\![1,N]\!])$-mesurable random variable $\Lambda=1+\max_{j\in[\![1,N]\!]}\|X_0^{j,N}\|$ and define

$$\text{for all } i\in[\![1,N]\!],\quad \tilde{X}_0^{i,N}=\frac{X_0^{i,N}}{\Lambda}.$$

We denote by $\tilde{F}_0^N$ the law of $(\tilde{X}_0^{1,N}, \cdots, \tilde{X}_0^{N,N})$ and we clearly have $\sup_{N\geq 5} \mathbb{E}(\|\tilde{X}_0^{1,N}\|^6) \leq 1$ and $\tilde{F}_0^N \in \mathcal{P}^*_{sym,1}$. We set $\tilde{X}_t^N = X^N_{\Lambda^2 t}/\Lambda$ and we see thanks to a change of variable and thanks to the scaling property of the Brownian motion that $(\tilde{X}_t^N)_{t\geq 0}$ is a $KS(2,N)$-process with inital law $\tilde{F}_0^N$. By hypothesis, we have that $\tilde{\tau}_3^{N,\ell_N} \to_{N\to\infty} \infty$ in probability with obvious notations. But since a.s. $\Lambda \geq 1$, we have a.s.

$$\text{for } N \text{ large enough,} \quad \beta_N = \tau_3^{N,\ell_N} \geq \Lambda^2 \tilde{\tau}_3^{N,\lfloor \Lambda^2 \ell_N \rfloor} \geq \tilde{\tau}_3^{N,\ell_N},$$

which proves the result.

Step 2. We are reduced to show the result when $\sup_{N\geq 5} \mathbb{E}[\|X_0^{1,N}\|^6] < \infty$.

Step 2.1. We set $\rho_N := \inf\{t \geq 0 : R_{[\![1,N]\!]}(X_t^N) \leq N^{1/2}\}$ and show $\mathbb{P}(\rho_N \leq t) \to 0$ as $N \to \infty$ for all $t > 0$. Considering $(Y_t)_{t\in[0,\tau_3^N)} = (R_{[\![1,N]\!]}(X_t^N)^{1/2})_{t\in[0,\tau_3^N)}$, Lemma 4.12 and the Itô's formula imply that for all $t \geq 0$,

$$\begin{aligned} Y_{t\wedge\rho_N} &= R_{[\![1,N]\!]}(X_0^N)^{1/2} + W_{t\wedge\rho_N} - \frac{1}{2}\int_0^{t\wedge\rho_N} \frac{\mathrm{d}s}{Y_s} \\ &\geq R_{[\![1,N]\!]}(X_0^N)^{1/2} + W_{t\wedge\rho_N} - \frac{N^{-1/4}}{2}(t\wedge\rho_N), \end{aligned}$$

where $(W_t)_{t\geq 0}$ is a 1-dimensional Brownian motion. For N large enough, we get

$$\begin{aligned} \mathbb{P}(\rho_N \leq t) \leq& \mathbb{P}\Big(R_{[\![1,N]\!]}(X_0^N)^{1/2} + \inf_{s\in[0,t]}(W_s - s/(2N^{1/4})) \leq N^{1/4}\Big) \\ \leq& \mathbb{P}\Big(\sup_{s\in[0,t]}(-W_s + s/(2N^{1/4})) \geq R_{[\![1,N]\!]}(X_0^N)^{1/2} - N^{1/4}\Big) \\ \leq& \mathbb{P}\Big(\sup_{s\in[0,t]} -W_s \geq R_{[\![1,N]\!]}(X_0^N)^{1/2} - N^{1/4} - t/(2N^{1/4})\Big) \\ \leq& \mathbb{P}\Big(\sup_{s\in[0,t]} -W_s > C_t N^{1/3}\Big) + \mathbb{P}\Big(R_{[\![1,N]\!]}(X_0^N) \leq N^{2/3}\Big), \end{aligned}$$

where C_t is a constant only depending on t. Since $(-W_t)_{t\geq 0}$ is a 1-dimensional Brownian motion, it suffices to show that the second term of the RHS goes to 0 as $N \to \infty$. Since f_0 is not a Dirac mass, we can fix $A > 0$ such that

$$\int_{\mathbb{R}^2} \psi_A(x) f_0(\mathrm{d}x) > 0,$$

where $\psi(y) = \|\varphi_A(y) - \int_{\mathbb{R}^2} \varphi_A(x) f_0(\mathrm{d}x)\|^2$ with $\varphi_A \in C_c(\mathbb{R}^2, \mathbb{R}^2)$ is a 1-Lipschitz bounded fonction such that $\varphi_A(x) = x$ for every $x \in B(0,A)$. Since φ_A is 1-Lipschitz, we have

$$\begin{aligned} \sum_{i=1}^N \Big\|\varphi_A(x^i) - \frac{1}{N}\sum_{k=1}^N \varphi_A(x^k)\Big\|^2 =& \frac{1}{2N}\sum_{1\leq i\neq j\leq N} \|\varphi_A(x^i) - \varphi_A(x^j)\|^2 \\ \leq& \frac{1}{2N}\sum_{1\leq i\neq j\leq N} \|x^i - x^j\|^2 = R_{[\![1,N]\!]}(x), \end{aligned}$$

so that

$$\begin{aligned}\mathbb{P}(R_{[\![1,N]\!]}(X_0^N)\le N^{2/3}) \le & \mathbb{P}\Big(\sum_{i=1}^N \Big\|\varphi_A(X_0^{i,N})-\frac{1}{N}\sum_{k=1}^N\varphi_A(X_0^{k,N})\Big\|^2\le N^{2/3}\Big)\\ \le & P_1^N+P_2^N,\end{aligned}$$

where

$$\begin{aligned}P_1^N &= \mathbb{P}\Big(\Big\|\int_{\mathbb{R}^2}\varphi_A(x)f_0(\mathrm{d}x)-\int_{\mathbb{R}^2}\varphi_A(x)\mu_0^N(\mathrm{d}x)\Big\|\ge\varepsilon\Big),\\ P_2^N &= \mathbb{P}\Big(\frac{2}{N}\sum_{i=1}^N\Big\|\varphi_A(X_0^{i,N})-\int_{\mathbb{R}^2}\varphi_A(x)f_0(\mathrm{d}x)\Big\|^2-2\varepsilon^2\le N^{-1/3}\Big),\end{aligned}$$

because $\int_{\mathbb{R}^2}\varphi_A(x)\mu_0^N(\mathrm{d}x)=N^{-1}\sum_{k=1}^N\varphi_A(X_0^{k,N})$, with $\varepsilon\in(0,(1/2)\int_{\mathbb{R}^2}\psi_A(x)f_0(\mathrm{d}x)]^{1/2}]$. By hypothesis on μ_0^N, we get that $\lim_{N\to\infty}P_1^N=0$, whence it suffices to show that $\lim_{N\to\infty}P_2^N=0$. It follows

$$\begin{aligned}P_2^N =&\mathbb{P}\Big(2\int_{\mathbb{R}^2}\psi_A(y)\mu_0^N(\mathrm{d}y)-2\varepsilon^2\le N^{-1/3}\Big)\\ \le&\mathbb{P}\Big(\Big|\int_{\mathbb{R}^2}\psi_A(y)\mu_0^N(\mathrm{d}y)-\int_{\mathbb{R}^2}\psi_A(y)f_0(\mathrm{d}y)\Big|\ge\varepsilon^2\Big)\\ &+\mathbb{P}\Big(2\int_{\mathbb{R}^2}\psi_A(y)f_0(\mathrm{d}y)-4\varepsilon^2\le N^{-1/3}\Big),\end{aligned}$$

and these both quantity tends to 0 as N tends to ∞ because $2\int_{\mathbb{R}^2}\psi(y)f_0(\mathrm{d}y)-4\varepsilon^2>0$, by definition of μ_0^N and since ψ_A is continous and bounded.

Step 2.2. We show that κ_N goes to infinity in probability as $N\to\infty$. We fix $t>0$ and we have

$$\mathbb{P}(\kappa_N\le t)\le\mathbb{P}(\rho_N\le t)+\mathbb{P}(\rho_N>t,\kappa_N\le t).$$

According to Step 2, it is sufficient to show that the last term of the previous inequality tends to 0. On the event $\{\rho_N>\kappa_N\}$, there exists $K\subset[\![1,N]\!]$ with $|K|=k_2$ such that $R_K(X_{\kappa_N}^N)\le 1/\ell_N$ and by definition of ρ_N we have

$$\begin{aligned}N^{1/2}\le R_{[\![1,N]\!]}(X_{\kappa_N}^N) &= \frac{1}{2N}\sum_{1\le i\ne j\le N}\|X_{\kappa_N}^{i,N}-X_{\kappa_N}^{j,N}\|^2\\ &=\frac{|K|}{N}R_K(X_{\kappa_N}^N)+\frac{1}{N}\sum_{i\in K,j\notin K}\|X_{\kappa_N}^{i,N}-X_{\kappa_N}^{j,N}\|^2+\frac{1}{2N}\sum_{i,j\notin K}\|X_{\kappa_N}^{i,N}-X_{\kappa_N}^{j,N}\|^2.\end{aligned}$$

Fixing $i_0=\min K$, using that $\|x+y\|^2\le 2(\|x\|^2+\|y\|^2)$ in the two last sums and that $R_K(X_{\kappa_N}^N)\le 1/\ell_N$ since $\kappa_N>\rho_N$, we get

$$\begin{aligned}N^{1/2} &\le\frac{|K|}{N\ell_N}+\frac{2|K^c|}{N}\sum_{i\in K}\|X_{\kappa_N}^{i,N}-X_{\kappa_N}^{i_0,N}\|^2+\frac{2|K|+2}{N}\sum_{j\notin K}\|X_{\kappa_N}^{i_0,N}-X_{\kappa_N}^{j,N}\|^2\\ &\le\frac{|K|}{N\ell_N}+\frac{4|K||K^c|}{N\ell_N}+\frac{2|K|+2}{N}\sum_{j\notin K}\|X_{\kappa_N}^{i_0,N}-X_{\kappa_N}^{j,N}\|^2.\end{aligned}$$

Using again that $\|x+y\|^2 \le 2(\|x\|^2+\|y\|^2)$, we deduce

$$N^{1/2} \le \frac{|K|}{N\ell_N} + \frac{4|K||K^c|}{N\ell_N} + \frac{8(|K|+1)|K^c|}{N} \max_{i\in K^c\cup\{i_0\}} \|X^{i,N}_{\kappa_N}\|^2,$$

which implies,

$$N^{1/2} \le \frac{9}{\ell_N} + 16 \max_{i\in K^c\cup\{i_0\}} \|X^{i,N}_{\kappa_N}\|^2,$$

since $|K| = k_2 \in [\![N-2, N-1]\!]$ according to Lemma 4.15. Since $\ell_N \ge N-4$ by construction, and by exchangeability, this gives that there exists $C>0$ such that for N large enough

$$\begin{aligned}\mathbb{P}(\rho_N > t, \kappa_N \le t) &\le \mathbb{P}\Big(\bigcup_{i\in[\![1,N]\!]} \{\|X^{i,N}_{t\wedge\kappa_N}\|^2 \ge CN^{1/2}\}\Big)\\ &\le N\mathbb{P}(\|X^{1,N}_{t\wedge\kappa_N}\|^2 \ge CN^{1/2}).\end{aligned}$$

Using the Markov inequality, this leads us to

$$\mathbb{P}(\rho_N > t, \kappa_N \le t) \le \frac{\mathbb{E}[\|X^{1,N}_{t\wedge\kappa_N}\|^6]}{C^3N^{1/2}},$$

which tends to 0 as $N\to\infty$ thanks to Lemma 4.16.

Step 2.3. Here we conclude, but we clearly have $\beta_N \to \infty$ in probability as N goes to infinity because recalling (4.21) and Step 2.2, for each $t \ge 0$ we get

$$\mathbb{P}(\beta_N \le t) \le \mathbb{P}(|\kappa_N - \beta_N| \le 1) + \mathbb{P}(\kappa_N \le t+1) \to_{N\to\infty} 0.$$

□

4.6 The critical case

Here we prove Theorem 4.6-(ii). We recall that $\theta = 2$, the process $(X^{i,N}_t)_{t\in[0,\tau^N_3), i\in[\![1,N]\!]}$ is a $KS(\theta,N)$-process issued from Theorem 4.5 and β_N has been defined in Proposition 4.11. We introduce $G : (\mathbb{R}^2)^3 \mapsto \mathbb{R}$ defined as follows. For $x,y,z \in (\mathbb{R}^2)^3$, we set $X = x-y$, $Y = y-z$ and $Z = z-x$ and put

$$G(x,y,z) = \Big(L(\|X\|^2)X + L(\|Y\|^2)Y + L(\|Z\|^2)Z\Big)\cdot\Big(\frac{X}{\|X\|^2}+\frac{Y}{\|Y\|^2}+\frac{Z}{\|Z\|^2}\Big),$$

if $0 \notin \{X,Y,Z\}$, where $L(r) = \log(1+1/r) - 1/(1+r)$. If now $0 \in \{X,Y,Z\}$ and $(X,Y,Z) \ne (0,0,0)$, we put $G(x,y,z) = \infty$. If finally $X=Y=Z=0$, then $G(x,y,z) = 0$.

The following result and its proof are highly linked with Proposition 4.10, replacing φ_a by $L(\cdot + a)$.

Proposition 4.17. *If $f_0 \in \mathcal{P}(\mathbb{R}^2)$ then for all $t>0$,*

$$\sup_{N\ge5} \mathbb{E}\Big[\int_0^{t\wedge\beta_N} G(X^{1,N}_s, X^{2,N}_s, X^{3,N}_s)\mathrm{d}s\Big] < \infty,$$

Démonstration. We fix $\eta \in (0,1]$ and we introduce $L_\eta(r) = L(\eta + r)$ for all $r \geq 0$. We apply the Itô formula to $\varphi_\eta(\|X^{1,N}_{t\wedge\beta_N} - X^{2,N}_{t\wedge\beta_N}\|^2)$ with $\varphi_\eta(r) = (r+\eta)\log(1+1/(\eta+r))$ whence $\varphi'_\eta(r) = L_\eta(r)$ and $\varphi''_\eta(r) = -[(\eta+r)(1+\eta+r)^2]^{-1}$. Taking the expectation, we find that

$$S^{N,\eta}_t := \mathbb{E}[\varphi_\eta(\|X^{1,N}_{t\wedge\beta_N} - X^{2,N}_{t\wedge\beta_N}\|^2)] = S^{N,\eta}_0 + \int_0^t \Big(A^{N,\eta}_s + \frac{2\theta}{N} B^{N,\eta}_s\Big)\mathrm{d}s, \tag{4.22}$$

where, setting $h_\eta(r) = -r/[(\eta+r)(1+\eta+r)^2]$,

$$\begin{aligned} A^{N,\eta}_s =& 4\mathbb{E}\Big[\mathbb{1}_{s\leq\beta_N}\Big(\Big(1-\frac{\theta}{N}\Big)L_\eta(\|X^{1,N}_s - X^{2,N}_s\|^2) + h_\eta(\|X^{1,N}_s - X^{2,N}_s\|^2)\Big)\Big] \\ \geq& 4\Big(1-\frac{\theta}{N}\Big)\mathbb{E}\Big[\mathbb{1}_{s\leq\beta_N} L_\eta(\|X^{1,N}_s - X^{2,N}_s\|^2)\Big] - 4 \end{aligned}$$

because h_η is bounded by 1, and where $B^{N,\eta}_s$ equals

$$\sum_{j=3}^N \mathbb{E}\Big[\mathbb{1}_{s\leq\beta_N}\Big(K(X^{1,N}_s - X^{j,N}_s) + K(X^{j,N}_s - X^{2,N}_s)\Big)\Big)\cdot(X^{1,N}_s - X^{2,N}_s)L_\eta(\|X^{1,N}_s - X^{2,N}_s\|^2)\Big],$$

so that by exchangeability, $B^{N,\eta}_s/(N-2)$ equals

$$\mathbb{E}\Big[\mathbb{1}_{s\leq\beta_N}\Big(K(X^{1,N}_s - X^{3,N}_s) + K(X^{3,N}_s - X^{2,N}_s)\Big)\Big)\cdot(X^{1,N}_s - X^{2,N}_s)L_\eta(\|X^{1,N}_s - X^{2,N}_s\|^2)\Big].$$

Symmetrizing by exchangeability and since $\theta = 2$, we get

$$A^{N,\eta}_s + \frac{2\theta}{N}B^{N,\eta}_s \geq \frac{4(N-2)}{3N}\mathbb{E}[\mathbb{1}_{s\leq\beta_N} F_{N,\eta}(X^{1,N}_s, X^{2,N}_s, X^{3,N}_s)] - 4, \tag{4.23}$$

where

$$\begin{aligned} F_{N,\eta}(x,y,z) =& L_\eta(\|x-y\|^2)[1 + (x-y)\cdot(K(x-z) + K(z-y))] \\ &+ L_\eta(\|y-z\|^2)[1 + (y-z)\cdot(K(y-x) + K(x-z))] \\ &+ L_\eta(\|z-x\|^2)[1 + (z-x)\cdot(K(z-y) + K(y-x))]. \end{aligned}$$

Setting $X = x-y$, $Y = y-z$, $Z = z-x$ and recalling that $K(X) = -\frac{X}{\|X\|^2}\mathbb{1}_{X\neq 0}$ we find for x, y, z distincts,

$$\begin{aligned} F_{N,\eta}(x,y,z) =& L_\eta(\|X\|^2)\Big[1 + X\cdot\Big(\frac{Z}{\|Z\|^2} + \frac{Y}{\|Y\|^2}\Big)\Big] \\ & L_\eta(\|Y\|^2)\Big[1 + Y\cdot\Big(\frac{X}{\|X\|^2} + \frac{Z}{\|Z\|^2}\Big)\Big] \\ &+ L_\eta(\|Z\|^2)\Big[1 + Z\cdot\Big(\frac{Y}{\|Y\|^2} + \frac{X}{\|X\|^2}\Big)\Big] \\ =& L_\eta(\|X\|^2)X\cdot\Big(\frac{X}{\|X\|^2} + \frac{Y}{\|Y\|^2} + \frac{Z}{\|Z\|^2}\Big) \\ &+ L_\eta(\|Y\|^2)Y\cdot\Big(\frac{X}{\|X\|^2} + \frac{Y}{\|Y\|^2} + \frac{Z}{\|Z\|^2}\Big) \\ &+ L_\eta(\|Z\|^2)Z\cdot\Big(\frac{X}{\|X\|^2} + \frac{Y}{\|Y\|^2} + \frac{Z}{\|Z\|^2}\Big) \\ =:& G_\eta(x,y,z), \end{aligned}$$

where we have put

$$G_\eta(x,y,z) = \Big(L_\eta(\|X\|^2)X + L_\eta(\|Y\|^2)Y + L_\eta(\|Z\|^2)Z\Big) \cdot \Big(\frac{X}{\|X\|^2} + \frac{Y}{\|Y\|^2} + \frac{Z}{\|Z\|^2}\Big).$$

Inserting this into (4.22) and (4.23), we find

$$S_t^{N,\eta} \geq S_0^{N,\eta} - 4t + \frac{4(N-2)}{3N}\mathbb{E}\Big[\int_0^{t\wedge\beta_N} G_\eta(X_s^{1,N}, X_s^{2,N}, X_s^{3,N})\mathrm{d}s\Big].$$

Observing that φ_η is bounded, we deduce that $\sup_{N\geq 5,\eta\in(0,1]} S_t^{N,\eta} < \infty$. Since moreover $S_0^{N,\eta} \geq 0$, we conclude that

$$\sup_{N\geq 5,\eta\in(0,1]} \mathbb{E}\Big[\int_0^{t\wedge\beta_N} G_\eta(X_s^{1,N}, X_s^{2,N}, X_s^{3,N})\mathrm{d}s\Big] < \infty. \tag{4.24}$$

Recall that G_η is nonnegative thanks to Lemma 4.9 because L_η is nonincreasing and positive. Finally, one easily shows, using Lemma 4.9 and that $r \mapsto L_\eta(r) - L_{\eta'}(r)$ is nonincreasing and nonnegative if $0 < \eta < \eta'$, that $\eta \to G_\eta(x,y,z)$ increases to $G(x,y,z)$ as η decreases to 0 for every x, y, $z \in \mathbb{R}^2$ distincts. We thus also deduce (4.17) from (4.24) by monotone convergence because a.s. $X_s^{1,N}$, $X_s^{2,N}$ and $X_s^{3,N}$ are distincts for a.e $s \in [0,\beta_N]$ according to Theorem 4.5. □

By Theorem 4.6-(i), we can find $(N_k)_{k\geq 0}$ and $(\mu_t)_{t\geq 0} \in C([0,\infty), \mathcal{P}(\mathbb{R}^2))$ such that $\lim_k N_k = \infty$ and $(\mu_t^{N_k,\beta_{N_k}})_{t\geq 0}$ goes to $(\mu_t)_{t\geq 0}$ in law as $k \to \infty$, where $C([0,\infty), \mathcal{P}(\mathbb{R}^2))$ is endowed with the uniform convergence on compact time intervals, $\mathcal{P}(\mathbb{R}^2)$ being endowed with the weak convergence topology. We obviously have $\mu_0 = f_0$.

The following Lemma shows that the dispersion of $(\mu_t)_{t\geq 0}$ increases with time, which implies that if f_0 is not equal to a Dirac mass, then for any $t \geq 0$, μ_t is not equal to a Dirac.

Lemma 4.18. *It holds that*

$$\textit{a.s. for all } t \geq 0, \ \textit{all } x_0 \in \mathbb{R}^2, \quad \int_{\mathbb{R}^2} \|x - x_0\|^2 \mu_t(\mathrm{d}x) \geq \int_{\mathbb{R}^2} \|x - x_0\|^2 f_0(\mathrm{d}x).$$

If the RHS is infinite, this result implies that the LHS is also infinite.

Démonstration. By the Skorokhod representation, we can find a probabilistic space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and some random variables $((\tilde\mu_t^{N_k})_{t\geq 0})_{k\geq 0}$ and $(\tilde\mu_t)_{t\geq 0}$ such that the law of $(\tilde\mu_t^{N_k})_{t\geq 0}$ equals the law of $(\mu_t^{N_k,\beta_{N_k}})_{t\geq 0}$ for all $k \geq 0$, the law of $(\tilde\mu_t)_{t\geq 0}$ equals the law of $(\mu_t)_{t\geq 0}$ and $(\tilde\mu_t^{N_k})_{t\geq 0}$ converges a.s. to $(\tilde\mu_t)_{t\geq 0}$ as $k \to \infty$. Since $(\mu_t)_{t\geq 0}$ and $(\tilde\mu_t)_{t\geq 0}$ have the same law, it suffices to show (i) and (ii) with $(\tilde\mu_t)_{t\geq 0}$ instead of $(\mu_t)_{t\geq 0}$. To this end we divide the proof in several steps.

Step 1. We show that for all $B \in \tilde{\mathcal{F}}$, all $x_0 \in \mathbb{R}^2$ and all $A > 0$, we have

$$\begin{aligned}\tilde{\mathbb{E}}\Big[\mathbb{1}_B \int_{\mathbb{R}^2} \varphi_A(\|x - x_0\|^2)\tilde\mu_t(\mathrm{d}x)\Big] \geq{}& \tilde{\mathbb{P}}(B) \int_{\mathbb{R}^2} \varphi_A(\|x - x_0\|^2) f_0(\mathrm{d}x) \\ &- Ct\Big(\Big(\frac{\sup_{N\geq 5, s\in[0,t]} \frac{1}{N}\sum_{i=1}^N \mathbb{E}[\psi(\|X_s^i - x_0\|^2)]}{\psi((A/2 - \|x_0\|^2)\vee 0)}\Big) \wedge \tilde{\mathbb{P}}(B)\Big),\end{aligned} \tag{4.25}$$

where $\varphi_A(r) = \chi(r/A)r$ with $\chi \in C_c^\infty(\mathbb{R}_+)$ such that $\chi(r) \in [0,1]$ for all $r \geq 0$, $\chi(r) = 1$ for all $r \in [0,1]$, and $\chi(r) = 0$ for all $r \geq 2$ and where we recall that ψ is a nondecreasing fonction such that $\lim_{r\to\infty} \psi(r) = \infty$ defined in Proposition 4.8.

We next introduce $\psi_A(x) = \varphi_A(\|x-x_0\|^2)$ and observe that ψ_A, $\nabla\psi_A$ and $\nabla^2\psi_A$ are bounded (uniformly in A) and that $\nabla\psi_A(x) = 2(x-x_0)\varphi'_A(\|x-x_0\|^2)$ and $\Delta\psi_A(x) = 4(\varphi'_A(\|x-x_0\|^2) + \|x-x_0\|^2\varphi''_A(\|x-x_0\|^2))$. We now apply Lemma (4.7)-(iii) with ψ_A, we multiply by $\mathbb{1}_B$ with $B \in \tilde{\mathcal{F}}$ and we take the expectation in order to get,

$$S_t^{A,k} = S_0^{A,k} + \tilde{\mathbb{E}}[\mathbb{1}_B M_t^{A,k}] + 2\tilde{\mathbb{E}}\Big[\mathbb{1}_B \int_0^t (I_s^{1,A,k} + I_s^{2,A,k})\mathrm{d}s\Big], \tag{4.26}$$

where

$$\begin{aligned} S_t^{A,k} &= \tilde{\mathbb{E}}\Big[\mathbb{1}_B \int_{\mathbb{R}^2} \varphi_A(\|x-x_0\|^2)\tilde{\mu}_t^{N_k}(\mathrm{d}x)\Big], \\ M_t^{A,k} &= \frac{1}{N_k}\sum_{i=1}^{N_k}\int_0^t \nabla\psi_A(X_s^{i,N_k})\mathrm{d}B_s^i, \\ I_s^{1,A,k} &= 2\int_{\mathbb{R}^2}[\varphi'_A(\|x-x_0\|^2) + \|x-x_0\|^2\varphi''_A(\|x-x_0\|^2)]\tilde{\mu}_s^{N_k}(\mathrm{d}x), \end{aligned}$$

and $I_s^{2,A,k}$ equals

$$\theta\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} K(x-y)\cdot[(x-x_0)\varphi'_A(\|x-x_0\|^2) - (y-x_0)\varphi'_A(\|y-x_0\|^2)]\tilde{\mu}_s^{N_k}(\mathrm{d}x)\tilde{\mu}_s^{N_k}(\mathrm{d}y).$$

Since $\varphi_A(r) = r$ for all $r \in [0,a]$ and since $\varphi'_A(\|x-x_0\|^2) + \|x-x_0\|^2\varphi''_A(\|x-x_0\|^2)$ is bounded,

$$\begin{aligned} I_s^{1,A,k} \geq & 2\int_{\|x-x_0\|^2\leq A} \tilde{\mu}_s^{N_k}(\mathrm{d}x) - C\int_{\|x-x_0\|^2\geq A}\tilde{\mu}_s^{N_k}(\mathrm{d}x) \\ =& 2\tilde{\mu}_s^{N_k}(\{x : \|x-x_0\|^2 \leq A\}) - C\tilde{\mu}_s^{N_k}(\{x : \|x-x_0\|^2 \geq A\}). \end{aligned} \tag{4.27}$$

Using again that $\varphi_A(r) = r$ for all $r \in [0,a]$, that $K(z) = -z/||z||^2\mathbb{1}_{\{z\neq 0\}}$ and that the Jacobian of $x \to (x-x_0)\varphi'_A(\|x-x_0\|^2) = \frac{1}{2}\nabla\psi_A(x)$ is uniformly bounded,

$$\begin{aligned} I_s^{2,A,k} \geq & -\theta\int_{\|x-x_0\|^2\leq A}\int_{\|y-x_0\|^2\leq A}\mathbb{1}_{\{x\neq y\}}\tilde{\mu}_s^{N_k}(\mathrm{d}x)\tilde{\mu}_s^{N_k}(\mathrm{d}y) \\ & - C\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathbb{1}_{\{\|x-x_0\|^2\geq A\}\cup\{\|y-x_0\|^2\geq A\}}\tilde{\mu}_s^{N_k}(\mathrm{d}x)\tilde{\mu}_s^{N_k}(\mathrm{d}y) \\ \geq & -\theta\tilde{\mu}_s^{N_k}\otimes\tilde{\mu}_s^{N_k}(\{x : \|x-x_0\|^2 \leq A\}\otimes\{y : \|y-x_0\|^2 \leq A\}) \\ & - 2C\tilde{\mu}_s^{N_k}(\{x : \|x-x_0\|^2 \geq A\}) \end{aligned}$$

where we used in the last inequality that $\mathbb{1}_{C\cup D} \leq \mathbb{1}_C + \mathbb{1}_D$. We easily conclude that

$$I_s^{2,A,k} \geq -\theta\tilde{\mu}_s^{N_k}(\{x : \|x-x_0\|^2 \leq A\}) - 2C\tilde{\mu}_s^{N_k}(\{x : \|x-x_0\|^2 \geq A\}). \tag{4.28}$$

Putting (4.26), (4.27) and (4.28) together, recalling that $\theta = 2$ and using the Cauchy-Schwarz inequality, we find

$$S_t^{A,k} \geq S_0^{A,k} - \sqrt{\mathbb{P}(B)}\sqrt{\tilde{\mathbb{E}}[(M_t^{A,k})^2]} - 3C\mathbb{E}\Big[\mathbb{1}_B\int_0^t \tilde{\mu}_s^{N_k}(\{\|x-x_0\|^2 \geq A\})\mathrm{d}s\Big]. \tag{4.29}$$

Finally, observing that

$$\tilde{\mu}_s^{N_k}(\{\|x-x_0\|^2 \geq A) \leq \tilde{\mu}_s^{N_k}\Big(\Big\{\|x\|^2 \geq \Big(\frac{A}{2}-\|x_0\|^2\Big)\vee 0\Big\}\Big)$$
$$=\tilde{\mu}_s^{N_k}\Big(\Big\{\psi(\|x\|^2) \geq \psi\Big(\Big(\frac{A}{2}-\|x_0\|^2\Big)\vee 0\Big)\Big\}\Big),$$

since ψ is nondecreasing, applying the Markov inequality, using exchangeability of the family $((X^{i,N}_{t\wedge\beta_N})_{t\geq 0}, i \in [\![1,N]\!])$ for all $N \geq 5$ and injecting in (4.29), we get

$$S_t^{A,k} \geq S_0^{A,k} - \sqrt{\tilde{\mathbb{P}}(B)}\sqrt{\tilde{\mathbb{E}}[(M_t^{A,k})^2]} \tag{4.30}$$
$$- Ct\Big(\Big(\frac{\sup_{N\geq 5, s\in[0,t]} \frac{1}{N}\sum_{i=1}^N \mathbb{E}[\psi(\|X^{i,N}_{s\wedge\beta_N}\|^2)]}{\psi((A/2-\|x_0\|^2)\vee 0)}\Big)\wedge \tilde{\mathbb{P}}(B)\Big). \tag{4.31}$$

We easily check that $M_t^{A,k} \to_{k\to\infty} 0$ in L^2. Moreover, since $F : \nu \in C([0,\infty),\mathcal{P}(\mathbb{R}^2)) \mapsto \int_{\mathbb{R}^2}\varphi_A(\|x-x_0\|^2)\nu_t(\mathrm{d}x)$ is continuous and bounded, it holds that

$$\lim_k S_t^{A,k} = \tilde{\mathbb{E}}\Big[\mathbb{1}_B \int_0^t \varphi_A(||x-x_0||^2)\tilde{\mu}_t(\mathrm{d}x)\Big].$$

Hence letting $k \to \infty$ in (4.30) and using that $\tilde{\mu}_0 = f_0$, we find (4.25).

Step 2. We conclude. We let A going to infinity in (4.25), the monotone convergence theorem gives us that for all $B \in \tilde{\mathcal{F}}$, all $x_0 \in \mathbb{R}^2$,

$$\tilde{\mathbb{E}}\Big[\mathbb{1}_B \int_{\mathbb{R}^2}\|x-x_0\|^2\tilde{\mu}_t(\mathrm{d}x)\Big] \geq \tilde{\mathbb{P}}(B)\int_{\mathbb{R}^2}\|x-x_0\|^2 f_0(\mathrm{d}x).$$

Choosing $B = B_{t,x_0} = \{\int_{\mathbb{R}^2}\|x-x_0\|^2\tilde{\mu}_t(\mathrm{d}x) < \int_{\mathbb{R}^2}\|x-x_0\|^2 f_0(\mathrm{d}x)\}$, we conclude that $\mathbb{P}(B_{t,x_0}) = 0$ (else we would have a contradiction). We conclude by continuity. that a.s., for all $t \geq 0$, all $x_0 \in \mathbb{R}^d$, $\int_{\mathbb{R}^2}\|x-x_0\|^2\tilde{\mu}_t(\mathrm{d}x) < \int_{\mathbb{R}^2}\|x-x_0\|^2 f_0(\mathrm{d}x)$. □

Proposition 4.19. *If* $\max_{x\in\mathbb{R}^2} f_0(\{x\}) < 1$ *i.e. if* f_0 *is not a Dirac mass, then a.s., for a.e. every* $t > 0$, μ_t *is a diffuse measure, i.e.*

$$\mathbb{E}\Big[\int_0^\infty \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathbb{1}_{\{x=y\}}\mu_t(\mathrm{d}x)\mu_t(\mathrm{d}y)\mathrm{d}t\Big] = 0.$$

Démonstration. Step 1. We first show that

$$\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G(x,y,z)\mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y)\mu_s(\mathrm{d}z)\mathrm{d}s\Big] < \infty.$$

According to Proposition 4.17, there exists $C > 0$ such that for all $k \geq 0$, all $M > 0$, all $\varepsilon > 0$,

$$\mathbb{E}\Big[\int_0^{t\wedge\beta_{N_k}} \chi_\varepsilon(x,y,z)(G(X_s^{1,N_k}, X_s^{2,N_k}, X_s^{3,N_k})\wedge M)\mathrm{d}s\Big] \leq C,$$

where $\chi_\varepsilon(x,y,z)$ is a smooth approximation of $\mathbb{1}_{D_\varepsilon}$ with $D_\varepsilon = \{(x,y,z) : \|X\|^2+\|Y\|^2+\|Z\|^2 \leq \varepsilon\}$ such that $\chi_{2\varepsilon} \leq \mathbb{1}_{D_\varepsilon} \leq \chi_\varepsilon$. We recall that we have set $X = x-y$, $Y = y-z$ and $Z = z-x$. This implies by exchangeability

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\chi_\varepsilon(x,y,z)(G(x,y,z)\wedge M)\mu_s^{N_k,\beta_{N_k}}(\mathrm{d}x)\mu_s^{N_k,\beta_{N_k}}(\mathrm{d}y)\mu_s^{N_k,\beta_{N_k}}(\mathrm{d}z)\mathrm{d}s\Big]\\
\leq&\mathbb{E}\Big[\int_0^{t\wedge\beta_{N_k}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\chi_\varepsilon(x,y,z)(G(x,y,z)\wedge M)\mu_s^{N_k,\beta_{N_k}}(\mathrm{d}x)\mu_s^{N_k,\beta_{N_k}}(\mathrm{d}y)\mu_s^{N_k,\beta_{N_k}}(\mathrm{d}z)\mathrm{d}s\Big]\\
&+M\mathbb{E}[t-t\wedge\beta_{N_k}]\\
\leq&\frac{(N_k-1)(N_k-2)}{N_k^2}C+M\frac{3N_k-2}{N_k^2}+M\mathbb{E}[t-t\wedge\beta_{N_k}].
\end{aligned}
$$

Since $(x,y,z)\mapsto\chi_\varepsilon(x,y,z)(G(x,y,z)\wedge M)$ is continuous and bounded, sending k to infinity gives

$$
\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\chi_\varepsilon(x,y,z)(G(x,y,z)\wedge M)\mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y)\mu_s(\mathrm{d}z)\mathrm{d}s\Big]\leq C,
$$

indeed, $\mathbb{E}[t-t\wedge\beta_{N_k}]\leq t\mathbb{P}(\beta_{N_k}\leq t)\to_{k\to\infty}0$ according to Proposition 4.11-(ii). We conclude using the monotone convergence theorem twice with $\varepsilon\to0$ and $M\to\infty$.

Step 2. We show that a.s., for almost every $s\geq0$, all $x\in\mathbb{R}^2$, $\mu_s(\{x\})\in\{0,1\}$. If it was not the case, it would exist with positive probability a bounded set $A\in\mathcal{B}(\mathbb{R}_+)$ with positive Lebesgue measure such that for all $s\in A$ there exists $a_s\in(0,1)$, $x_s\in\mathbb{R}^2$ and a measure g_s such that $\mu_s=a_s\delta_{x_s}+g_s$ and $g_s(\{x_s\})=0$. On this event, we would have for all $s\in A$,

$$
\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}G(x,y,z)\mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y)\mu_s(\mathrm{d}z)\geq a_s^2\int_{\mathbb{R}^2}G(x_s,x_s,z)g_s(\mathrm{d}z)=\infty,
$$

Indeed, recall that $G(x,x,z)=\infty$ if $z\neq x$. We get a contradiction with Step 1 by integrating in time and taking the expectation. Notice that the measurability of $s\to a_s$, $s\to x_s$ and $s\to g_s$ are not required to justify this computation.

Step 3. By Lemma 4.18 we see that a.s., for all $t>0$, μ_t is not a (full) Dirac measure. Indeed, if with positive probability there exists $t>0$ such that μ_t is a Dirac measure, say δ_{x_0} with $x_0\in\mathbb{R}^2$, then we have $0=\int_{\mathbb{R}^2}\|x-x_0\|^2f_t(\mathrm{d}x)\geq\int_{\mathbb{R}^2}\|x-x_0\|^2f_0(\mathrm{d}x)$ which implies $f_0=\delta_{x_0}$ and this is forbiden since $\max_{x\in\mathbb{R}^2}f_0(\{x\})<1$ by assumption.

Step 4. Gathering Steps 2 and 3, we conclude that a.s., for a.e. $t\geq0$, $\sup_{x\in\mathbb{R}^2}\mu_t(\{x\})=0$, whence the result. □

We finally give the

Proof of Theorem 4.6-(ii). Recall that $\theta=2$, $f_0\in\mathcal{P}(\mathbb{R}^2)$ and that $(\mu^{N,\beta_N})_{N\geq5}$ is the corresponding family of empirical processes. We know by Theorem 4.6-(i), that the family $\{(\mu_t^{N,\beta_N})_{t\geq0},N\geq5\}$ is tight, so we can consider $(N_k)_{k\geq0}$ and a random variable $(\mu_t)_{t\geq0}$ belonging to $C([0,\infty),\mathcal{P}(\mathbb{R}^2))$ such that $\lim_k N_k=\infty$ and $(\mu_t^{N_k,\beta_{N_k}})_{t\geq0}$ goes to $(\mu_t)_{t\geq0}$ in law as $k\to\infty$. Moreover we have $\mu_0=f_0$ since by hypothesis, μ_0^N goes weakly to f_0 in probability as $N\to\infty$. It suffices to prove that $(\mu_t)_{t\geq0}$ satisfies (4.6), which we have already seen in Proposition 4.19, and is a weak solution to (4.1)-(4.2), which we now do.

We apply Lemma 4.7-(iii) to $\varphi\in C_b^2(\mathbb{R}^2)$ and get

$$
I_t^1(\mu^{N_k,\beta_{N_k}})-I_t^2(\mu^{N_k,\beta_{N_k}})=M_k(t)+R_k(t),
$$

where

$$M_k(t) = \frac{1}{N_k} \int_0^t \sum_{i=1}^{N_k} \nabla\varphi(X_s^{i,N_k}) \cdot \mathrm{d}B_s^i,$$

for all $\nu \in C([0,\infty), \mathcal{P}(\mathbb{R}^2))$,

$$I_t^1(\nu) = \int_{\mathbb{R}^2} \varphi(x)\nu_t(\mathrm{d}x) - \int_{\mathbb{R}^2} \varphi(x)\nu_0(\mathrm{d}x) - \frac{1}{2}\int_0^t \int_{\mathbb{R}^2} \Delta\varphi(x)\nu_s(\mathrm{d}x)\mathrm{d}s,$$
$$I_t^2(\nu) = \frac{\theta}{2}\int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} K(x-y)\cdot[\nabla\varphi(x) - \nabla\varphi(y)]\nu_s(\mathrm{d}x)\nu_s(\mathrm{d}y)\mathrm{d}s,$$

and

$$R_k(t) = -\frac{1}{2}\int_{t\wedge\beta_{N_k}}^t \int_{\mathbb{R}^2} \Delta\varphi(x)\mu_s^{N_k,\beta_{N_k}}(\mathrm{d}x)\mathrm{d}s,$$
$$+\frac{\theta}{2}\int_{t\wedge\beta_{N_k}}^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} K(x-y)\cdot[\nabla\varphi(x) - \nabla\varphi(y)]\mu_s^{N_k,\beta_{N_k}}(\mathrm{d}x)\mu_s^{N_k,\beta_{N_k}}(\mathrm{d}y)\mathrm{d}s.$$

Using that $||K(x)|| \le ||x||^{-1}\mathbb{1}_{\{x\neq 0\}}$ and that $D^2\varphi$ is bounded, one easily checks that

$$\mathbb{E}[|R_k(t)|] \le C\mathbb{E}[t - t\wedge\beta_{N_k}] \le Ct\mathbb{P}(\beta_{N_k} \le t),$$

which tends to 0 as $k \to \infty$ by Proposition 4.11.

Hence we conclude exactly as in Step 2 of the proof of Theorem 4.4-(ii) that a.s., for each $t \ge 0$, $I_t^1(\mu) = I_t^2(\mu)$, which precisely means that μ is a weak solution to (4.1)-(4.2). Observe that we may also use here that a.s., $\mu_s \otimes \mu_s(D) = 0$ for a.e. $s \ge 0$, where $D = \{(x,y) \in (\mathbb{R}^2)^2 : x = y\}$, by Proposition 4.19. □

Chapitre 5

A post-explosion model for the supercritical Keller-Segel particle system

We present here informal ideas of a well-engaged project. Let $N \geq 2$ and $\theta \geq 2$, we set $k_0 = \lceil 2N/\theta \rceil$.

Definition 5.1. *We say that a continuous stochastic process* $(X_t^N)_{t\geq 0} := (X_t^{1,N}, \dots X_t^{N,N})_{t\geq 0}$ *with values in* $(\mathbb{R}^2)^N$ *is an extended* $KS(\theta, N)$*-process if a.s. for all starting point* $x \in (\mathbb{R}^2)^N$ *:*

- *for all* $K \subset [\![1, N]\!]$ *with* $|K| \geq k_0$*, if it exists* $t_0 \geq 0$ *such that* $R_K(X_{t_0}^N) = 0$*, then for* $t \geq t_0$*,* $R_K(X_t^N) = 0$*.*
- *We set for all* $t \geq 0$*,*

$$\tilde{\mathcal{K}}_t = \{K \subset [\![1, N]\!] : |K| \geq k_0 \quad \textit{and} \quad R_K(X_t^N) = 0 \textit{ and for all } i \notin K, R_{K\cup\{i\}}(X_t^N) > 0\},$$

$$\mathcal{K}_t = \tilde{\mathcal{K}}_t \cup \bigcup_{i\in[\![1,N]\!]\setminus\cup_{K\in\tilde{\mathcal{K}}_t}} \{\{i\}\}$$

We define $(\sigma_k)_{k\geq 0}$ *by induction by setting* $\sigma_0 = 0$ *and for all* $k \geq 0$*,*

$$\sigma_{k+1} = \inf\{t \geq \sigma_k : \mathcal{K}_t \neq \mathcal{K}_{\sigma_k}\}.$$

We set for all $k \geq 0$*,* $\mathcal{K}^k = \mathcal{K}_{\sigma_k}$*. Then for all* $k \geq 0$*, all* $t \in [\sigma_k, \sigma_{k+1})$*, writting* $\mathcal{K}^k = (K^{1,k}, \dots, K^{|\mathcal{K}^k|,k})$*, we get, at least in the theory of Dirichlet forms sense,*

$$\textit{for all } i \in [\![1, |\mathcal{K}^k|]\!],\ \mathrm{d}S_{K^{i,k}}(X_t^N) = \frac{\mathrm{d}W_t^{i,k}}{\sqrt{|K^{i,k}|}} - \frac{\theta}{N}\sum_{j\neq i}|K^{j,k}|\frac{S_{K^{i,k}}(X_t^N) - S_{K^{j,k}}(X_t^N)}{\|S_{K^{i,k}}(X_t^N) - S_{K^{j,k}}(X_t^N)\|^2}\mathrm{d}t,$$

where $((W_t^{i,k})_{t\geq 0})_{i\in[\![1,|\mathcal{K}^k|]\!]}$ *is a family of independant* 2*-dimensional Brownian motion.*

We set

$$\mathbb{S} = \{x \in H : \|x\|^2 = 1\} \quad \text{where} \quad H = \{x \in (\mathbb{R}^2)^N : S_{[\![1,N]\!]}(x) = 0\}.$$

We observe that the particle system defined on $[\sigma_k, \sigma_{k+1})$ is a version of the Keller-Segel particle system where we allow particle to have masses and we apply the interaction law by taking in account the masses. This model was already proposed by Fournier-Jourdain [20]. The idea is that, following the intuition presented in Chapter 2, when particles collide, if the sum of the masses of the particles involved in the collision exceed k_0, then the particles stay stick together forever. If the sum of the masses is strictly less than k_0, then particles separate instantaneously.

We also introduce the extended spherical Keller-Segel process which will be crucial for our purpose. As in Chapter 2 Proposition 2.10, one has to understand this as the normalization of the extended $KS(\theta, N)$-process, and the masses of the particles is taking in account in the geometry we use to describe the problem. Since we did not investigate a lot yet in the study of this process, there could be some errors in the conjecture of its expression.

Definition 5.2. *We say that a continuous stochastic process $(U_t^N)_{t\geq 0} := (U_t^{1,N}, \dots U_t^{N,N})_{t\geq 0}$ with values in $\mathbb{S}$ is an extended $SKS(\theta, N)$-process if a.s. for all starting point $u \in \mathbb{S}$:*

- *for all $K \subsetneq [\![1, N]\!]$ with $|K| \geq k_0$, if it exists $t_0 \geq 0$ such that $R_K(U_{t_0}^N) = 0$, then for $t \geq t_0$, $R_K(U_t^N) = 0$.*
- *We set for all $t \geq 0$,*

$$\tilde{\mathcal{K}}_t^U = \{K \subset [\![1, N]\!] : |K| \geq k_0 \quad \text{and} \quad R_K(U_t^N) = 0 \text{ and for all } i \notin K, R_{K\cup\{i\}}(U_t^N) > 0\},$$
$$\mathcal{K}_t^U = \tilde{\mathcal{K}}_t^U \cup \bigcup_{i \in [\![1,N]\!] \setminus \cup_{K \in \tilde{\mathcal{K}}_t^U}} \{\{i\}\}.$$

We define $(\sigma_k^U)_{k\geq 0}$ by induction by setting $\sigma_0^U = 0$ and for all $k \geq 0$,

$$\sigma_{k+1}^U = \inf\{t \geq \sigma_k^U : \mathcal{K}_t^U \neq \mathcal{K}_{\sigma_k^U}^U\}.$$

We set for all $k \geq 0$, $\mathcal{K}^{U,k} = \mathcal{K}_{\sigma_k^U}^U$. Then for all $k \geq 0$, all $t \in [\sigma_k^U, \sigma_{k+1}^U)$, setting $\mathcal{K}^{U,k} = (K^{1,k}, \dots, K^{|\mathcal{K}^k|,k})$, and $(S_{\mathcal{K}^{U,k}}(U_t^N))_{t\geq 0} = ((S_{K^{i,k}}(U_t^N))_{i\in[\![1,|\mathcal{K}^{U,k}|]\!]})_{t\geq 0}$, we get, at least in the theory of Dirichlet forms sense,

$$\mathrm{d}S_{\mathcal{K}^{U,k}}(U_t^N) = \pi_H^k \pi^k_{S_{\mathcal{K}^{U,k}}(U_t^N)^\perp}(A_k \mathrm{d}W_t^k + b_k(S_{\mathcal{K}^{U,k}}(U_t^N)))\mathrm{d}t - \frac{2N-3}{2} S_{\mathcal{K}^{U,k}}(U_t^N)\mathrm{d}t,$$

where A_k is a diagonal $2N \times 2N$ matrix such that for all $i \in [\![1, N]\!]$, $(A_k)_{2i-1,2i-1} = (A_k)_{2i,2i} = |K^{i,k}|^{-1/2}$, $(W_t^k)_{t\geq 0}$ is a $2N$-dimensional Brownian motion, for the scalar product

$$(\cdot, \cdot)_k : (x, y) \in (\mathbb{R}^2)^{|\mathcal{K}^{U,k}|} \times (\mathbb{R}^2)^{|\mathcal{K}^{U,k}|} \mapsto \sum_{i=1}^{|\mathcal{K}^{U,k}|} |K^{i,k}| x^i \cdot y^i,$$

π_H^k is the orthogonnal projection on $H = vect(1, \dots, 1)^\perp$, for all x, $y \in (\mathbb{R}^2)^N$, $\pi^k_{x^\perp}(y) = y - \|x\|_k^{-2}(x \cdot y)_k x$ with $\|\cdot\|_k^2 = (\cdot, \cdot)_k$, and for all $x \in (\mathbb{R}^2)^N$, $b_k(x) = (b_k^1(x), \dots, b_k^{|\mathcal{K}^k|}(x))$ where

$$\text{for all } i \in [\![1, |\mathcal{K}^k|]\!], \quad b_k(x) = -\frac{\theta}{N} \sum_{j\neq i} |K^{j,k}| \frac{S_{K^{i,k}}(x) - S_{K^{j,k}}(x)}{\|S_{K^{i,k}}(x) - S_{K^{j,k}}(x)\|^2}.$$

We want to show that the extended $KS(\theta, N)$-process is unique and a natural extension of the particle system after the explosion time. To this end, we define for all $\varepsilon > 0$ the regularized process $(X_t^{N,\varepsilon})_{t\geq 0} = (X^{1,N,\varepsilon}, \dots, X_t^{N,N,\varepsilon})_{t\geq 0}$ checking the classically well-posed SDE :

$$\text{for all } i \in [\![1, N]\!], \quad \mathrm{d}X_t^{i,N,\varepsilon} = \mathrm{d}B_t^i - \frac{\theta}{N} \sum_{j\neq i} \frac{X_t^{i,N,\varepsilon} - X_t^{j,N,\varepsilon}}{\|X_t^{i,N,\varepsilon} - X_t^{j,N,\varepsilon}\|^2 + \varepsilon^2} \mathrm{d}t,$$

where $(B_t^1, \dots, B_t^N)_{t\geq 0}$ is a $2N$-dimensional Brownian motion. We call such a process a $KS_\varepsilon(\theta, N)$-process. We want to show the following :

Theorem 5.3. *The sequence $(X_t^{N,\varepsilon})_{t\geq 0}$ converges in law to an extended $KS(\theta, N)$-process.*

What motivates this approximation is the fact that it informally corresponds to give to particles a thickness of size ε.

5.1 Regularized clusters

A usefull tool to prove this result is a well notion of cluster for the $KS_\varepsilon(\theta, N)$-process. Indeed, the definition of $(\mathcal{K}_t)_{t\geq 0}$ is not suitable to this case since the interaction is bounded for ε being fixed : according to the Girsanov's theorem, trajectories of this process are the same as those of a $2N$-dimensional Brownian motion for fixed horizon of time, which prevents particles from colliding. However, we expect to observe emergence of clusters of particle with dispersion of the order of ε, which would be the analogous of collisions in the regularized case. We define for all $\nu > 0$, all $K \subset [\![1, N]\!]$ and all $x = (x^1, \dots, x^N) \in C(\mathbb{R}_+, (\mathbb{R}^2)^N)$ the sequences of stopping time $(\tau_k^{K,\nu}(x))_{k\geq 0}$ and $(\tilde{\tau}_k^{K,\nu}(x))_{k\geq 0}$ by induction by setting $\tilde{\tau}_0^{K,\nu}(x) = 0$ and for all $k \geq 0$,

$$\tau_{k+1}^{K,\nu}(x) = \inf\Big\{t \geq \tilde{\tau}_k^{K,\nu,\varepsilon} : R_K(x(t)) \leq \nu^{1+1/|K|} \quad \text{and} \quad \min_{i\notin K} R_{K\cup\{i\}}(x(t)) \geq 2\nu^{1+1/(|K|+1)}\Big\},$$
$$\tilde{\tau}_{k+1}^{K,\nu}(x) = \inf\Big\{t \geq \tau_{k+1}^{K,\nu}(x) : R_K(x(t)) \geq 2\nu^{1+1/|K|} \quad \text{or} \quad \min_{i\notin K} R_{K\cup\{i\}}(x(t)) \leq \nu^{1+1/(|K|+1)}\Big\}.$$

Observe that if there exists $k \geq 0$ such that $t \in [\tau_k^{K,\nu}(x), \tilde{\tau}_k^{K,\nu}(x))$, then if ν is small enough, there exists a constant $c > 0$ such that for all $i \in K$, all $j \notin K$, $\|x^i(t) - x^j(t)\|^2 \geq c\nu^{1+1/(|K|+1)}$. Indeed, if particles indexed by K are close to each other, and when any particle indexed by K^c is added, the total dispersion is bigger, then it means that particles indexed by K are far away from particles indexed by K^c.

Finally, we set for all $t \geq 0$,

$$\mathcal{K}^\nu(t, x) = \Big\{K \subset [\![1, N]\!] : t \in \cup_{k\geq 0}[\tau_k^{K,\nu}(x), \tilde{\tau}_k^{K,\nu}(x))\Big\},$$

which is a regularized verion of $(\mathcal{K}_t)_{t\geq 0}$. We say that $\mathcal{K}(t, x)$ is the configuration according to ν of x at time t. Moreover, we define the sequence $(\sigma_k^\nu(x))_{k\geq 0}$ by induction by setting $\sigma_0^\nu(x) = 0$ and for all $k \geq 0$,

$$\sigma_{k+1}^\nu(x) = \inf\{t \geq 0 : \mathcal{K}^\nu(t, x) \neq \mathcal{K}^\nu(\sigma_k^\nu(x), x)\}.$$

Finally, we set for all $k \geq 0$, $\sigma_k^{\nu,\varepsilon} = \sigma_k^\nu(X^{N,\varepsilon})$ and $\mathcal{K}_k^{\nu,\varepsilon} = \mathcal{K}^\nu(\sigma_k^{\nu,\varepsilon}, X^{N,\varepsilon})$.

5.2 Tightness

Here we explain informally the reasons why the sequence $((X_t^{N,\varepsilon})_{t\geq 0})_{\varepsilon>0}$ should be tight in $\mathcal{P}(C(\mathbb{R}_+,(\mathbb{R}^2)^N))$. We proceed by induction on N, the case $N=1$ being clear since for all $\varepsilon>0$, $(X_t^{N,\varepsilon})_{t\geq 0}$ is a 2-dimensional Brownian motion.

Let $N\geq 1$. We assume we showed the tightness for all $k\in [\![1,N]\!]$ and we explain why the sequence $((X_t^{N+1,\varepsilon})_{t\geq 0})_{\varepsilon>0}$ should be tight.

The first idea is that for any fixed $T>0$, $\nu>0$, $\limsup_{\varepsilon>0}\mathbb{P}(\sigma_M^{\nu,\varepsilon}\leq T)\to 0$ as $M\to\infty$, which means that there is not too much change of configuration. Indeed there are two kind of change of configuration :

- (a) either $|\mathcal{K}_{k+1}^{\nu,\varepsilon}|\leq|\mathcal{K}_k^{\nu,\varepsilon}|$.
- (b) or $|\mathcal{K}_{k+1}^{\nu,\varepsilon}|>|\mathcal{K}_k^{\nu,\varepsilon}|$ and in this case, there must have a cluster in $\mathcal{K}_k^{\nu,\varepsilon}$ that separate in many pieces.

Since $(|\mathcal{K}_k^{\nu,\varepsilon}|)_{k\geq 0}$ can only take a finite number of value because it belongs to $[\![1,N]\!]$, the number of change of configuration of type (a) and (b) should offset each other. In particular this means that on the event $\{\sigma_M^{\nu,\varepsilon}\leq T\}$ with M large enough, we will observe a lot of change of configuration of type (b). What makes this event unlikely is the fact that it takes time for a cluster separated from the other to split in several pieces. To be more precise, once a cluster is far from the other, the dispersion of this cluster behaves like a squared Bessel process, see Chapter 2. Thus, the time for the dispersion for being large enough is the time spent by a squared Bessel process for being large enough. Since this should occur a lot of independant times, the large law of numbers make it difficult to happen during the finite time interval $[0,T]$.

Once we know that with high probability there exists $M>0$ such that $\sigma_M^{\nu,\varepsilon}\geq T$, it is informally sufficient to look at the tightness on the time intervals $[\sigma_k^{\nu,\varepsilon},\sigma_{k+1}^{\nu,\varepsilon})$ for all $k\geq 0$.

- If $\mathcal{K}_k^{\nu,\varepsilon}\neq\{[\![1,N+1]\!]\}$, then there are several clusters far from each other, and so particles in different clusters see their interactions between each other being bounded. According to the Girsanov's theorem, even if it means to change the probability by an absolutely continuous one with uniform lower and upper bound on the density, this process behaves like several independant regularized Keller-Segel particle system, and this is tight according to the induction hypothesis since these particles system are composed of less than N particles each.
- If $\mathcal{K}_k^{\nu,\varepsilon}=\{[\![1,N+1]\!]\}$, then during this time intervals, $(R_{[\![1,N+1]\!]}(X_t^{N,\varepsilon}))_{t\geq 0}$ is small, so in a sense, since all the particle are at the same position, it suffices to know $(S_{[\![1,N+1]\!]}(X_t^{N,\varepsilon}))_{t\geq 0}$ in order to understand the behaviour of $(X_t^{N,\varepsilon})_{t\geq 0}$. However since the drift is odd, we have that for all $\varepsilon>0$, $(S_{[\![1,N+1]\!]}(X_t^{N,\varepsilon}))_{t\geq 0}$ is a Brownian motion and so is clearly tight.

5.3 Unicity of the extended $KS(\theta,N)$-process.

We give an informal proof to explain why there exists a unique extended $KS(\theta,N)$-process and a unique $SKS(\theta,N)$-process for all $N\geq 1$ by induction. The case where $N=1$ is clear since a $KS(\theta,N)$-process is a 2-dimensional Brownian motion. We assume that we proved the result for $k\in[\![1,N]\!]$ with $N\geq 1$ and we explain why there must exist a unique $KS(\theta,N+1)$-process in three steps.

Step 1 Fix $\nu > 0$. We explain why there exists a unique extended $KS(\theta, N+1)$ process until

$$\tau^\nu = \inf\{t \geq 0 : R_{[\![1,N+1]\!]}(X_t^{N+1}) \leq \nu\}.$$

We proceed as follow :

- If $\mathcal{K}_0 = \{[\![1, N+1]\!]\}$, then it is over since $\tau^\nu = 0$ a.s.
- Otherwise, we can define $((X_t^{|K|})_{t\geq 0})_{K\in\mathcal{K}_0}$ a family of independant extended $KS(\theta|K|/(N+1), |K|)$-processes starting at $(x^i)_{i\in K}$, and we kill the whole process when there is a change of configuration according to ν.
- we define a measure $\mathbb{Q}$ which intuitively correspond to the measure of the change of probability given by the Girsanov's theorem in order to add the interactions missing between particles which are in different subsets of $\mathcal{K}_0$, we just built the process under $\mathbb{Q}$ until the time of the first change of configuration according to ν.
- It suffices to iterate the idea until the new configuration acording to ν equals $\{[\![1, N+1]\!]\}$.

According to the induction hypothesis, we get in addition the unicity of the process until τ^ν.

Step 2 We explain why there exists a unique extended $SKS(\theta, N+1)$ process. Roughly we use the ideas of the decomposition of Chapter 2 from Proposition 2.10. Since the extended $SKS(\theta, N+1)$-process can be expressed thanks to the $KS(\theta, N+1)$-process until its first collision between the N particles, Step 1.1 allows us to conclude.

Step 3 Since the Step 2 is sufficient to show that there exists a unique $KS(\theta, N+1)$-process in the case where $k_0 \leq N$, it is sufficient to conclude the Step 1 in the case where $k_0 > N$. According to the idea of Chapter 2 Proposition 2.10, $(X_t^{N+1})_{t\geq 0}$ can be viewed as a decomposition of three processes, see Proposition 2.10. We can define $(M_t)_{t\geq 0}$ a 2-dimensional Brownian motion with diffusion coefficient $(N+1)^{-1/2}$ starting at $S_{[\![1,N+1]\!]}(x)$ and $(D_t)_{t\geq 0}$ an independant squared Bessel process with dimension $d_{\theta,N}(N)$ starting at $R_{[\![1,N+1]\!]}(x)$. Then conditionnally on $(D_t)_{t\geq 0}$, for each excursion open interval (a, b) of $(D_t)_{t\geq 0}$, we fix $t_0 \in (a, b)$ and set for all $t \in (a, b)$, $A_t = \int_{t_0}^t D_s^{-1} \mathrm{d}s$ and $(\rho_t)_{t\geq 0}$ its generalized inverse. Then we set for all $t \in (a, b)$, $\tilde{U}_t^{N+1} = U_{A_t}^{N+1}$ with $(U_t^{N+1})_{t\geq 0}$ an eternal extended $SKS(\theta, N+1)$-process starting from $-\infty$ with the invariant distribution of the process, which exists since it is an ergodic process because $k_0 > N$, see Chapter 2. It is worthy to emphase that roughly this is necessary for this process to be stationnary during excursions except the first one since it is an eternal process, the law of $(\tilde{U}_t^{N+1})_{t\geq 0}$ is thus entirely determined. Finally, setting for all $t \geq 0$, $X_t^{N+1} = M_t + \sqrt{D_t}\tilde{U}_t^{N+1}$, we built an extended $KS(\theta, N)$-process and its law is entirely determined.

5.4 Characterization of the limit process

We show by induction on N that for all $N \geq 1$, $(X_t^{N,\varepsilon})_{t\geq 0}$ converges in law to the extended $KS(\theta, N)$-process as $\varepsilon \to 0$. If $N = 1$, then the result is clear since for all $\varepsilon > 0$, $(X_t^{N,\varepsilon})_{t\geq 0}$ is a 2-dimensional Brownian motion. We assume that the result has been proven for all $k \in [\![1, N]\!]$ for some $N \geq 1$ and we prove that $(X_t^{N+1,\varepsilon})_{t\geq 0}$ converges in law to $(X_t^{N+1})_{t\geq 0}$ as $\varepsilon \to 0$ where $(X_t^{N+1})_{t\geq 0}$ is the extended $KS(\theta, N+1)$-process.

According to the previous section, it should exists a continuous stochastic process $(Z_t^{N+1})_{t\geq 0}$ with values in $(\mathbb{R}^2)^{N+1}$ and a subsequence $(\varepsilon_p)_{p\geq 0}$ such that $\varepsilon_p \to 0$ as $p \to \infty$ and $(X_t^{N+1,\varepsilon_p})_{t\geq 0} \to$

$(Z_t^{N+1})_{t\geq 0}$ as $p\to\infty$. It is sufficent to show that $(Z_t^{N+1})_{t\geq 0}$ and $(X_t^{N+1})_{t\geq 0}$ has the same law. We split the explanation in several steps.

Step 1. We explain that for all $\nu>0$, all $k\geq 0$, $x\in C(\mathbb{R}_+,(\mathbb{R}^2)^N)\mapsto \sigma_k^\nu(x)$ is continuous $\mathcal{L}((Z_t^{N+1})_{t\geq 0})$-a.e. This is informally true because σ_k^ν is continuous on the elements of $C(\mathbb{R}_+,(\mathbb{R}^2)^N)$ such that for all $K\subset [\![1,N+1]\!]$, $R_K\circ x$ does not have a local extremum equal to $\nu^{1+1/|K|}$ or $2\nu^{1+1/|K|}$ while $\min_{i\notin K} R_{K\cup\{i\}}\circ x$ is larger than $\nu^{1+1/(|K|+1)}$. If we assume that $(Z_t^{N+1})_{t\geq 0}$ is an extended $KS(\theta,N+1)$-process and if $R_K(Z_t^{N+1}))_{t\geq 0}$ has a local extremum at $t>0$ equal to $\nu^{1+1/|K|}$ or $2\nu^{1+1/|K|}$ while $\min_{i\notin K} R_{K\cup\{i\}}(Z_t^{N+1})\geq \nu^{1+1/(|K|+1)}$, then locally in time, particles indexed by K are far away from the other particles. Thus it means that the trajectories of $(R_K(Z_t^{N+1}))_{t\geq 0}$ locally in time look like the trajectories of a squared Bessel process (see Chapter 2), which conclude the argumentation since a squared Bessel process a.s does not have local extremum equal to a precise positive value. Of course this is irrelevant since we assume what we want to show, but using the Portmanteau Theorem, we can estimate the probability that $(Z_t^{N+1})_{t\geq 0}$ belongs to some sets by being led by this idea and make it rigorous.

Step 2. Informally, we show that for all $k\geq 0$, conditionally to the event $\{\mathcal{K}_k^\nu\neq [\![1,N+1]\!]\}$, we have

$$\text{the law of } (Z_t^{N+1})_{t\in[\sigma_k^\nu,\sigma_{k+1}^\nu)} \text{is equal to the law of } (X_t^{N+1})_{t\geq 0} \text{ starting at } Z_{\sigma_k^\nu}^{N+1}. \tag{5.1}$$

From Step 1, we deduce that for all $k\geq 0$, $(X_t^{N+1,\varepsilon_p})_{t\in[\sigma_k^{\nu,\varepsilon_p},\sigma_{k+1}^{\nu,\varepsilon_p})}$ converges in law to $(Z_t^{N+1})_{t\in[\sigma_k^\nu,\sigma_{k+1}^\nu)}$ as $p\to\infty$, where for all $k\geq 0$, $\sigma_k^\nu=\sigma_k^\nu(Z^{N+1})$.

Fix $k\geq 0$. We do the assumption that on the event $\{\mathcal{K}_k^\nu\neq [\![1,N+1]\!]\}$, for all p large enough we have $\mathcal{K}_k^{\nu,\varepsilon_p}=\mathcal{K}_k^\nu$, which will not happen but this makes the reasonning clearer. We write $\mathcal{K}_k^\nu=(K^1,\dots,K^m)$ and on the time interval $[\sigma_k^{\nu,\varepsilon_p},\sigma_{k+1}^{\nu,\varepsilon_p})$, for all $i\in[\![1,m]\!]$, particles indexed by K^i are far away from the other. Since the interactions between particles in different sets of $\mathcal{K}_k^\nu$ are bounded, one can use again the Girsanov's theorem to obtain the existence of a probability measure $\mathbb{Q}^{\nu,p,k}$ such that $(X_t^{N,\varepsilon_p})_{t\in[\sigma_k^{\nu,\varepsilon_p},\sigma_k^{\nu,\varepsilon_p})}$ under $\mathbb{Q}^{\nu,p,k}$ has the law of $((Y_t^{1,|K^1|,\varepsilon_p})_{t\geq 0},\dots,(Y_t^{m,|K^m|,\varepsilon_p})_{t\geq 0})$ where for all $i\in[\![1,m]\!]$, $(Y_t^{i,|K^i|,\varepsilon_p})_{t\geq 0}$ is a $KS_{\varepsilon_p}(\theta|K^i|/N,|K^i|)$-process starting at $(X^{j,N,\varepsilon_p}_{\sigma_k^{\nu,\varepsilon_p}})_{j\in K^i}$ and, recalling the definition of σ_1^ν, killed at $\sigma_1^\nu(((Y_t^{1,|K^1|,\varepsilon_p})_{t\geq 0},\dots,(Y_t^{m,|K^m|,\varepsilon_p})_{t\geq 0}))$. However, by induction hypothesis we get that for all $i\in[\![1,m]\!]$, the law of $(Y_t^{i,|K^i|,\varepsilon_p})_{t\geq 0}$ starting at $(X^{j,N,\varepsilon_p}_{\sigma_k^{\nu,\varepsilon_p}})_{j\in K^i}$ converges in law to $(X_t^{|K^i|})_{t\geq 0}$ starting at $(Z^{j,N}_{\sigma_k^\nu})_{j\in K^i}$ as $p\to\infty$. We assume that $\mathbb{Q}^{\nu,p,k}$ converges in a nice sense to $\mathbb{Q}^{\nu,k}$ as $p\to\infty$, with $\mathbb{Q}^{\nu,k}$ being the probability measure obtained by the Girsanov's theorem applied to $(X_t^{N+1})_{t\in[\sigma_k^\nu,\sigma_{k+1}^\nu)}$ in order to suppress the interactions between particles which are not in a same set in $\mathcal{K}_k^\nu$. By this way, we obtain the desired result.

We now split the proof in two parts

Step 3. If $k_0>N+1$. Recall the definition of $(\tau_k^{[\![1,N+1]\!],\nu})_{k\geq 1}$ and $(\tilde{\tau}_k^{[\![1,N+1]\!],\nu})_{k\geq 0}$. Thanks to (5.1), we get that for all $k\geq 0$, the law of $(R_{[\![1,N+1]\!]}(Z_t^{N+1}))_{t\in[\tilde{\tau}_k^\nu,\tau_{k+1}^\nu)}$ is equal to the law of a squarred Bessel process with dimension $d_{\theta,N+1}(N+1)>0$ starting at $R_{[\![1,N+1]\!]}(Z^{N+1}_{\tilde{\tau}_k^\nu})$ and stopped when it reaches $\nu^{1+1/(N+1)}$. However, the law of Z_0^{N+1} is known and for all $k\geq 1$, $R_{[\![1,N+1]\!]}(Z^{N+1}_{\tilde{\tau}_k^\nu})=2\nu^{1+1/(N+1)}$. All in all, sending $\nu\to 0$ we get that $(R_{[\![1,N+1]\!]}(Z_t^{N+1}))_{t\geq 0}$ is distributed as a squared Bessel process with dimension $d_{\theta,N+1}(N+1)>0$ starting at $R_{[\![1,N+1]\!]}(Z_0^{N+1})$. Moreover, since for all $\varepsilon > 0$, $(S_{[\![1,N+1]\!]}(X_t^{N+1,\varepsilon}))_{t\geq 0}$ is a Brownian motion with a diffusion coefficient equal to $(N+1)^{-1/2}$, we get that this is the same for $(S_{[\![1,N+1]\!]}(Z_t^{N+1}))_{t\geq 0}$. Thus, it suffices to characterize the law of $(\tilde{U}_t)_{t\geq 0} := (\tilde{U}_t^1, \dots, \tilde{U}_t^{N+1})_{t\geq 0}$ knowing $(S_{[\![1,N+1]\!]}(Z_t^{N+1}), R_{[\![1,N+1]\!]}(Z_t^{N+1}))_{t\geq 0}$ where for all $i \in [\![1, N+1]\!]$,

$$\tilde{U}_t^i = \frac{Z_t^{i,N+1} - S_{[\![1,N+1]\!]}(Z_t^{N+1})}{\sqrt{R_{[\![1,N+1]\!]}(Z_t^{N+1})}}.$$

Since $O = \{t \geq 0 : R_{[\![1,N+1]\!]}(Z_t^{N+1}) > 0\}$ is an open set of $\mathbb{R}_+$, we get that there exists two sequences $(a_i)_{i\geq 0}$ and $(b_i)_{i\geq 0}$ such that O is a disjoint union of the intervals (a_i, b_i). Let $i \geq 0$ such that $a_i \neq 0$, we fix $t_0 \in (a_i, b_i)$ and consider for all $t \in (a_i, b_i)$, $A_t = \int_{t_0}^t (R_{[\![1,N+1]\!]}(Z_s^{N+1}))^{-1}\mathrm{d}s$ and $(\rho_t)_{t\in\mathbb{R}}$ its generalized inverse. Indeed, according to Chapter 2, $(A_t)_{t\geq 0}$ is a bijection between (a_i, b_i) and $\mathbb{R}$. We define

$$(U_t)_{t\in\mathbb{R}} = (\tilde{U}_{\rho_t})_{t\in\mathbb{R}}.$$

According to (5.1) and Chapter 2-Proposition 2.10, we know that for all s, $t \in \mathbb{R}$, $(U_t)_{t\in[s,t]}$ is an extended $SKS(\theta, N+1)$-process starting at U_s and stopped at $t-s$. This process is recurrent positive and eternal, and we assume that this is sufficient to imply that this is a stationnary process. (In fact, we wonder if we need here to have some uniformity with respect to the initial condition to have rigourously this result. It may be necessary to prove that the process is Feller in some sense.) It implies that the law of $(U_t)_{t\geq 0}$ is known. In the case of the excursion interval contaning 0 which is written $[0, b)$ with $b > 0$, we define the same kind of process but with $A_t = \int_0^t D_s^{-1}\mathrm{d}s$, this implies that ρ is a bijection between $\mathbb{R}_+$ and $[0, b)$ and we know the law of U_0 in this case. All in all, the law of $(\tilde{U}_t)_{t\geq 0}$ is entirely characterized.

Step 2.4. If $k_0 \leq N+1$. Recalling the definition of $\tau_1^{[\![1,N+1]\!],\nu}$, thanks to (5.1) we know the law of $(Z_t^{N+1})_{t\in[0,\tau_1^{[\![1,N+1]\!],\nu}]}$. Letting ν going to 0, we get that $(Z_t^{N+1})_{t\geq 0}$ and $(X_t^{N+1})_{t\geq 0}$ have the same law until the first time of $[\![1, N+1]\!]$-collision. It suffices to show that setting $\tau_1 = \inf\{t \geq 0 : R_{[\![1,N+1]\!]}(Z_t^{N+1}) = 0\}$, then for all $t \geq 0$, $R_{[\![1,N+1]\!]}(Z_{\tau_1+t}^{N+1}) = 0$.

To this end, since we already know that the process is regular in the sense that $((X_t^{N+1,\varepsilon})_{t\geq 0})_{\varepsilon>0}$ is tight, it is sufficient to prove that for all $a > 0$, all $t > 0$,

$$\int_0^t \mathbb{1}_{R_{[\![1,N+1]\!]}(X_{\tau_\varepsilon+s}^{N+1,\varepsilon})>a}\mathrm{d}s \to 0 \quad \text{as } \varepsilon \to 0, \tag{5.2}$$

with $\tau_\varepsilon = \inf\{t \geq 0 : R_{[\![1,N+1]\!]}(X_t^{N+1,\varepsilon}) \leq \varepsilon\}$.

We observe that setting for all $t \geq 0$, $Y_t = \varepsilon^{-1/2} X_{\tau_\varepsilon+\varepsilon t}^{N+1,\varepsilon} = (Y_t^1, \dots Y_t^N)_{t\geq 0}$, the process $(Y_t)_{t\geq 0}$ satisfies the SDE :

$$\text{for all } i \in [\![1, N+1]\!], \quad Y_t^i = X_{\tau_\varepsilon}^{N+1,\varepsilon} + B_t^i - \frac{\theta}{N}\int_0^t \sum_{j\neq i} \frac{Y_s^i - Y_s^j}{\|Y_s^i - Y_s^j\|^2 + 1}\mathrm{d}s,$$

where $(B_t)_{t\geq 0} = (B_t^1, \dots, B_t^{N+1})_{t\geq 0}$ is a $2(N+1)$-dimensional Brownian motion. Thanks to a change of variable in (5.2) we are reduced to show that

$$\varepsilon \int_0^{\varepsilon^{-1}t} \mathbb{1}_{R_{[\![1,N+1]\!]}(Y_s)>a\varepsilon^{-1}}\mathrm{d}s \to 0 \quad \text{as } \varepsilon \to 0.$$

Consider $(V_t)_{t\geq 0} = (\pi_H(Y_t))_{t\geq 0}$ where π_H is the orthgonnal projection on

$$H = \{x \in (\mathbb{R}^2)^{N+1} : S_{[\![1,N+1]\!]}(x) = 0\}.$$

One can show that $(V_t)_{t\geq 0}$ is a positive recurrent process with an invariant probability measure ν. The initial condition $V_0 = \varepsilon^{-1/2}\pi_H(X^{N+1,\varepsilon}_{\tau_\varepsilon})$ depends on ε but is tight since

$$\varepsilon^{-1}\|\pi_H(X^{N+1,\varepsilon}_{\tau_\varepsilon})\|^2 = \varepsilon^{-1}R_{[\![1,N+1]\!]}(X^{N+1,\varepsilon}_{\tau_\varepsilon}) = 1,$$

so we can hope that we can adapt in a way or in an other the following intuition which work for a fixed initiale condition $x \in H$:

We fix $A > 0$. For $\varepsilon > 0$ small enough

$$\limsup_{\varepsilon\to 0} \varepsilon \int_0^{\varepsilon^{-1}t} \mathbb{1}_{R_{[\![1,N+1]\!]}(Y_s)>a\varepsilon^{-1}} \mathrm{d}s \leq \limsup_{\varepsilon\to\infty} \varepsilon \int_0^{\varepsilon^{-1}t} \mathbb{1}_{\|V_s\|^2>A} \mathrm{d}s.$$

We conclude thanks to the ergodic theorem that for all $A > 0$,

$$\limsup_{\varepsilon\to 0} \varepsilon \int_0^{\varepsilon^{-1}t} \mathbb{1}_{R_{[\![1,N+1]\!]}(V_s)>a\varepsilon^{-1}} \mathrm{d}s \leq \nu(B_H(0,A)^c).$$

Letting $A \to \infty$ gives us the result by the monotone convergence theorem.

Chapitre 6

Numerical Simulations

Abstract. We present here some numerical simulation in link with Chapter 2.

6.1 The supercritical particle system near explosion

We show some simulations of the particle system studied in Chapter 2. This particle system consists of N point particles in the plane, subjected to Brownian excitation and binary attraction in $\theta x/(N\|x\|^2)$, where x stands for the (vectorial) relative position between two particles and where θ is a scalar parameter. This system is critical in that

- if $\theta < 2$, the system is subcritical and no cluster does emerge ;
- if $\theta > 2$, the system is supercritical and a macroscopic cluster does emerge.

In Chapter 2, we describe the intricate behavior of the particle system, in the supercritical case, before explosion, and in particular which kinds of point collisions occur. According to the precise values of N and θ, there may be either two or three possible kinds of non-sticky collisions.

In all the simulations below, we use an Euler scheme with time step $\mathrm{d}t$ and a smoothed interaction, replacing $\theta x/(N\|x\|^2)$ by $\theta x/(N(\|x\|^2+\varepsilon^2))$.

In the supercritical case, we carry on the simulation after the macroscopic cluster has emerged, which makes sense thanks to the smoothing parameter ε, even if this is not covered by the theory.

We take $N = 23$ particles and $\theta = 2.7$. The theory says that there are binary, 16-ary and 17-ary non-sticky point collisions, and that the emergence of a cluster containing precisely 18 particles makes explode the system. There are no 3-ary, ..., 15-ary collisions.

The c++ code, using the time step $\mathrm{d}t = 10^{-9}$ and the regularization parameter $\varepsilon = 10^{-8}$, writes, every 10^3 time steps,

$$\text{current time} \quad \log_{10}(Z[2]), \quad \log_{10}(Z[3]), \dots, \quad \log_{10}(Z[N]),$$

where $Z[i]$ is defined from the particle configuration $(X[1], ..., X[N]) \in (\mathbb{R}^2)^N$ (at some given time) as follows.

- For $k = 1, \dots, N$, and $i = 2, \dots, N$, let $R[k, i]$ be the square of the distance between $X[k]$ and its $(i-1)$-th nearest neighbor. If e.g. $N = 4$ and $||X[2] - X[3]|| < ||X[2] - X[1]|| < ||X[2]-X[4]||$, then $R_{2,2} = ||X[2]-X[3]||^2$, $R_{2,3} = ||X[2]-X[1]||^2$ and $R_{2,4} = ||X[2]-X[4]||^2$.

- Put $Z[i] = \min\{R[1,i], \dots, R[N,i]\}$ for each $i = 2, \dots, N$.

Observe that $Z[i] = 0$ if and only if there are i particles at the same place.

On Figure 6.1, we show the result of a particurarly favorable simulation. We start with i.i.d. $\mathcal{N}(0, 0.2I_2)$ positions. We plot $2i + \log_{10}(Z[i])$ as a function of time, for $i = 2, \dots, 23$. We add $2i$ only to separate the curves.

- It seems that there are many binary collisions : the purple curve (the lowest one) dips very often.
- It seems that there are a few 3-ary and 4-ary collisions : the green and blue curves have a few dives. This must be numerical.
- It seems that there are no 5-ary to 15-ary collisions : the corresponding curves all dip (for the first time) simpultaneously, and the red curve (labeled 16) also dips at the same time. Since at this time the red curve dips and the black one (labeled 17) does not, this seems to really correspond to a 16-ary collision.
- A few later, we observe a few non-sticky 17-ary collisions (the black curve labeled 17 dips but the purple curve labeled 18 does not).
- A few time later, it seems that some 18-cluster appears forever (the purple curve dips and remains low forever). This is not completely clear, but clearly once the 19-cluster (green curve) has dipped, it remains low forever.
- Finally, the remaining particles join this cluster, one by one.

Figure 6.2 is a zoom of Figure 6.1 around the explosion time.

We next show on Figure 6.3 a less favorable simulation (with the same parameters and initial conditions). Figure 6.4 is a zoom of Figure 6.3 around the explosion time. Here there are some defaults : it seems there is no 16-ary collision (but rather something like a 15-ary one) and the 18-cluster (see the purple curve) manages to separate for a short but clearly positive time. Since such a singular particle system is difficult to simulate, it is not surprising to observe such defaults.

6.2 Subcritical and supercritical illustrations

Next, we show some animations, produced by a R code. We consider $N = 100$ particles and start with four clouds of particles, we take the time step $\mathrm{d}t = 0.00001$ and the regularization parameter $\varepsilon = 0.0001$. We use different values for θ.

In Figure 6.5, we consider the subcritical case $\theta = 1.5$ at time $t = 0$. The theory predicts that there are only binary non-sticky point collisions. We indeed observe that the particles are rather well spread.

In Figure 6.6, we consider the supercritical case $\theta = 2.898$, the theory predicts that there are only binary, 68 and 69-ary non-sticky point collisions, and that the emergence of a cluster containing precisely 70 particles makes explode the system. We observe that the particles begin by spreading out, and then they indeed collapse.

FIGURE 6.1 – A favorable simulation

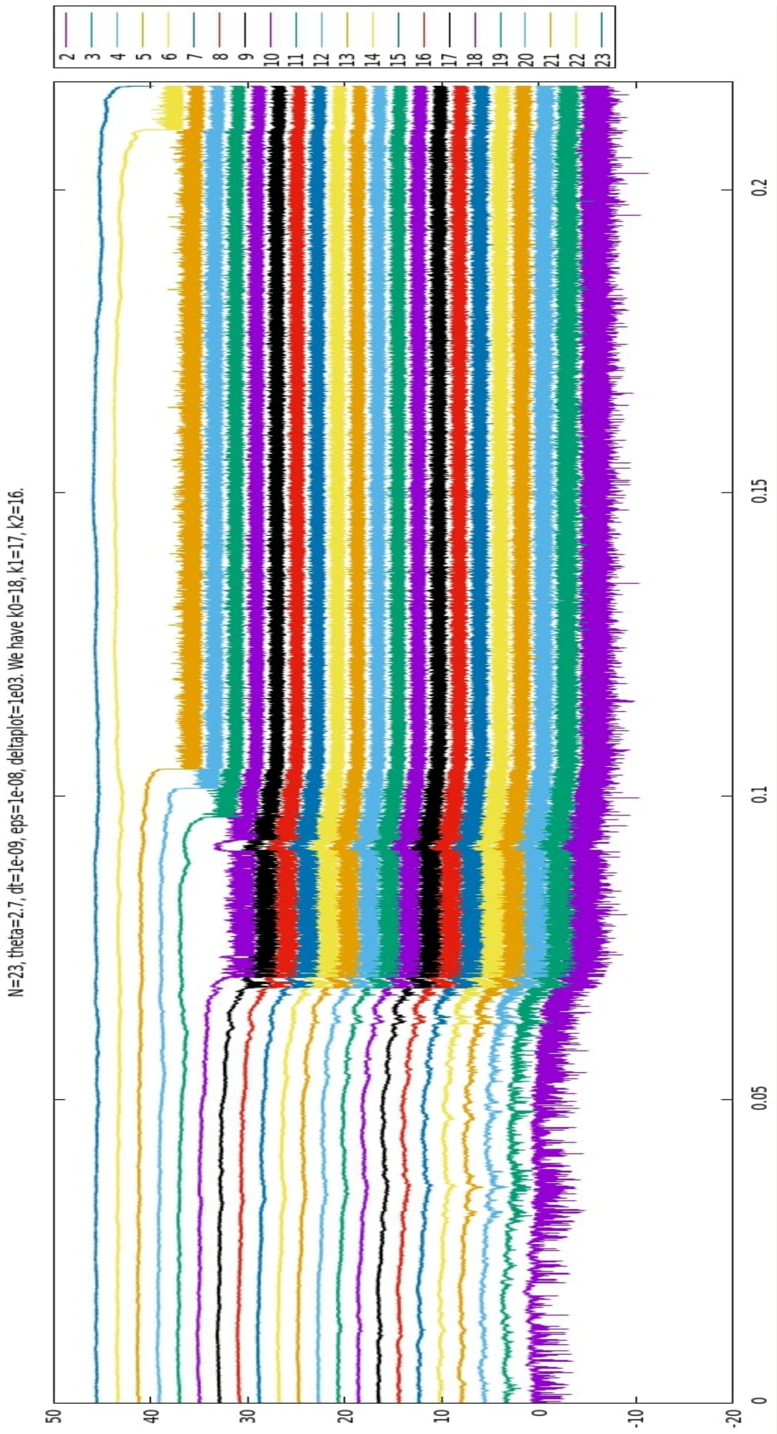


FIGURE 6.2 – Zoom of Figure 6.1

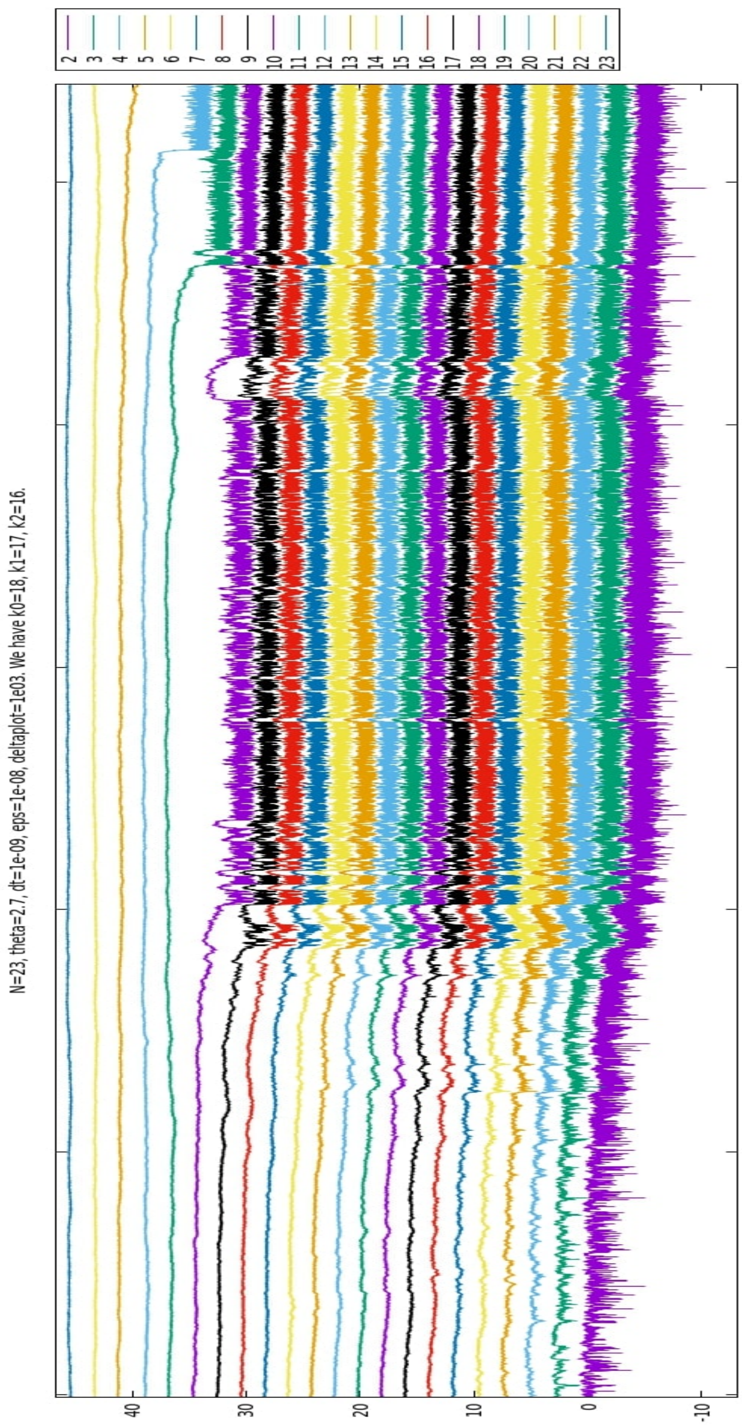


Figure 6.3 – A less favorable simulation

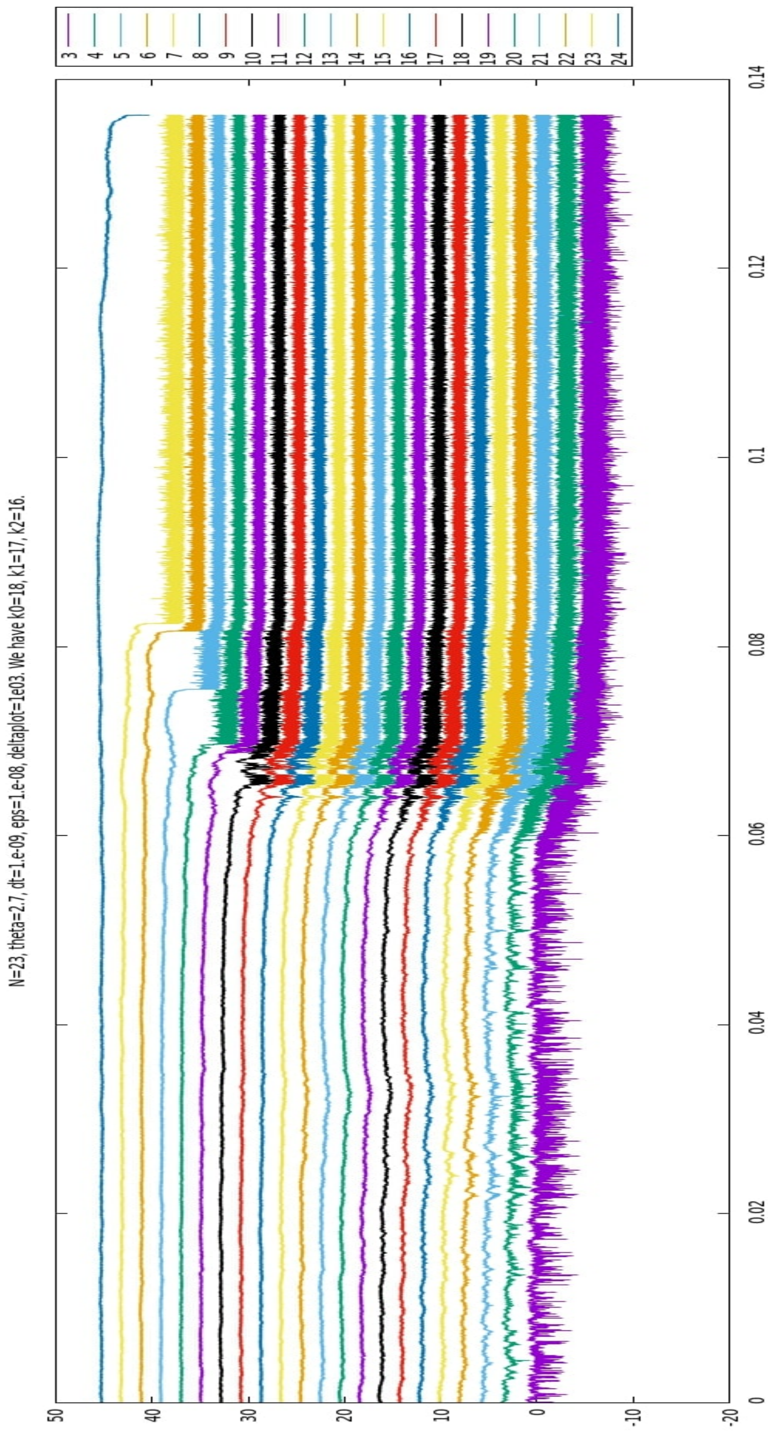


Figure 6.4 – Zoom of Figure 6.3

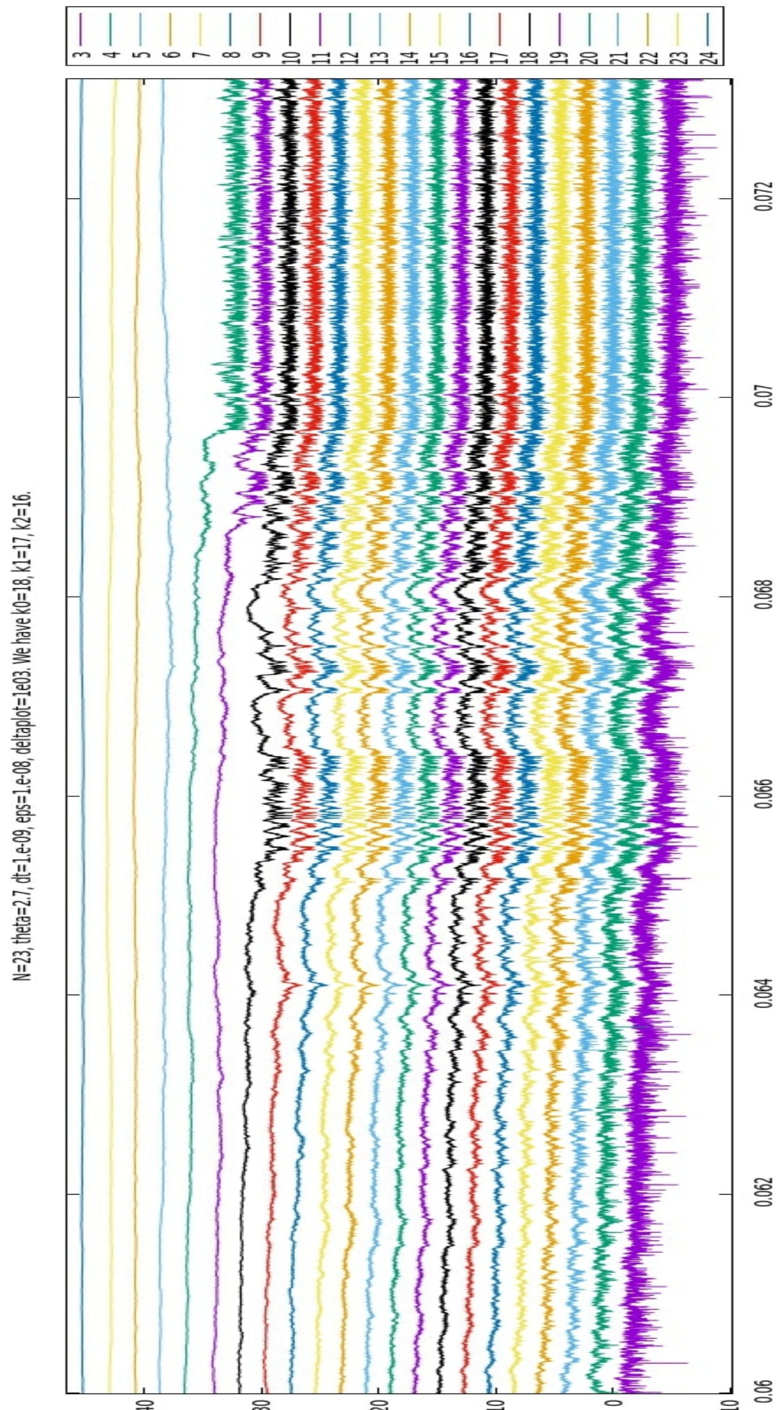


FIGURE 6.5 – Subcritical case

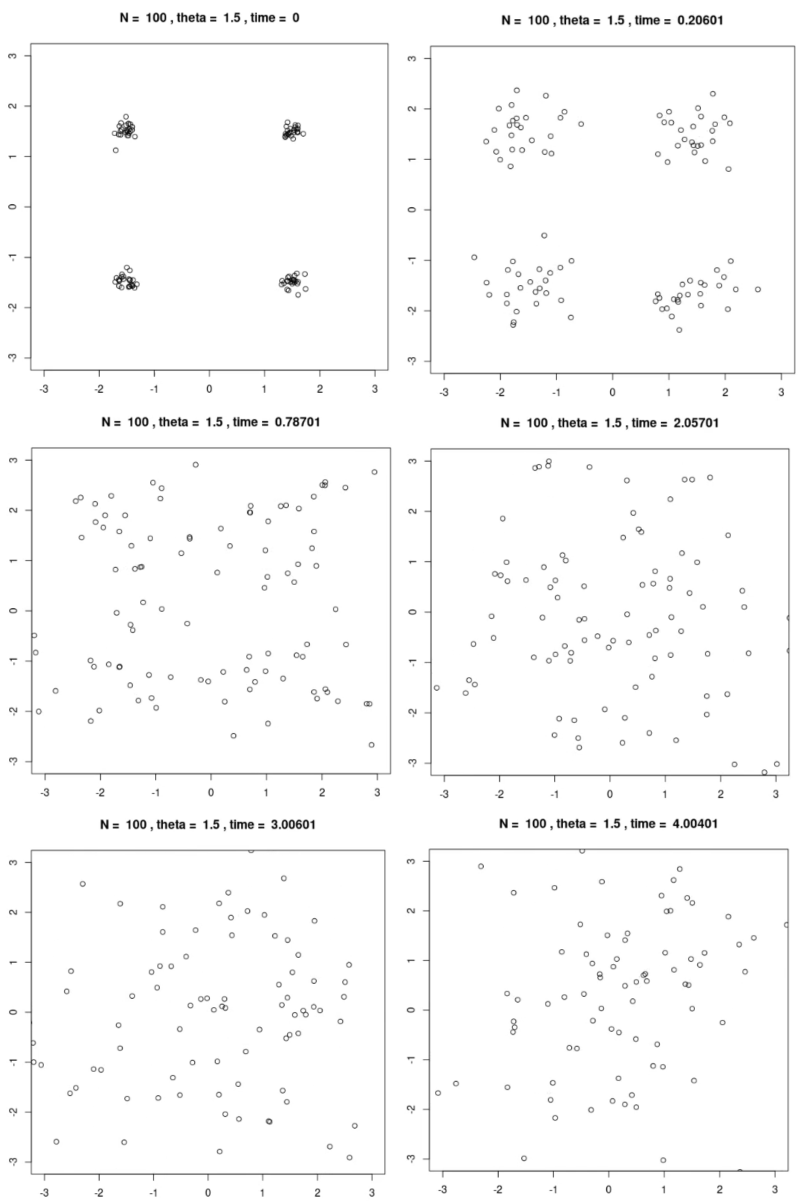
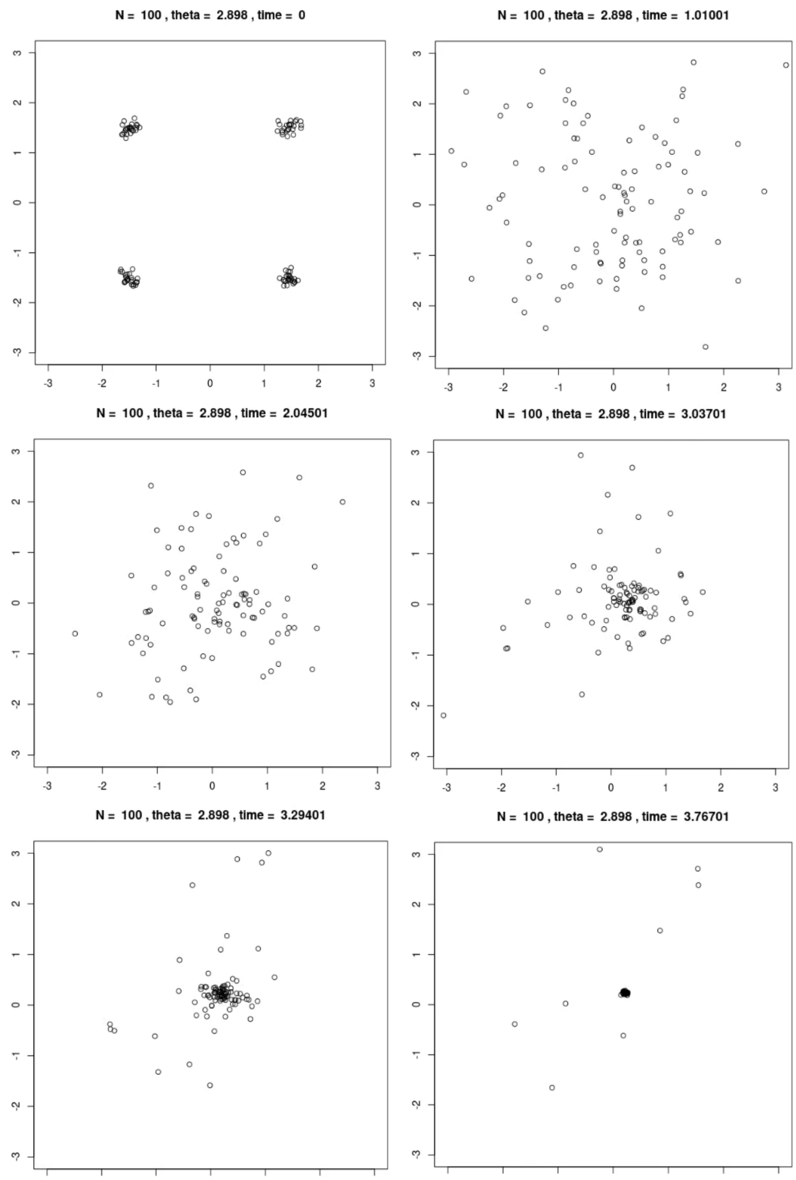


Figure 6.6 – Supercritical case



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